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On the Structure of Spectra of Modulated Travelling Waves

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Abstract. Modulated travelling waves are solutions to reaction–diffusion equations that are time–periodic in an appropriate moving coordinate frame. They may arise through Hopf bifurcations or essential instabilities from pulses or fronts. In this article, a framework for the stability analysis of such solutions is presented: the reaction–diffusion equation is cast as an ill–posed elliptic dynamical system in the spatial variable acting upon time–periodic functions. Using this formulation, points in the resolvent set, the point spectrum, and the essential spectrum of the linearization about a modulated travelling wave are related to the existence of exponential dichotomies on appropriate intervals for the associated spatial elliptic eigenvalue problem. Fredholm properties of the linearized operator are characterized by a relative Morse–Floer index of the elliptic equation. These results are proved without assumptions on the asymptotic shape of the wave. Analogous results are true for the spectra of travelling waves to parabolic equations on unbounded cylinders. As an application, we study the existence and stability of modulated spatially–periodic patterns with long–wavelength that accompany modulated pulses.

1. Introduction

1.1. Motivation

We consider systems of reaction–diffusion equations posed on the real line or on unbounded cylinders. An important class of solutions to such equations are travelling waves that move with a certain constant wave speed c without changing their shape. Hence, travelling waves are stationary in a coordinate frame that moves with velocity c. Another interesting class of solutions are modulated waves that are time–periodic in an appropriate moving frame. Such waves may bifurcate from travelling waves via a Hopf bifurcation (when a pair of isolated complex–conjugate eigenvalues crosses the imaginary axis) or through an essential instability (when a part of the continuous spectrum crosses the imaginary axis [39]). In this article, we investigate the stability properties of modulated waves. Since modulated waves are time–periodic in an appropriate moving frame, they are fixed points of the temporal period map of the

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underlying partial differential equation (PDE) in the moving frame. Their linearized stability is therefore determined by the linearization of the temporal period map about the modulated wave. The reader may wonder why we write an article about the stability issue if all one has to do is to investigate the spectrum of a certain linear operator given by the linearized period map. The reason is that there are often other issues involved besides proving stability of a given primary modulated wave. We give three examples:

First, any comparison with experiments or numerical simulations requires that the underlying unbounded domain is replaced by a bounded domain supplemented with appropriate boundary conditions. It is then important to determine in what way the existence and stability properties of the modulated wave depend upon the boundary conditions. In particular, one has to compare the spectra of the linearized operators posed on two quite different domains (one is compact, the other one is not). For pulses and fronts, such a comparison has been carried out in [5, 43].

A second related issue is to determine the existence and stability of other modulated waves that are related to the primary wave. Suppose, for instance, that the primary wave is localized so that it converges to zero as the spatial variable ξ tends to $\pm\infty$. In this situation, it is then of interest to investigate modulated waves that consist of several well separated copies of the primary wave. Since such waves are not close to the primary wave in any L^p or L^∞ norm, it is not clear how their stability can be investigated using the stability properties of the primary wave. Of particular importance is the stability of modulated spatially–periodic waves that consist of infinitely many equidistant copies of the primary modulated pulse. We emphasize that not even the existence of such waves has been proved previously.

The third issue is related to the presence of essential spectrum. As mentioned before, modulated waves can bifurcate from travelling waves at the onset of instability. In particular, modulated waves bifurcate from pulses near parameter values where a part of the essential spectrum associated with the pulse crosses the imaginary axis [39]. To demonstrate linear stability of the bifurcating modulated pulses, one has to locate the essential spectrum of the modulated wave. This, however, is not sufficient to ensure stability of the modulated wave: it is possible that discrete eigenvalues move out of the essential spectrum and destabilize the modulated wave. For travelling waves, this issue arises naturally when considering conservation laws [13] or dissipative perturbations of integrable PDEs such as the nonlinear Schrödinger equation [16]. The difficulty in locating discrete eigenvalues near the essential spectrum is the lack of Fredholm properties for the linearized operator near the essential spectrum.

In summary, the issues raised above are all related to changing either the domain on which the PDE is posed or the underlying wave (or both). If the domains were close to each other and if the waves of interest were close in some norm, then a regular perturbation analysis could be used to relate the spectra of the relevant linear operators. In all the cases listed above, however, they are either not close or the operators lack the Fredholm properties needed for a perturbation analysis; it is then not clear how to show closeness of the relevant spectra.

The key observation is that all the aforementioned problems are related to changes along the spatial direction. Truncating the real line (or the unbounded cylinder) to a finite but large interval or replacing the underlying wave by several well separated copies of it happens in the spatial direction. In either case, the relevant objects are close to each other locally in the spatial variable ξ but not necessarily uniformly in $\xi \in \mathbb{R}$. In the time variable t, we always have periodicity since we consider modulated waves. The key to taking the changes in the spatial directions into account is to regard the PDE as a dynamical system in the unbounded spatial direction acting on functions that depend on time together with periodic boundary conditions in the time variable. This approach will allow us to address, and successfully resolve, the issues mentioned above. We begin by outlining this strategy in more detail for travelling waves.

The spatial–dynamics approach to elliptic equations has been introduced by Kirchgässner [19, 20] to investigate the existence of stationary waves of small amplitude. Since then, there has been a number of contributions using this approach. Stability of stationary patterns of small amplitude has been studied, for instance, in [6, 28]. In [15], spatial dynamics has been exploited to establish the existence of modulated waves of small amplitude. The stability of modulated waves of small amplitude to equations of Ginzburg–Landau type has recently been studied in [46, 47] using again spatial dynamics.

1.2. Review: spectra of travelling waves

To be specific, consider a reaction–diffusion equation

$$u_t \;=\; Du_{xx} + f(u)\,, \quad x \in \mathbb{R} \tag{1.1}$$

on the real line, where $u \in \mathbb{R}^n$, D is a diagonal matrix with positive entries, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth nonlinearity.

We first consider travelling–wave solutions $u(x,t) = q(x - ct)$ to (1.1). In an appropriate moving coordinate frame $\xi = x - ct$, travelling waves $q(\xi)$ are bounded solutions to the ordinary differential equation (ODE)

$$-cq_\xi \;=\; Dq_{\xi\xi} + f(q)\,. \tag{1.2}$$

In other words, they are stationary, i. e. time–independent, solutions of

$$u_t \;=\; Du_{\xi\xi} + cu_\xi + f(u)\,, \quad \xi \in \mathbb{R}\,. \tag{1.3}$$

As such, the spectral stability of q is determined by the linearization

$$v_t \;=\; Dv_{\xi\xi} + cv_\xi + f_u(q(\xi))v\,, \quad \xi \in \mathbb{R}$$

of (1.3) about q. The associated eigenvalue problem is given by the linear non–autonomous ODE

$$Dv_{\xi\xi} + cv_\xi + f_u(q(\xi))v \;=\; \lambda v \tag{1.4}$$

that can be cast as the first–order ODE

$$\begin{pmatrix} v_\xi \\ w_\xi \end{pmatrix} \;=\; \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\lambda - f_u(q(\xi))) & -cD^{-1} \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} \;=\; A(\xi) \begin{pmatrix} v \\ w \end{pmatrix}. \tag{1.5}$$

Hence, for travelling waves, the eigenvalue problem can be cast as a dynamical system in the spatial variable ξ. There are two related concepts that are relevant when determining the spectrum of the linearization about q.

The first concept are Fredholm properties that can be used to characterize qualitatively different points in the spectrum. Denote by $\mathcal{L}$ the operator that is defined by the left–hand side of (1.4) acting on the space $L^2(\mathbb{R}, \mathbb{C}^n)$. By definition, a complex number λ is in the essential spectrum of $\mathcal{L}$ if either $\mathcal{L} - \lambda$ is not Fredholm or else $\mathcal{L} - \lambda$ is Fredholm with non–zero index. The first case occurs for values of λ where the Fredholm index of $\mathcal{L} - \lambda$ changes; typically, the range of $\mathcal{L} - \lambda$ is then not closed. In the case where $\mathcal{L} - \lambda$ is Fredholm with index zero, it suffices to find the null space of $\mathcal{L} - \lambda$ in order to decide whether λ is in the resolvent set or in the point spectrum of $\mathcal{L}$.

The second concept we shall introduce are exponential dichotomies of the first–order equation (1.5). Note that bounded solutions of (1.5) correspond to eigenfunctions of the operator $\mathcal{L}$. We focus on the situation where any bounded solution to (1.5) actually decays exponentially to zero as ξ tends to $\pm\infty$. Roughly speaking (see Definition 2.1 for a precise statement), we say that (1.5) has exponential dichotomies on $\mathbb{R}^+$ and on $\mathbb{R}^-$ if there are two subspaces $E^{\mathrm{s}}_+(\lambda)$ and $E^{\mathrm{u}}_-(\lambda)$ of $\mathbb{C}^{2n}$ with the following properties: any solution $(v, w)(\xi)$ of (1.5) with $(v, w)(0) \in E^{\mathrm{s}}_+(\lambda)$ decays to zero exponentially as $\xi \to \infty$, while $(v, w)(\xi)$ grows exponentially as $\xi \to \infty$ whenever $(v, w)(0) \notin E^{\mathrm{s}}_+(\lambda)$. Similarly, any solution with $(v, w)(0) \in E^{\mathrm{u}}_-(\lambda)$ decays to zero exponentially as $\xi \to -\infty$, while $(v, w)(\xi)$ grows exponentially as $\xi \to -\infty$ whenever $(v, w)(0) \notin E^{\mathrm{u}}_-(\lambda)$. Finally, we say that (1.5) has an exponential dichotomy on $\mathbb{R}$ provided it has exponential dichotomies on $\mathbb{R}^+$ and on $\mathbb{R}^-$ such that $E^{\mathrm{s}}_+(\lambda) \oplus E^{\mathrm{u}}_-(\lambda) = \mathbb{C}^{2n}$.

The relation between these two concepts is as follows. It has been demonstrated in [14, Appendix to Section 5] and [29, 30] that $\mathcal{L} - \lambda$ is invertible if, and only if, (1.5) has an exponential dichotomy on $\mathbb{R}$. More generally, $\mathcal{L} - \lambda$ is Fredholm if, and only if, (1.5) has exponential dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$. In this situation, the Fredholm index of $\mathcal{L} - \lambda$ is given by

$$\mathrm{ind}(\mathcal{L} - \lambda) \;=\; \dim E^{\mathrm{u}}_-(\lambda) + \dim E^{\mathrm{s}}_+(\lambda) - 2n\,.$$

In fact, exponential dichotomies of (1.5) on $\mathbb{R}$ can be used to construct a Green's function for the operator $\mathcal{L} - \lambda$. In the case where $\mathcal{L} - \lambda$ is Fredholm with index zero, λ is in the point spectrum of $\mathcal{L}$ if, and only if, the intersection of $E^{\mathrm{s}}_+(\lambda)$ and $E^{\mathrm{u}}_-(\lambda)$ is non–trivial, since this intersection is isomorphic to the null space of $\mathcal{L} - \lambda$. Hence, we can detect isolated eigenvalues by monitoring the distance between the stable and unstable subspaces $E^{\mathrm{s}}_+(\lambda)$ and $E^{\mathrm{u}}_-(\lambda)$, respectively, of (1.5) as λ varies in a region in the complex plane where $\mathcal{L} - \lambda$ is Fredholm with index zero. The Evans function $E(\lambda)$ measures precisely the aforementioned distance; see, for instance, [1].

Fredholm properties are important since they can, for instance, be used to carry out a perturbation analysis using Lyapunov–Schmidt reduction that relies on Fredholm's alternative. Such an approach is, however, restricted to operators that are close to each other in norm. The advantage of exponential dichotomies is that they can be used for a perturbation analysis where the operators are close to each other in a much weaker sense. Coming back to the issues that we raised at the beginning of the introduction, we are interested in the stability of waves that consist of several copies of a given primary travelling wave as well as the stability of waves in situations where the essential spectrum touches the imaginary axis. In both cases, dichotomies have been used successfully to determine the stability of waves.

An example for the first situation are periodic waves that accompany pulses. Suppose that $q(\xi)$ is a homoclinic orbit of (1.2). Generically, there exists then a family $p_L(\xi)$ of periodic solutions to (1.2) that are close to the homoclinic orbit $q(\xi)$ in phase space; the periodic solutions are parametrized by their large period $2L$. Given that the pulse $q(\xi)$ is stable, it is then of interest to derive stability criteria for the long–wavelength periodic waves $p_L(\xi)$ on the real line. In the operator norm, however, the linearizations (1.4) about $q(\xi)$ and $p_L(\xi)$ are not close. Still, Gardner [11] showed that the spectra for the pulse and the periodic waves are close. The reason is that the coefficients of the associated ODEs (1.5) are close for $|\xi| \leq L$. In [42], we located the spectrum of the periodic waves accurately using exponential dichotomies. This result allowed us to decide upon the linear stability of spatially periodic waves that accompany a stable pulse.

Examples for the second situation where the essential spectrum touches the imaginary axis arise naturally for dissipative perturbations of integrable PDEs such as the nonlinear Schrödinger equation. As mentioned above, it is possible that discrete eigenvalues move out of the essential spectrum upon adding small perturbations to the operator. Since the essential spectrum is close to the imaginary axis, these discrete eigenvalues may destabilize the travelling wave; see [17] for an example. It is therefore important to detect such eigenvalues. Since they pop out of the essential spectrum where Fredholm properties of $\mathcal{L}$ break down, it is difficult to find these eigenvalues. Often, however, dichotomies can be continued into the essential spectrum; they do not correspond to exponentially decaying solutions but to solutions that are bounded or grow with a small exponential rate. Utilizing dichotomies, it is then possible to detect points in the essential spectrum where discrete eigenvalues can move out upon adding a small perturbation. We refer to [16] for details; see also [13].

1.3. Outline of the approach for modulated waves

In this article, we are interested in the structure of the spectra of modulated waves $\tilde{q}(x,t)$ to (1.3) that satisfy

$$\tilde{q}(x, t+T) \;=\; \tilde{q}(x-cT, t)$$

for some temporal period T and all $t, x \in \mathbb{R}$. Equivalently, we may require that, in an appropriate moving frame given by $\xi = x - ct$, we have $q(\xi, t+T) = q(\xi, t)$ for all $t, \xi \in \mathbb{R}$ where $q(\xi, t) := \tilde{q}(\xi + ct, t)$. The stability properties of modulated waves are determined by the spectrum of the linearized time–T map associated with (1.3). Our strategy is to characterize the points λ in the spectrum of the linearized time–T map by certain properties of the ill–posed first–order system

$$\begin{pmatrix} v_\xi \\ w_\xi \end{pmatrix} \;=\; \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\partial_t + \alpha - f_u(q(\xi,\,\cdot))) & -cD^{-1} \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix},$$

where $\lambda = \mathrm{e}^{\alpha T}$, and (v, w) are T–periodic in t for every $\xi \in \mathbb{R}$. In other words, rather than investigating the temporal period map associated with a modulated wave directly, we employ a dynamical–systems approach by using the spatial variable ξ as an evolution variable.

Our goal is then to present a general framework for studying the spectral stability of waves in the aforementioned situations. In particular, we relate Fredholm properties of the linearized operator to the existence of exponential dichotomies of the associated first–order system. The main problem in establishing such a relation is that the first–order system is ill–posed. Hence, it is not clear whether the stable and unstable spaces $E^{\mathrm{s}}_{+}(\lambda)$ and $E^{\mathrm{u}}_{-}(\lambda)$ exist. In any case, both of them will be infinite–dimensional.

We emphasize that the very same approach is also applicable to travelling waves in parabolic equations

$$u_t \;=\; u_{xx} + \Delta u + f(u)\,, \quad (x, y) \in \mathbb{R} \times \Omega \tag{1.6}$$

on unbounded cylindrical domains with bounded cross–section $\Omega \subset \mathbb{R}^N$ (we remark that the approach works also if Ω is unbounded). Here, Δ is the Laplace operator in the y–variable. In a moving frame, travelling waves are then solutions $u(\xi, y)$ to the elliptic problem

$$u_{\xi\xi} + \Delta u + c u_\xi + f(u) \;=\; 0\,, \quad (\xi, y) \in \mathbb{R} \times \Omega\,.$$

The associated linearized eigenvalue–problem is given by

$$v_{\xi\xi} + \Delta u + c v_\xi + f_u(q(\xi, y))v \;=\; \lambda v$$

that can formally be written as

$$\begin{pmatrix} v_\xi \\ w_\xi \end{pmatrix} \;=\; \begin{pmatrix} 0 & \mathrm{id} \\ -\Delta + \lambda - f_u(q(\xi,\,\cdot\,)) & -c \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}.$$

This first–order system is ill–posed and does not admit a semiflow or evolution. For pulses and fronts to (1.6), Fredholm properties of the linearized operator were established in [48]; see also [24]. Our contribution is a characterization in terms of exponential dichotomies that is, for instance, useful for the numerical calculation of isolated eigenvalues. In addition, our results apply to travelling waves with tails that are periodic in ξ; in fact, we do not use any information about the asymptotic shape of the wave.

Exponential dichotomies can be used efficiently in Lyapunov–Schmidt reductions; they also facilitate the construction of Green's functions to the linearized elliptic equation. In particular, using Lyapunov–Schmidt reduction, an Evans function can be constructed for parabolic equations on unbounded cylinders or for modulated waves, at least locally (e. g. near given points in the point or essential spectrum). Exploiting Galerkin approximations as in Section 4 below, it is also possible to construct a global Evans function so that topological, index–type arguments become applicable.

We demonstrate these ideas by calculating the spectrum of periodic modulated waves with large spatial period that accompany modulated pulses. As a result, we show that time–periodic forcing of pulses may change the interaction between consecutive humps in a wave train drastically: long–wavelength periodic orbits close to the pulse can destabilize under weak time–periodic forcing, depending on their spatial wavelength. This phenomenon is caused by oscillatory behavior at the tails of the periodically–forced pulse and can be interpreted physically as a locking phenomenon. We show that the oscillations at the tails indicate weakly stable time–periodic eigenfunctions of the asymptotic state. On the other hand, we argue that this phenomenon should not be present for pulses in singularly perturbed systems. In particular, we prove that long–wavelength patterns that accompany fast pulses in a periodically excited FitzHugh–Nagumo system are stable.

Modulated waves can emerge via Hopf bifurcations from travelling waves. In [39], we have shown how Hopf bifurcations can be analyzed by casting (1.3) as an ill–posed dynamical system in the spatial variable acting upon time–periodic functions. The advantage of this approach compared with a standard center–manifold reduction is that it is also applicable in the case where the essential spectrum crosses the imaginary axis [39]. The linearized stability of the bifurcating modulated waves has been investigated in [40]. To demonstrate linearized stability, we had to show that discrete eigenvalues do not move out of the essential spectrum; as mentioned above, this proof relies on the fact that Fredholm properties imply the existence of exponential dichotomies; see [40]. Here, we concentrate on the abstract properties of the linearization about a modulated wave that is not necessarily close to a bifurcation point and does not necessarily approach a stationary or periodic pattern.

The paper is organized as follows. The main results on the spectra of modulated travelling waves are stated in Section 2. The corresponding results for travelling waves on cylinders are formulated in Section 3. These results are then proved in Sections 4, 5 and 6. In Section 8, we present applications to the spectra of spatially–periodic waves with long wavelength. Finally, we conclude with a discussion in Section 9.

2. Spectra of modulated waves

2.1. The parabolic equation: temporal dynamics

Suppose that $q(\xi,t)$ is a bounded and smooth modulated wave that satisfies (1.3); in particular, we have $q(\xi,t+T)=q(\xi,t)$. The linearized equation about q is given by

$$v_t \;=\; Dv_{\xi\xi}+cv_\xi+a(\xi,t)v\,, \tag{2.1}$$

where $a(\xi,t):=f_u(q(\xi,t))$ is bounded and smooth.

The nonlinear equation (1.3) is well–posed on a variety of Banach spaces including, for instance, $L^\infty(\mathbb{R},\mathbb{R}^n)$, $C^0_{\mathrm{unif}}(\mathbb{R},\mathbb{R}^n)$, $L^p(\mathbb{R},\mathbb{R}^n)$ or the space $L^p_w(\mathbb{R},\mathbb{R}^n)$ for a given weight function $w(\xi)$. For the last two spaces, we may have to assume that the nonlinearity f satisfies certain growth and sign conditions. The choice of the underlying space for the nonlinear equation (1.3) certainly affects the dynamics of modulated waves in that it restricts their shape and their nonlinear stability properties. In this paper, however, we are mainly interested in spectral stability, i. e. in the linearized equation (2.1). For the sake of clarity, we consider (2.1) on the space $X=L^2(\mathbb{R},\mathbb{C}^n)$ where it is well–posed. Similar results can be derived in other spaces provided certain maximal regularity properties hold.

Associated with (2.1) posed on the space $X=L^2(\mathbb{R},\mathbb{C}^n)$ is the linear evolution operator $\Phi_{t,s}:X\to X$ that maps the ξ–profile $v(\,\cdot\,,s)$ of the solution at time s to the profile $v(\,\cdot\,,t)$ at time t. Note that we have $\Phi_{t+T,s+T}=\Phi_{t,s}$ due to time–periodicity of q. Hence, for the purpose of determining the stability of the linearized equation, it suffices to investigate the period map $\Phi := \Phi_{T,0}$. The eigenvalue equation for Φ is then given by

$$v(\xi,T) \;=\; \lambda v(\xi,0)\,, \quad \xi\in\mathbb{R}\,,$$

where $v(\xi,t)$ satisfies (2.1) for $0<t<T$.

The spectrum of Φ regarded as an operator on X is denoted by $\Sigma\subset\mathbb{C}$. We may divide Σ into two disjoint subsets

$$\Sigma \;=\; \Sigma_{\mathrm{ess}}\,\dot{\cup}\,\Sigma_{\mathrm{point}}\,.$$

The set Σ_{point} denotes the point spectrum of Φ, i.e. the union of eigenvalues λ for which $\Phi-\lambda$ is Fredholm with index zero. Its complement Σ_{ess} in Σ is the essential spectrum. The pure point spectrum Σ_{ppoint} is the set of isolated eigenvalues with finite multiplicity; it is contained in Σ_{point}. Note that the point spectrum may be empty; the essential spectrum is always non–empty since the equation is posed on the unbounded real line. We remark that there are other, slightly different, definitions of the essential spectrum.

2.2. The elliptic formulation: spatial dynamics

We write the eigenvalue problem for Φ formally as a dynamical system in the ξ–variable. We obtain

$$\tilde v_\xi \;=\; \widetilde w\,, \quad \widetilde w_\xi \;=\; D^{-1}(\tilde v_t - c\,\widetilde w - a(\xi,t)\tilde v)$$

together with the boundary conditions

$$\tilde v(\xi,T) \;=\; \lambda\tilde v(\xi,0)\,, \quad \widetilde w(\xi,T) \;=\; \lambda\widetilde w(\xi,0)\,,$$

where $\lambda\in\mathbb{C}$ denotes a prospective eigenvalue of Φ. In order to remove the dependence on λ from the boundary conditions, we introduce the new variables

$$v(\xi,t) \;=\; \mathrm{e}^{-\alpha t}\tilde v(\xi,t)\,, \quad w(\xi,t) \;=\; \mathrm{e}^{-\alpha t}\widetilde w(\xi,t)$$

for $\lambda\neq 0$, where $\alpha\in\mathbb{C}$ is chosen such that $\mathrm{e}^{\alpha T}=\lambda$. In other words, α is the temporal Floquet exponent that belongs to the prospective temporal eigenvalue λ. The transformed equation is given by

$$v_\xi \;=\; w\,, \quad w_\xi \;=\; D^{-1}(v_t+\alpha v - cw - a(\xi,t)v) \tag{2.2}$$

with periodic boundary conditions

$$v(\xi,T) \;=\; v(\xi,0)\,, \quad w(\xi,T) \;=\; w(\xi,0)\,.$$

We consider (2.2) on the Hilbert space $Y=H^{\frac12}(\mathbb{R}/T\mathbb{Z})\times L^2(\mathbb{R}/T\mathbb{Z})$ of T–periodic functions of t with values in $\mathbb{C}^n$. The reader may think of Y as the phase space for (2.2). We write (2.2) in the abstract form

$$\frac{\mathrm{d}}{\mathrm{d}\xi}V \;=\; A(\xi)V \tag{2.3}$$

where $V(\xi) = (v,w)(\xi) \in Y$ for every ξ and

$$A(\xi) \;=\; \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\partial_t + \alpha - a(\xi,t)) & -cD^{-1} \end{pmatrix} \;:\; Y \;\longrightarrow\; Y \tag{2.4}$$

is closed and densely defined with domain $Y^1 = H^1(\mathbb{R}/T\mathbb{Z}) \times H^{\frac{1}{2}}(\mathbb{R}/T\mathbb{Z})$ for every $\xi \in \mathbb{R}$. Note, however, that the initial–value problem $V(0) = V_0$ on Y is ill–posed which can be seen after setting $\alpha = c = 0$, $a = 0$, and $D = \mathrm{id}$: the spectrum of the linear operator $A(\xi)$ on the right–hand side of (2.2) then consists of the points $\pm\sqrt{\mathrm{i}k}$, $k \in \mathbb{Z}$, that have unbounded positive and negative real part.

We say that a function $V = (v,w) \in C^0(J,Y)$ is a solution of (2.2) on an interval J if, for any ξ in the interior of J, $V(\xi)$ is continuous with values in Y^1, differentiable in ξ as a function into Y, and satisfies (2.2) in Y.

Definition 2.1. [32, Section 2.1.] Let $J = \mathbb{R}^+$, $\mathbb{R}^-$ or $\mathbb{R}$. Equation (2.2) is said to have an *exponential dichotomy on* J if there exist positive constants K and η and a strongly continuous family of projections $P : J \to \mathrm{L}(Y)$ such that the following is true.

(a) *Stability.* For any $\zeta \in J$ and $V_0 \in Y$, there exists a solution $\varphi^{\mathrm{s}}(\xi;\zeta)V_0$ of (2.2) that is defined for $\xi \geq \zeta$ in J, is continuous in (ξ,ζ) for $\xi \geq \zeta$ and differentiable in (ξ,ζ) for $\xi > \zeta$, and we have $\varphi^{\mathrm{s}}(\zeta;\zeta)V_0 = P(\zeta)V_0$ as well as

$$|\varphi^{\mathrm{s}}(\xi;\zeta)V_0|_Y \;\leq\; K\mathrm{e}^{-\eta\,|\xi-\zeta|}\,|V_0|_Y$$

for all $\xi \geq \zeta$ such that $\xi, \zeta \in J$.

(b) *Instability.* For any $\zeta \in J$ and $V_0 \in Y$, there exists a solution $\varphi^{\mathrm{u}}(\xi;\zeta)V_0$ of (2.2) that is defined for $\xi \leq \zeta$ in J, is continuous in (ξ,ζ) for $\xi \leq \zeta$ and differentiable in (ξ,ζ) for $\xi < \zeta$, and we have $\varphi^{\mathrm{u}}(\zeta;\zeta)V_0 = (\mathrm{id} - P(\zeta))V_0$ as well as

$$|\varphi^{\mathrm{u}}(\xi;\zeta)V_0|_Y \;\leq\; K\mathrm{e}^{-\eta\,|\xi-\zeta|}\,|V_0|_Y$$

for all $\xi \leq \zeta$ such that $\xi, \zeta \in J$.

(c) *Invariance.* The solutions $\varphi^{\mathrm{s}}(\xi;\zeta)V_0$ and $\varphi^{\mathrm{u}}(\xi;\zeta)V_0$ satisfy

$$\begin{aligned} \varphi^{\mathrm{s}}(\xi;\zeta)V_0 \in \mathrm{R}(P(\xi)) \quad &\text{for all} \quad \xi \;\geq\; \zeta \quad \text{with} \quad \xi\,,\,\zeta \in J\,, \\ \varphi^{\mathrm{u}}(\xi;\zeta)V_0 \in \mathrm{N}(P(\xi)) \quad &\text{for all} \quad \xi \;\leq\; \zeta \quad \text{with} \quad \xi\,,\,\zeta \in J\,. \end{aligned}$$

Finally, we require that the solution operators $\varphi^{\mathrm{s}}(\xi;\zeta)$ and $\varphi^{\mathrm{u}}(\xi;\zeta)$ are linear on Y.

Note that this definition is almost the same as the usual definition of exponential dichotomies for ODEs. The difference is that, due to ill–posedness of the initial–value problem, the existence of solutions in forward or backward time on the range $\mathrm{R}(P(\xi))$ or the null space $\mathrm{N}(P(\xi))$, respectively, is an important part of the definition.

Note also that we assumed differentiability with respect to the initial time ζ. Often, this property is a consequence of differentiability in time; we do not know, however, whether that is always the case. The reason that we require differentiability in the initial time is that we want to prove that the adjoint equation of (2.3) has an exponential dichotomy whenever (2.3) does; see Lemma 5.1. Again, this may hold without having differentiability in ζ, but we do not know how to prove it.

2.3. Relative Morse indices

If the elliptic equation (2.2) has an exponential dichotomy on $\mathbb{R}^+$, $\mathbb{R}^-$ or $\mathbb{R}$, we expect that both the range $\mathrm{R}(P(0))$ and the null space $\mathrm{N}(P(0))$ are infinite–dimensional, since this is true for the principal part of the equation with $a = 0$; see [39]. In order to count dimensions, we introduce the reference equation

$$v_\xi \;=\; w\,, \quad w_\xi \;=\; D^{-1}(v_t + v - cw)\,. \tag{2.5}$$

Note that the time–T map of the associated parabolic equation

$$v_t \;=\; Dv_{\xi\xi} + cv_\xi - v$$

is a contraction. A direct computation using Fourier series expansions in Y shows that (2.5) has an exponential dichotomy on $\mathbb{R}$; see [39] and the example below. It also follows that the associated projections $P(\xi)$ are in fact independent of ξ; we denote them by P_{ref}. Initial values in the range $\mathrm{R}(P_{\mathrm{ref}})$ of P_{ref} lead to solutions of (2.5) that are bounded on $\mathbb{R}^+$; similarly, initial values in the null space $\mathrm{N}(P_{\mathrm{ref}})$ of P_{ref} correspond to solutions that are bounded on $\mathbb{R}^-$.

Theorem 2.2. *Fix $\alpha \in \mathbb{C}$. Suppose that* (2.2) *has an exponential dichotomy on $J = \mathbb{R}^+$ with projections $P(\xi)$ for this value of α; see Definition* 2.1. *The restriction $P(\xi) : \mathrm{R}(P_{\mathrm{ref}}) \to \mathrm{R}(P(\xi))$ is then Fredholm, and its Fredholm index* $\mathrm{ind}(P)$ *is independent of ξ. If the exponential dichotomy $P(\xi)$ is defined on $J = \mathbb{R}$, then the restriction* $(\mathrm{id} - P_{\mathrm{ref}}) : \mathrm{N}(P(\xi)) \to \mathrm{N}(P_{\mathrm{ref}})$ *is also Fredholm with the same index.*

Definition 2.3. Fix $\alpha \in \mathbb{C}$, and suppose that (2.2) has an exponential dichotomy on $\mathbb{R}^+$ with projections $P_+(\xi)$. The *relative Morse index of* (2.2) *at* $+\infty$ is then defined by $i_+ = \mathrm{ind}(P_+)$; it is the Fredholm index of $P_+(\xi) : \mathrm{R}(P_{\mathrm{ref}}) \to \mathrm{R}(P_+(\xi))$. Analogously, if (2.2) has an exponential dichotomy on $\mathbb{R}^-$ with projections $P_-(\xi)$, then the associated *relative Morse index at* $-\infty$ is defined as the Fredholm index $i_- = \mathrm{ind}(P_-)$ of the operator $(\mathrm{id} - P_{\mathrm{ref}}) : \mathrm{N}(P_-(\xi)) \to \mathrm{N}(P_{\mathrm{ref}})$.

Recall that the indices $i_\pm$ are well–defined by Theorem 2.2. Note, however, that they depend upon the choice of the reference equation. Choosing a different reference equation might lead to a shift of all relative Morse indices by the same integer. This is the reason why we call the indices $i_\pm$ the relative Morse indices. The absolute Morse index, i. e. the dimension of $\mathrm{R}(P(\xi))$ or $\mathrm{N}(P(\xi))$, would be infinite and of poor interest. We refer to Section 2.6 for an equivalent definition for operators with asymptotically periodic coefficients that uses spectral–flow ideas from Floer theory.

Example 2.4. Suppose that $a(\xi, t) = a(\xi)$ does not depend upon t and $a(\xi) \to a_\pm$ as $\xi \to \pm\infty$. We may then restrict (2.2) to the $2n$–dimensional subspace of Y that consists of all t–independent functions. On this subspace, (2.2) is an ODE, and the existence of exponential dichotomies on $\mathbb{R}^\pm$ is equivalent to the hyperbolicity of the asymptotic equations where $a(\xi)$ is replaced by the constants $a_\pm$. Hyperbolicity of the asymptotic ODEs reduces to the condition

$$\det\left(Dk^2 - c\mathrm{i}k - \alpha + a_\pm\right) \;\neq\; 0$$

for all $k \in \mathbb{R}$. Similarly, for any $l \in \mathbb{Z}$, we may consider solutions of (2.2) that are of the form $(u, v)(\xi)\, \mathrm{e}^{2\pi \mathrm{i} l t/T}$. The resulting ODE on $\mathbb{C}^{2n}$ has dichotomies if

$$\det\big(Dk^2 - c\mathrm{i}k - \alpha + a_\pm + 2\pi\mathrm{i}l/T\big) \;\neq\; 0$$

for all $k \in \mathbb{R}$. In fact, the existence of dichotomies for the full problem on the phase space Y is equivalent to the condition that the ODEs on the Fourier subspaces have exponential dichotomies for all l; the relevant computations can be found in [39, Lemma 3.1].

2.4. Exponential dichotomies and elliptic boundary–value problems

In this section, we seek solutions to the eigenvalue problem (2.2) as elements in the null space of the operator

$$\begin{aligned} \mathcal{T} : H^1(\mathbb{R}, Y) \cap L^2\big(\mathbb{R}, Y^1\big) \;&\longrightarrow\; L^2(\mathbb{R}, Y) \\ (v, w) \;&\longmapsto\; \big(v_\xi - w, w_\xi - D^{-1}(v_t + \alpha v - cw - a(\xi,t)v)\big)\,. \end{aligned} \tag{2.6}$$

Recall the definitions $Y = H^{\frac12}\big(S^1\big) \times L^2\big(S^1\big)$ and $Y^1 = H^1\big(S^1\big) \times H^{\frac12}\big(S^1\big)$ of the phase space Y and the domain Y^1, respectively, of the closed operator $A(\xi): Y \to Y$ that has been defined in (2.4). Here, $S^1 = \mathbb{R}/T\mathbb{Z}$. We often write

$$\mathcal{T} \;=\; \frac{\mathrm{d}}{\mathrm{d}\xi} - A(\xi)\,.$$

Note that the adjoint operator $A(\xi)^*$ exists and is again closed and densely defined with the same domain Y^1 in Y; see Section 6.2 for an explicit representation of $A(\xi)^*$. Hence, we can consider the adjoint equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi} V \;=\; -A(\xi)^* V\,. \tag{2.7}$$

Hypothesis (U1). If $(v, w)(\xi)$ is a solution on $\mathbb{R}$, bounded uniformly in ξ with values in Y, of either (2.2) or its adjoint equation (2.7) such that $(v, w)(\xi_0) = 0$ for some $\xi_0 \in \mathbb{R}$, then $(v, w)(\xi)$ vanishes identically.

In other words, we need a weak uniqueness property of the ill–posed Cauchy problem associated with (2.2) and its adjoint equation (2.7).

Remark 2.5. Our regularity assumptions on the coefficients $a(\xi, t)$ actually imply that Hypothesis (U1) is always met. Indeed, parabolic regularity ensures that bounded solutions actually belong to

$$W^{2,1}_{2,\mathrm{loc}}\big(\mathbb{R} \times S^1\big) \;=\; \big\{u;\; u, u_t, u_x, u_{xx} \in L^2(Q) \text{ for any compact } Q \subset \mathbb{R} \times S^1\big\}\,.$$

Hence, once $V = (v, w) \in \mathrm{N}(\mathcal{T})$, the hypotheses of [7, Theorem 2.1] are satisfied. As a consequence of [7, Theorem 2.1], we have that either $v(x, t) = 0$ vanishes identically or else there is an integer $m \geq 0$ such that

$$\begin{aligned} \lim_{\epsilon \to 0} \epsilon^{-m} v\big(x_0 + \epsilon y, \epsilon^2 s\big) \;&=\; h(y, s)\,, \\ \lim_{\epsilon \to 0} \epsilon^{-m+1} v_x\big(x_0 + \epsilon y, \epsilon^2 s\big) \;&=\; h_y(y, s) \end{aligned} \tag{2.8}$$

uniformly in (y, s) in bounded subsets of $\mathbb{R} \times \mathbb{R}$. Furthermore, it follows from [7, Section 3] that $h(y, -1)$ is then a Hermite polynomial. In particular, $h(y, -1)$ has only simple zeros.

Suppose therefore that $V = (v, w) \in \mathrm{N}(\mathcal{T})$ with $v(x_0, \cdot\,) = v_x(x_0, \cdot\,) = 0$, then either $V = 0$ or else

$$\begin{aligned} \lim_{\epsilon\to 0} \epsilon^{-m} v\big(x_0, \epsilon^2 s\big) &= 0 = h(0, s)\,, \\ \lim_{\epsilon\to 0} \epsilon^{-m+1} v_x\big(x_0, \epsilon^2 s\big) &= 0 = h_y(0, s) \end{aligned}$$

upon setting $y = 0$ in (2.8). This, however, is not possible since $h(y, -1)$ has only simple zeros. Thus, we conclude that $V = 0$.

We then have the following theorem.

Theorem 2.6. *Fix $\alpha \in \mathbb{C}$. Assume that* (U1) *is met. The operator $\mathcal{T}$ is Fredholm if, and only if,* (2.2) *has exponential dichotomies on $\mathbb{R}^+$ and on $\mathbb{R}^-$. If $\mathcal{T}$ is Fredholm, its index is given by* $\mathrm{ind}(\mathcal{T}) = i_- - i_+$, *where $i_\pm$ are the relative Morse indices of* (2.2) *at $\pm\infty$ according to Definition* 2.3. *Furthermore, the operator $\mathcal{T}$ is invertible if, and only if,* (2.2) *has an exponential dichotomy on $\mathbb{R}$.*

Palmer [29] proved this theorem when $Y^1 = Y = \mathbb{R}^n$. For evolutionary equations that admit semiflows, similar results have been demonstrated, for instance, in [30, 22]. In the context of elliptic equations, however, the proof of Theorem 2.6 requires the construction of a Green's function that is carried out in Section 5.3 below.

The spaces H^1 and L^2 in the independent variable $\xi \in \mathbb{R}$ are somewhat arbitrary. The property we need is a maximal–regularity result for solutions to the inhomogeneous linearized equation.

We emphasize that the first–order operator $\mathcal{T}$ is Fredholm (invertible) if, and only if, the second–order operator

$$\begin{aligned} \widetilde{\mathcal{T}} \;:\; H^2(\mathbb{R}, Y) \cap L^2\big(\mathbb{R}, Y^1\big) &\longrightarrow L^2(\mathbb{R}, Y) \\ v &\longmapsto Dv_{\xi\xi} - v_t - \alpha v + cv_\xi + a(\xi, t)v \end{aligned}$$

is Fredholm (invertible), and their Fredholm indices are equal. This is a consequence of the proof of Theorem 2.8 below in Section 6.1; see in particular the transformation from (6.4) to (6.6) and vice versa.

Remark 2.7. As we shall see in the proof, Hypothesis (U1) is only needed for the "only if"–part. The existence of exponential dichotomies always implies Fredholm properties.

2.5. Spectra of modulated travelling waves

The next theorem gives the structure of the spectrum associated with the linearization about a modulated travelling wave.

Theorem 2.8. *Let $\lambda \in \mathbb{C}$ with $\lambda \neq 0$ and choose $\alpha \in \mathbb{C}$ so that $\mathrm{e}^{\alpha T} = \lambda$. Assume that* (U1) *is met. We then have the following alternatives.*

i) λ *is in the resolvent set of* Φ *if, and only if,* (2.2) *has an exponential dichotomy on* $\mathbb{R}$.

ii) λ *is in the point spectrum* $\Sigma_{\rm point}$ *if, and only if,* (2.2) *has exponential dichotomies on* $\mathbb{R}^+$ *and* $\mathbb{R}^-$ *with the same relative Morse index and* $\dim \mathrm{N}(\mathcal{T}) > 0$.

iii) λ *is in the essential spectrum* $\Sigma_{\rm ess}$ *if either* (2.2) *does not have exponential dichotomies on* $\mathbb{R}^+$ *or* $\mathbb{R}^-$ *or else if it does but the relative Morse indices on* $\mathbb{R}^+$ *and* $\mathbb{R}^-$ *differ.*

Finally, the pure point spectrum is characterized as follows. Let $C \subset \mathbb{C}$ *be a connected component of* $\mathbb{C} \setminus \Sigma_{\rm ess}$*; in particular,* $\Phi - \lambda$ *is Fredholm with index zero for any* $\lambda \in C$. *If there is a point* λ_0 *such that* $\Phi - \lambda_0$ *is invertible, then* $C \cap \Sigma_{\rm point} \subset \Sigma_{\rm ppoint}$.

Note that $\lambda = 0$, which is the only complex number excluded in the above theorem, belongs to the essential spectrum. In fact, $\lambda = 0$ is an accumulation point of $\Sigma_{\rm ess}$.

For $\lambda \in \Sigma_{\rm ppoint}$, choose the temporal Floquet exponent α so that $\lambda = \mathrm{e}^{\alpha T}$, and consider the operator $\mathcal{T}$ for this value of α. It is then a consequence of the proof of Theorem 2.8 in Section 6.3 that $\dim \mathrm{N}(\mathcal{T}) = \dim \mathrm{N}(\Phi - \lambda)$ so that the geometric multiplicity of λ is determined by $\mathcal{T}$. A similar statement is true for the algebraic multiplicity of λ as an eigenvalue of Φ. We define the multiplicity of α as follows: Assume that $\mathrm{N}(\mathcal{T}) = \mathrm{span}\{U_1(x)\}$. We say that α has multiplicity ℓ if there are solutions $U_j(x)$ to

$$\frac{\mathrm{d}}{\mathrm{d}\xi} U_j \;=\; A(\xi) U_j + B U_{j-1}$$

for $j = 2, \ldots, \ell$ but no solution to $U' = A(x)U + BU_\ell$, where

$$B \;=\; \begin{pmatrix} 0 & 0 \\ D^{-1} & 0 \end{pmatrix}.$$

In general, we say that α has multiplicity ℓ if the sum of the multiplicities of a maximal set of linearly independent elements in $\mathrm{N}(\mathcal{T})$ is equal to ℓ. We then have that algebraic and geometric multiplicities of individual eigenvalues of Φ and the corresponding Floquet exponents are equal. In other words, the Jordan structure of an eigenvalue λ of Φ is determined by properties of the associated operator $\mathcal{T}$. The proof is not difficult but tedious, and we omit it. We refer to the proof of Theorem 8.4 in Section 8.1 for a partial proof; see also [12, 36] for related results.

2.6. Asymptotically constant or periodic coefficients

In this section, we consider the linearized equation

$$v_t \;=\; D v_{\xi\xi} + c v_\xi + a(\xi, t) v$$

in the case where $a(\xi, t)$ is asymptotically constant or periodic in ξ.

Hypothesis (P). Assume that there are two differentiable functions $a_\pm(\xi, t)$ and non–zero constants $p_\pm$ that satisfy

$$a_\pm(\xi + p_\pm, t) \;=\; a_\pm(\xi, t)\,, \quad a_\pm(\xi, t+T) \;=\; a_\pm(\xi, t)$$

for all ξ and t so that $|a(\xi, t) - a_\pm(\xi, t)| \to 0$ as $\xi \to \pm\infty$.

Of interest are then the asymptotic elliptic equations

$$v_\xi \;=\; w\,, \quad w_\xi \;=\; D^{-1}(v_t + \alpha v - cw - a_\pm(\xi,t)v) \tag{2.9}$$

with $(v,w) \in Y$.

We have to replace Hypothesis (U1) by the following stronger Hypothesis (U2).

Hypothesis (U2). If $(v,w)(\xi)$ is a solution on $[\xi_0,\infty)$ or on $(-\infty,\xi_0]$, bounded uniformly in ξ with values in Y, of either (2.2) or its adjoint equation such that $(v,w)(\xi_0) = 0$, then $(v,w)(\xi)$ vanishes identically.

Remark 2.9. Similarly to Hypothesis (U1), the uniqueness assumption in Hypothesis (U2) is always satisfied. First observe that a solution on $[\xi_0,\infty)$ can be extended to a solution in $W^{2,1}_{2,\mathrm{loc}}(\mathbb{R}\times S^1)$, setting $V(x) \equiv 0$ for all $x < 0$. We may now follow the arguments in Remark 2.5, and conclude that $V(x)$ vanishes for all $x \in \mathbb{R}$.

Proposition 2.10. *Let $\lambda \in \mathbb{C}$ with $\lambda \neq 0$, choose $\alpha \in \mathbb{C}$ so that $\lambda = \mathrm{e}^{\alpha T}$, and consider* (2.2) *for this value of α. Assume that* (P) *and* (U2) *are met. The following is then true. Equation* (2.2) *does not admit exponential dichotomies on $\mathbb{R}^+$ if, and only if,* (2.9) *for $a_+(\xi,t)$ has a spatial Floquet exponent on the imaginary axis. Similarly, the non–existence of dichotomies of* (2.2) *on $\mathbb{R}^-$ and the existence of a purely imaginary spatial Floquet exponent to* (2.9) *for $a_-(\xi,t)$ are equivalent.*

Here, by definition, (2.9) for $a_+(\xi,t)$ has a spatial Floquet exponent on the imaginary axis if there is a solution $U(\xi) = (v,w)(\xi)$ to (2.9) such that $U(p_+) = \mathrm{e}^{\mathrm{i}\beta}U(0)$ for some $\beta \in \mathbb{R}$.

Remark 2.11. It is a consequence of [27, Theorem 2.3] and Theorem 2.6 that (2.9) has a purely imaginary spatial Floquet exponent if, and only if, it has a solution that is bounded on $\mathbb{R}$. Note that this statement is not quite as obvious as it may sound since we need the existence of exponential dichotomies (i.e. a Green's function) for (2.9) to demonstrate its validity.

Corollary 2.12. *If the Hypotheses* (P) *and* (U2) *are met, then the spectra Σ_2 and Σ_∞ of the operator Φ considered on $X = L^2(\mathbb{R},\mathbb{C}^n)$ and on $C^0_{\mathrm{unif}}(\mathbb{R},\mathbb{C}^n)$, respectively, are the same.*

If the coefficients $a(\xi,t)$ are asymptotically constant in ξ, i.e., if $a_\pm(\xi,t) = a_\pm(t)$, then the relative Morse index can be computed via a linear homotopy to the reference equation: it is given by the number of eigenvalues, counted with multiplicity, of the operator

$$\begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\partial_t + \alpha v - a_\pm(t)v) & -cD^{-1} \end{pmatrix}$$

on the space Y that cross the imaginary axis during the homotopy.

Similarly, for asymptotically periodic coefficients, the relative Morse index can be computed by counting the spatial Floquet multipliers of the linearization

$$\begin{pmatrix} u_\xi \\ v_\xi \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\partial_t + \alpha v - a_\pm(\xi,t)v) & -cD^{-1} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

that cross the unit circle during the homotopy. This procedure is reminiscent of the construction of Floer homology; see, for instance, [33].

3. Spectra of travelling waves to parabolic equations on unbounded cylinders

In this section, we investigate travelling waves to the parabolic equation

$$u_t = u_{xx} + \Delta u + f(u), \quad (x,y) \in \mathbb{R} \times \Omega \tag{3.1}$$

on unbounded cylindrical domains where the bounded cross–section $\Omega \subset \mathbb{R}^m$ has a smooth boundary. Here, Δ is the Laplace operator in the y–variable, and we impose appropriate boundary conditions such as Neumann or Dirichlet conditions on $\partial\Omega$. The operator Δ is then a sectorial operator in the space $X = L^2(\Omega)$, and we denote its dense domain by X^1; also, the associated fractional power spaces are denoted by X^α. For Dirichlet and Neumann boundary conditions on $\partial\Omega$, we have $X^1 = H^2(\Omega) \cap H^1_0(\Omega)$ and $X^1 = \{u \in H^2(\Omega);\ \partial_\nu u|_{\partial\Omega} = 0\}$, respectively. In a moving frame, travelling waves to (3.1) satisfy the elliptic problem

$$u_{\xi\xi} + \Delta u + cu_\xi + f(u) = 0, \qquad (\xi, y) \in \mathbb{R} \times \Omega.$$

The linearization of (3.1), in a moving frame, about a travelling wave $q(\xi, y)$ is

$$\mathcal{L}v = v_{\xi\xi} + \Delta u + cv_\xi + f_u(q(\xi,y))v.$$

We are interested in the operator

$$\begin{aligned} \mathcal{T} : H^1(\mathbb{R}, Y) \cap L^2(\mathbb{R}, Y^1) &\longrightarrow L^2(\mathbb{R}, Y) \\ (v, w) &\longmapsto (v_\xi - w, w_\xi + \Delta v - \lambda v + f_u(q(\xi,y))v + cw) \end{aligned} \tag{3.2}$$

associated with the eigenvalue problem $(\mathcal{L} - \lambda)v = 0$. Here, $Y = X^{\frac{1}{2}} \times X$ and $Y^1 = X^1 \times X^{\frac{1}{2}}$. The equation $\mathcal{T}(v, w) = 0$, written as a first–order equation, is given by

$$\begin{pmatrix} v_\xi \\ w_\xi \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ -\Delta + \lambda - f_u(q(\xi,\cdot)) & -c \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}. \tag{3.3}$$

We again need uniqueness of the ill–posed Cauchy problem associated with (3.3) and its adjoint equation, where the latter is defined as in (2.7).

Hypothesis (U3). If $(v, w)(\xi)$ is a solution on $\mathbb{R}$, bounded uniformly in ξ with values in Y, of either (3.3) or its adjoint equation such that $(v, w)(\xi_0) = 0$, then $(v, w)(\xi)$ vanishes identically.

We remark that Hypothesis (U3) is satisfied whenever $f : X^{\frac{1}{2}} \to X$ is analytic.

Theorem 3.1. *Fix $\lambda \in \mathbb{C}$. Assume that* (U3) *is met. The operator $\mathcal{T}$ is Fredholm if, and only if,* (3.3) *has exponential dichotomies on $\mathbb{R}^+$ and on $\mathbb{R}^-$. Furthermore, the operator $\mathcal{T}$ is invertible if, and only if,* (3.3) *has an exponential dichotomy on $\mathbb{R}$.*

We emphasize that, in contrast to the results in [48, 32], there are no assumptions on the asymptotic behavior of the wave as $\xi \to \pm\infty$.

Fix $\lambda \in \mathbb{C}$, then $\mathcal{T}$ is Fredholm if, and only if, $\mathcal{L} - \lambda$ is Fredholm, with the same index. Furthermore, the algebraic and geometric multiplicity of an eigenvalue λ of the operator $\mathcal{L}$ are determined by the operator $\mathcal{T}$, used with that value of λ.

If the travelling wave $q(\xi, y)$ has tails that are asymptotically constant or periodic in ξ, then the characterization of the essential spectrum given in Proposition 2.10 is also true for (3.1).

4. Proof of Theorem 2.2

First, assume that (2.2) has an exponential dichotomy on $\mathbb{R}^+$. We use Galerkin approximations to reduce the elliptic equation to a finite–dimensional problem. Functions in Y can be represented by their Fourier series. We denote by Q_m the L^2–orthogonal projection that assigns the Fourier series truncated at order m to a function in $L^2(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^n)$:

$$Q_m u \;=\; \sum_{k=-m}^{m} \mathrm{e}^{2\pi \mathrm{i} k t/T} u_k\,,$$

where $u_k \in \mathbb{C}^n$ is defined by

$$u_k \;=\; \frac{1}{T}\int_0^T u(t)\mathrm{e}^{-2\pi \mathrm{i} k t/T}\mathrm{d}t$$

for any $k \in \mathbb{Z}$. Consider the approximating equation

$$\text{(4.1)} \qquad v_\xi \;=\; w\,, \quad w_\xi \;=\; D^{-1}(v_t + v - cw - Q_m(a(\xi,\cdot\,) - \alpha + 1)v)\,.$$

Since $a(\xi, t)$ is bounded and smooth and $Q_m v \to v$ as $m \to \infty$, it is easy to see that the operator

$$H^{\frac{1}{2}}(\mathbb{R}/T\mathbb{Z}) \;\longrightarrow\; L^2(\mathbb{R}/T\mathbb{Z})\,, \quad v \;\longmapsto\; (\mathrm{id} - Q_m)(a(\xi,\cdot\,) - \alpha + 1)v$$

converges to zero in norm as $m \to \infty$. Hence, the approximating equation (4.1) is a small perturbation of the full equation (2.2).

On account of the robustness of exponential dichotomies [32], the approximating equation also has exponential dichotomies provided $m \geq m_0$ is sufficiently large. We denote the corresponding projections by $P_m(\xi)$. Again by robustness, $P_m(\xi) \to P(\xi)$ in $\mathrm{L}(Y)$, uniformly in ξ, for $m \to \infty$; though not needed here, we remark that the proof in [32] even gives compactness of the difference $P_m(\xi) - P(\xi)$.

The operator $\mathrm{diag}(Q_m, Q_m)$ acts diagonally on $Y = H^{\frac{1}{2}} \times L^2$. Let $Y = Y_m \oplus Y_m^\perp$ where $Y_m := \mathrm{diag}(Q_m, Q_m)Y$ and $Y_m^\perp := (\mathrm{id} - \mathrm{diag}(Q_m, Q_m))Y$ are the range and null space, respectively, of $\mathrm{diag}(Q_m, Q_m)$ on Y. Note that the space Y_m has dimension $2(2m+1)n$. The approximating equation (4.1) restricted to $Y_m^\perp$ decouples

$$v_\xi \;=\; w\,, \quad w_\xi \;=\; D^{-1}(v_t + v - cw)\,. \tag{4.2}$$

Hence, (4.2) and the reference equation (2.5) restricted to $Y_m^\perp$ coincide. Note that it is here where we used that the wave speeds c appearing in (2.5) and (4.2) are equal.

Collecting these findings, we see that the projection of the dichotomy to the approximating equation (4.1) has a tridiagonal structure

$$P_m(\xi) \;=\; \begin{pmatrix} P_m^{(11)}(\xi) & P_m^{(12)}(\xi) \\ 0 & P_m^{(22)}(\xi) \end{pmatrix} \;:\; Y_m \oplus Y_m^\perp \;\longrightarrow\; Y_m \oplus Y_m^\perp\,.$$

The projection P_{ref} of the reference equation is diagonal

$$P_{\mathrm{ref}} \;=\; \begin{pmatrix} P_{\mathrm{ref}}^{(11)} & 0 \\ 0 & P_{\mathrm{ref}}^{(22)} \end{pmatrix}.$$

Since the equations (4.2) and the reference equation (2.5) coincide on $Y_m^\perp$, we conclude that $\mathrm{R}\big(P_{\mathrm{ref}}^{(22)}\big) = \mathrm{R}\big(P_m^{(22)}(\xi)\big)$, and both ranges are equal to the set of initial values to bounded solutions on $Y_m^\perp$ for $\xi \geq 0$. But then we obtain

$$P_m(\xi)\big|_{\mathrm{R}(P_{\mathrm{ref}})} \;=\; P_m(\xi)\big|_{\mathrm{R}(P_{\mathrm{ref}}^{(11)})\times\mathrm{R}(P_{\mathrm{ref}}^{(22)})} \;=\; \begin{pmatrix} P_m^{(11)}(\xi) & P_m^{(12)}(\xi) \\ 0 & \mathrm{id} \end{pmatrix}$$

on $\mathrm{R}\big(P_{\mathrm{ref}}^{(11)}\big) \times \mathrm{R}\big(P_{\mathrm{ref}}^{(22)}\big)$, and the restriction $P_m(\xi) : \mathrm{R}(P_{\mathrm{ref}}) \to \mathrm{R}(P_m(\xi))$ is therefore Fredholm as a finite–dimensional extension of the identity with index independent of ξ.

Next, we prove that $P(\xi) : \mathrm{R}(P_{\mathrm{ref}}) \to \mathrm{R}(P(\xi))$ is a Fredholm operator with the same index. Note that

$$P(\xi)(\mathrm{id} + (P_m(\xi) - P(\xi))) \;=\; (\mathrm{id} + (P(\xi) - P_m(\xi)))P_m(\xi)\,.$$

Since $P_m(\xi) \to P(\xi)$ in norm, it follows that

$$P(\xi) \;=\; (\mathrm{id} + (P(\xi) - P_m(\xi)))\, P_m(\xi)\, (\mathrm{id} + (P_m(\xi) - P(\xi)))^{-1}\,.$$

Hence, $P(\xi)$ is the composition of the Fredholm operator $P_m(\xi)$ with invertible operators and therefore itself Fredholm with the same index. Since $P(\xi)$ and $P_m(\xi)$ are close uniformly in ξ, the above argument applies to all values of ξ and proves that $P(\xi)$ is a Fredholm operator for any ξ and its index is independent of ξ.

This proves the first part of Theorem 2.2. The statements about exponential dichotomies on $J = \mathbb{R}^-$ are proved analogously. It remains to show that the indices of $P_+(\xi) : \mathrm{R}(P_{\mathrm{ref}}) \to \mathrm{R}(P_+(\xi))$ and $(\mathrm{id} - P_{\mathrm{ref}}) : \mathrm{N}(P_-(\xi)) \to \mathrm{N}(P_{\mathrm{ref}})$ coincide whenever (2.2) has an exponential dichotomy on $\mathbb{R}$. To prove this claim, we use the expression for $P_m(\xi)$ and the analogous expression for the approximating dichotomy on $\mathbb{R}^-$. The indices at $\pm\infty$ can then be reduced to indices of $P_m^{(11)}$ and $P_{\mathrm{ref}}^{(11)}$ and the analogous expressions on $\mathbb{R}^-$. The claim is then a straightforward consequence of the corresponding results for finite–dimensional spaces. This completes the proof of the theorem.

Remark 4.1. In [32, Equation (3.20)], we showed that, given an exponential dichotomy with projection $P(\xi)$ on $\mathbb{R}^+$, we can construct a new dichotomy to any given prescribed closed subspace $E_+^{\mathrm{u}}(0)$ that satisfies $\mathrm{R}(P(0)) \oplus E_+^{\mathrm{u}}(0) = Y$. In other words, only the range of the projection to a dichotomy on $\mathbb{R}^+$ is uniquely determined, whereas the null space is arbitrary. By a suitable choice of this null space for $P_m(\xi)$, we can achieve that $P_{\mathrm{ref}}^{(22)} = P_m^{(22)}(\xi)$. The difference $P_{\mathrm{ref}} - P_m(\xi)$ is then compact for any ξ. Since this difference converges in norm to $P_{\mathrm{ref}} - P(\xi)$, we see that $P_{\mathrm{ref}} - P(\xi)$ is also compact.

5. Proof of Theorem 2.6

5.1. Dichotomies for the adjoint equation

As before, we write (2.2) in the more compact form $V_\xi = A(\xi)V$. The adjoint equation is then abstractly given by $W_\xi = -A(\xi)^*W$ where the adjoint of the closed and densely defined operators $A(\xi)$ is taken with respect to the inner product in Y. We refer to Section 6.2 for a more explicit representation of $A(\xi)^*$.

Lemma 5.1. *Suppose that* (2.2) *has an exponential dichotomy on J with constants K and η and evolution operators $\varphi^{\mathrm{s,u}}$. The adjoint equation then has an exponential dichotomy with the same constants and evolution operators $\hat{\varphi}^{\mathrm{s,u}}$ given by*

$$\hat{\varphi}^{\mathrm{s}}(\xi;\zeta) \;=\; \varphi^{\mathrm{u}}(\zeta;\xi)^*\,, \quad \hat{\varphi}^{\mathrm{u}}(\xi;\zeta) \;=\; \varphi^{\mathrm{s}}(\zeta;\xi)^*\,,$$

where the adjoint is again taken with respect to the inner product in Y.

Proof. We compute

$$\begin{aligned} 0 &= \frac{\partial}{\partial\tau}\,\varphi^{\mathrm{s,u}}(\xi;\zeta)V \\ &= \frac{\partial}{\partial\tau}\,(\varphi^{\mathrm{s,u}}(\xi;\tau)\varphi^{\mathrm{s,u}}(\tau;\zeta)V) \\ &= \frac{\partial}{\partial\tau}\,(\varphi^{\mathrm{s,u}}(\xi;\tau))\varphi^{\mathrm{s,u}}(\tau;\zeta)V + \varphi^{\mathrm{s,u}}(\xi;\tau)\,\frac{\partial}{\partial\tau}\,\varphi^{\mathrm{s,u}}(\tau;\zeta)V \\ &= \frac{\partial}{\partial\tau}\,(\varphi^{\mathrm{s,u}}(\xi;\tau))\varphi^{\mathrm{s,u}}(\tau;\zeta)V + \varphi^{\mathrm{s,u}}(\xi;\tau)A(\tau)\varphi^{\mathrm{s,u}}(\tau;\zeta)V\,. \end{aligned}$$

Passing to the limit $\zeta \to \tau$, we obtain

$$\frac{\partial}{\partial\tau}\,\varphi^{\mathrm{s,u}}(\xi;\tau)V \;=\; -\varphi^{\mathrm{s,u}}(\xi;\tau)A(\tau)V\,.$$

Taking the adjoint proves the Lemma. □

As mentioned in Section 2.2, the statement of the lemma could be true without assuming differentiability with respect to the initial time; see, for instance, [14, Proof of Theorem 7.3.1] and [32] for arguments that work for asymptotically constant or periodic coefficients.

5.2. The existence of dichotomies implies that $\mathcal{T}$ is Fredholm

We begin the proof of Theorem 2.6 by showing that the existence of exponential dichotomies on $\mathbb{R}^\pm$ implies that $\mathcal{T}$ is Fredholm. The proof of this statement is quite similar to the one given in [29] for ODEs.

First, we show that $\mathcal{T}$ is a closed operator with domain $\mathrm{D}(\mathcal{T}) = H^1(\mathbb{R},Y) \cap L^2\big(\mathbb{R},Y^1\big)$. Let $\mathcal{T}_{\mathrm{ref}}$ be the operator that corresponds to the reference equation (2.5); $\mathcal{T}_{\mathrm{ref}}$ is then given by (2.6) with $\alpha = 1$ and $a = 0$. An explicit computation using Fourier series shows that

$$\mathcal{T}_{\mathrm{ref}} \;:\; \mathrm{D}(\mathcal{T}_{\mathrm{ref}}) \;=\; H^1(\mathbb{R},Y)\cap L^2\big(\mathbb{R},Y^1\big) \;\hookrightarrow\; L^2(\mathbb{R},Y) \;\longrightarrow\; L^2(\mathbb{R},Y)$$

is invertible. In particular, $\mathcal{T}_{\mathrm{ref}}$ is a closed operator in $L^2(\mathbb{R},Y)$ with domain $\mathrm{D}(\mathcal{T}_{\mathrm{ref}}) = H^1(\mathbb{R},Y)\cap L^2\big(\mathbb{R},Y^1\big)$. Hence, it follows from [18, Theorem IV.1.1] that $\mathcal{T}$ is also closed with the same domain since $\mathcal{T} - \mathcal{T}_{\mathrm{ref}}$ is bounded.

As a preparation for the following arguments, we claim that we can assume that $P_+(0) - P_-(0)$ is compact. Note that only $\mathrm{R}(P_+(0))$ and $\mathrm{N}(P_-(0))$ are uniquely determined; see [32, (3.20)]. Following the arguments in the proof of Theorem 2.2 and Remark 4.1, we see that $P_+(0)$ and $P_-(0)$ are limits of projections $P_m^+(0)$ and $P_m^-(0)$, respectively, for the Galerkin approximation (4.1) of (2.2). Furthermore, the projections $P_m^+(0)$ and $P_m^-(0)$ restricted to $Y_m^\perp$ coincide with the projection P_{ref} restricted to $Y_m^\perp$. Hence, $P_+(0) - P_-(0)$ is the limit, in norm, of degenerate operators with finite–dimensional range and thus compact.

Consider the null space of $\mathcal{T}$. Due to Lemma 5.1, the adjoint equation has an exponential dichotomy, and the associated evolution operators are the adjoints of the evolution operators of the original equation. As a consequence, any bounded solution to (2.2) lies in $\mathrm{R}(P_+(0)) \cap \mathrm{N}(P_-(0))$ and in fact decays exponentially as $\xi \to \pm\infty$. The fact that $P_+(0) - P_-(0)$ is compact implies that $P_-(0) + (\mathrm{id} - P_+(0))$ is Fredholm. Hence, $\dim \mathrm{N}(P_-(0) + (\mathrm{id} - P_+(0))) < \infty$ and, since $\mathrm{R}(P_+(0)) \cap \mathrm{N}(P_-(0)) \subset \mathrm{N}(P_-(0) + (\mathrm{id} - P_+(0)))$, we conclude that $\mathrm{N}(\mathcal{T})$ is finite–dimensional.

In the next step, we prove that the range of $\mathcal{T}$ is closed. We use an integral representation for mild solutions U of $\mathcal{T}U = G$. Suppose that $G_l \in \mathrm{R}(\mathcal{T})$ so that $\mathcal{T}U_l = G_l$ for some U_l and $G_l \to G$ in $L^2(\mathbb{R},Y)$. Recall that $\varphi_\pm^{\mathrm{s,u}}$ are the evolution operators appearing in Definition 2.1 of exponential dichotomies on $\mathbb{R}^\pm$. For $\xi \geq 0$ and $\xi \leq 0$, define

$$\begin{aligned} U_l^+(\xi) \;&=\; \varphi_+^{\mathrm{s}}(\xi;0)V_l^+ + \int_0^\xi \varphi_+^{\mathrm{s}}(\xi;\zeta)G_l(\zeta)\,\mathrm{d}\zeta + \int_\infty^\xi \varphi_+^{\mathrm{u}}(\xi;\zeta)G_l(\zeta)\,\mathrm{d}\zeta\,, \\ U_l^-(\xi) \;&=\; \varphi_-^{\mathrm{u}}(\xi;0)V_l^- + \int_0^\xi \varphi_-^{\mathrm{u}}(\xi;\zeta)G_l(\zeta)\,\mathrm{d}\zeta + \int_{-\infty}^\xi \varphi_-^{\mathrm{s}}(\xi;\zeta)G_l(\zeta)\,\mathrm{d}\zeta\,, \end{aligned} \tag{5.1}$$

respectively, where the initial values $V_l^\pm$ are chosen according to

$$V_l^+ \;=\; P_+(0)U_l(0)\,, \quad V_l^- \;=\; (\mathrm{id} - P_-(0))U_l(0)\,.$$

Note that $U_l(0)$ is well–defined since $U_l \in \mathrm{D}(\mathcal{T})$ and $H^1(\mathbb{R},Y) \hookrightarrow C^0(\mathbb{R},Y)$. Observe that the functions $U_l^\pm$ are mild solutions of $\mathcal{T}U_l^\pm = G_l$ on $L^2(\mathbb{R}^\pm,Y)$. We claim that $U_l^+(\xi) = U_l(\xi)$ for $\xi \geq 0$. Note that the difference $U_l^+(\xi) - U_l(\xi)$ satisfies the homogeneous equation for $\xi \in \mathbb{R}^+$ and is bounded uniformly in $\xi \in \mathbb{R}^+$. In addition, the initial condition $U_l^+(0) - U_l(0)$ is contained in the unstable space $\mathrm{N}(P_+(0)) = \mathrm{R}(\mathrm{id} - P_+(0))$. Multiplying by a suitable bounded solution of the adjoint equation, we obtain a contradiction since the scalar product of solutions of a linear equation and its adjoint equation is constant in ξ. Thus, we have that $U_l^+(\xi) = U_l(\xi)$ for $\xi \in \mathbb{R}^+$. The same argument shows that $U_l^-(\xi) = U_l(\xi)$ for $\xi \in \mathbb{R}^-$. Using $U_l^\pm(0) = U_l(0)$ and setting $\xi = 0$ in the above integral equation, we obtain

$$\begin{aligned} V_l^+ &= P_+(0)\left[V_l^- + \int_{-\infty}^0 \varphi_-^{\mathrm{s}}(0;\zeta) G_l(\zeta)\,\mathrm{d}\zeta\right], \\ V_l^- &= (\mathrm{id} - P_-(0))\left[V_l^+ + \int_{\infty}^0 \varphi_+^{\mathrm{u}}(0;\zeta) G_l(\zeta)\,\mathrm{d}\zeta\right] \end{aligned}$$

or, in a more compact form,

$$V_l^+ = P_+(0)V_l^- + h_l^-, \quad V_l^- = (\mathrm{id} - P_-(0))V_l^+ + h_l^+$$

where $h_l^- \in \mathrm{R}(P_+(0))$ and $h_l^+ \in \mathrm{N}(P_-(0))$. We may write this as $\mathcal{K}\big(V_l^+, V_l^-\big) = \big(h_l^-, h_l^+\big)$, where the linear operator $\mathcal{K}$ is defined by

$$\begin{aligned} \mathcal{K} : \mathrm{R}(P_+(0)) \times \mathrm{N}(P_-(0)) &\longrightarrow \mathrm{R}(P_+(0)) \times \mathrm{N}(P_-(0)), \\ (V^+, V^-) &\longmapsto \big(V^+ - P_+(0)V^-, V^- - (\mathrm{id} - P_-(0))V^+\big). \end{aligned}$$

Below, we prove that $\mathcal{K}$ is Fredholm. For the moment, we assume that $\mathcal{K}$ is Fredholm and complete the proof that $\mathcal{T}$ is closed. The solution $\big(V_l^+, V_l^-\big)$ of $\mathcal{K}\big(V_l^+, V_l^-\big) = \big(h_l^-, h_l^+\big)$ is unique and bounded in terms of $h_l^\pm$, up to elements in the null space $\mathrm{N}(\mathcal{K})$. Since $G_l \to G$ in $L^2(\mathbb{R}, Y)$, we have $h_l^\pm \to h^\pm$ in Y. Thus, possibly after subtracting appropriate elements in $\mathrm{N}(\mathcal{K})$, the sequence $\big(V_l^+, V_l^-\big)$ converges to some element $\big(V^+, V^-\big)$ in Y. Therefore, the solutions U_l converge to an element U in $L^2(\mathbb{R}, Y)$ that is obtained by substituting $\big(V^+, V^-\big)$ for $\big(V_l^+, V_l^-\big)$ and G for G_l into the integral equation (5.1).

It remains to prove that $\mathcal{K}$ is Fredholm. We begin by calculating its null space. The equations $V^+ = P_+(0)V^-$ and $V^- = (\mathrm{id} - P_-(0))V^+$ imply that $P_+(0)P_-(0)V^+ = 0$. Since $P_+(0)$ and $\mathrm{id} - P_-(0)$ differ from P_{ref} and $\mathrm{id} - P_{\mathrm{ref}}$ by compact operators, see Remark 4.1, we have

$$\begin{aligned} P_+(0)P_-(0) = P_+(0) - P_+(0)(\mathrm{id} - P_-(0)) &= P_+(0) - P_{\mathrm{ref}}(\mathrm{id} - P_{\mathrm{ref}}) + \mathrm{cpt.} \\ &= P_+(0) + \mathrm{cpt.} \end{aligned}$$

Therefore, $P_+(0)P_-(0)$ restricted to $\mathrm{R}(P_+(0))$ is Fredholm. This shows that $\mathrm{N}(\mathcal{K})$ is finite–dimensional. In the next step, we prove that the range $\mathrm{R}(\mathcal{K})$ is closed. Thus, consider

$$V_l^+ - P_+(0)V_l^- = h_l^-, \quad V_l^- - (\mathrm{id} - P_-(0))V_l^+ = h_l^+$$

and assume that $h_l^\pm \to h^\pm$ in Y. It follows that $P_+(0)P_-(0)V_l^+ = h_l^- + P_+(0)h_l^+$ converges to $h^- + P_+(0)h^+$. Thus, $V_l^+ \to V^+$ possibly after subtracting elements in the null space of the Fredholm operator $P_+(0)P_-(0)\big|_{\mathrm{R}(P_+(0))}$. We conclude that

$V_l^- = (\mathrm{id} - P_-(0))V_l^+ + h_l^+$ converges to $(\mathrm{id} - P_-(0))V^+ + h^+$ which proves that $\mathrm{R}(\mathcal{K})$ is closed. Applying the aforementioned arguments to the adjoint operator, it is not too hard to show that the complement of the range is finite–dimensional. Therefore, $\mathcal{K}$ is Fredholm as claimed above.

Next, we establish that the range of $\mathcal{T}$ has finite codimension. Regarding $\mathcal{T}$ as a closed operator on the Hilbert space $L^2(\mathbb{R}, Y)$, we can compute its adjoint $\mathcal{T}^*$ which is again a closed operator on $L^2(\mathbb{R}, Y)$. The null space $\mathrm{N}(\mathcal{T}^*)$ of $\mathcal{T}^*$ has finite dimension since $P_-(0)^* + (\mathrm{id} - P_+(0)^*)$ is Fredholm. Hence, $\mathrm{R}(\mathcal{T})$ has finite codimension since $\mathrm{R}(\mathcal{T})^\perp = \mathrm{N}(\mathcal{T}^*)$. Therefore, $\mathcal{T}$ is Fredholm.

The formula for the index follows from the finite–dimensional analogue since, as in the preceding section, we can use Galerkin approximations to reduce to a finite–dimensional problem.

Hence, we have proved that the existence of exponential dichotomies on $\mathbb{R}^\pm$ implies that $\mathcal{T}$ is Fredholm. It remains to show that, if there exists an exponential dichotomy on $\mathbb{R}$, then $\mathcal{T}$ is invertible. From the discussion above, it follows that $\mathcal{T}$ is Fredholm with index zero if it has an exponential dichotomy on $\mathbb{R}$. In addition, due to [32, Theorem 2], $\mathcal{T}$ has trivial null space. Thus, $\mathcal{T}$ is invertible.

5.3. Fredholm properties of $\mathcal{T}$ imply the existence of dichotomies

In this section, we show that, if the operator $\mathcal{T}$ defined in (2.6) is Fredholm, then the elliptic problem (2.2) has exponential dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$. The construction of the dichotomies is carried out by seeking solutions to the homogeneous equation $V' = A(\xi)V$ for $\xi \in \mathbb{R}$ with "boundary" conditions at $\xi = \xi_0$ for any fixed ξ_0.

We may assume that $\alpha = 0$ by incorporating the parameter α into the function $a(\xi, t)$. We write (2.2) in the more compact form

$$\frac{\mathrm{d}}{\mathrm{d}\xi} U \;=\; A(\xi)U \tag{5.2}$$

as a differential equation on the Hilbert space Y. Define

$$\mathcal{H} \;=\; L^2(\mathbb{R}, Y)\,, \quad \mathcal{D} \;=\; \mathrm{D}(\mathcal{T}) \;=\; H^1(\mathbb{R}, Y) \cap L^2\big(\mathbb{R}, Y^1\big)\,.$$

Throughout the remainder of the paper, we identify $\mathcal{H}$ and $\mathcal{H}^*$ so that

$$\mathcal{D} \subset \mathcal{H} \;=\; \mathcal{H}^* \subset \mathcal{D}^*.$$

Note that

$$\mathcal{T} \;=\; \frac{\mathrm{d}}{\mathrm{d}\xi} - A(\xi) \;:\; \mathcal{D} \;\longrightarrow\; \mathcal{H}$$

is bounded. We regard $\mathcal{T}$ as a closed, densely defined operator on $L^2(\mathbb{R}, Y)$. The adjoint operator $\mathcal{T}^* = -\frac{\mathrm{d}}{\mathrm{d}\xi} - A(\xi)^*$ is also closed and densely defined. The domain $\mathrm{D}(\mathcal{T}^*)$ of $\mathcal{T}^*$ is equal to $\mathcal{D} = H^1(\mathbb{R}, Y) \cap L^2\big(\mathbb{R}, Y^1\big)$ since the arguments at the beginning of Section 5.2 apply to the adjoint operator; we also refer to Section 6.2 below where we calculate the adjoint operator $A(\xi)^*$ explicitly. As a consequence, the operator $\mathcal{T}^*$ restricted to its domain $\mathcal{D}$ is bounded and Fredholm with index $-\,\mathrm{ind}(\mathcal{T})$. Let

$$(\mathcal{T}^*)^{\mathrm{ad}} \;:\; \mathcal{H} \;=\; \mathcal{H}^* \;\longrightarrow\; \mathcal{D}^*$$

be the adjoint of $\mathcal{T}^*$, considered as a bounded operator from $\mathcal{D}$ to $\mathcal{H}$. In particular, $(\mathcal{T}^*)^{\mathrm{ad}}U = G$ with $U \in \mathcal{H}$ and $G \in \mathcal{D}^*$ means that $((\mathcal{T}^*)\psi, U) = (\psi, G)$ for all $\psi \in \mathcal{D}$, where $(\,\cdot\,,\,\cdot\,)$ denotes the dual pairing between $\mathcal{D}$ and $\mathcal{D}^*$. More explicitly, we have

$$-\int_{\xi\in\mathbb{R}} (\psi_\xi + A(\xi)^*\psi, U)_Y \,\mathrm{d}\xi \;=\; \int_{\xi\in\mathbb{R}} (\psi, G)_Y \,\mathrm{d}\xi \tag{5.3}$$

for all $\psi \in \mathcal{D}$, and the pairings in (5.3) are understood as scalar products in Y in the sense of distributions. Note that the operator $(\mathcal{T}^*)^{\mathrm{ad}}$ restricted to $\mathcal{D}$ is equal to $\mathcal{T}$.

5.3.1. The operator $\mathcal{T}$ is invertible

First, we consider the case where $\mathcal{T}$ is invertible.

Lemma 5.2. *Suppose that $\mathcal{T}$ is invertible. There is then a constant $C > 0$ with the following property. Let $G(\xi,t) = G_0(t)\delta(\xi)$ for some $G_0 \in Y$, where $\delta(\,\cdot\,) \in H^{-1}(\mathbb{R})$ is the usual δ–distribution. The equation $(\mathcal{T}^*)^{\mathrm{ad}}U = G$ has then a unique solution $U \in \mathcal{H}$. The restrictions of U to $\mathbb{R}^\pm$ belong to $C^0(\mathbb{R}^\pm, Y)$ and are differentiable; in particular, the limits $U_+ = \lim_{\xi\searrow 0} U(\xi)$ and $U_- = \lim_{\xi\nearrow 0} U(\xi)$ exist in Y, and we have $U_+ - U_- = G_0$. Furthermore, U satisfies $|U|_{L^\infty(\mathbb{R},Y)} + |U|_{\mathcal{H}} \le C\,|G_0|_Y$.*

Proof. Note that G is a bounded linear functional on $H^1(\mathbb{R}, Y)$ acting through $G(V) = (G_0, V(0))_Y$. Hence, $G \in \mathcal{D}^*$. Since $\mathcal{T}$ is invertible, so is $\mathcal{T}^*$. Consequently, $(\mathcal{T}^*)^{\mathrm{ad}}$ is invertible, and the equation $(\mathcal{T}^*)^{\mathrm{ad}}U = G$ has a unique solution $U \in \mathcal{H}$.

We construct the solution U in a different fashion. Write $A(\xi) = A_{\mathrm{ref}} + B(\xi)$ with

$$A_{\mathrm{ref}} \;=\; \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\partial_t + \mathrm{id}) & 0 \end{pmatrix}, \quad B(\xi) \;=\; \begin{pmatrix} 0 & 0 \\ -D^{-1}(a(\xi,t) - \mathrm{id}) & -cD^{-1} \end{pmatrix}.$$

In particular, $B(\xi) \in C^0(\mathbb{R}, \mathrm{L}(Y))$.

First, we solve the equation

$$\left(\mathcal{T}^*_{\mathrm{ref}}\right)^{\mathrm{ad}} V \;=\; \left(\frac{\mathrm{d}}{\mathrm{d}\xi} - A_{\mathrm{ref}}\right) V \;=\; G\,.$$

The equation $\frac{\mathrm{d}}{\mathrm{d}\xi}V = A_{\mathrm{ref}}V$ has exponential dichotomies; these can be calculated explicitly using Fourier series. Therefore, there is a continuous projection P_{ref} defined on Y so that $A_{\mathrm{ref}}P_{\mathrm{ref}}$ and $-A_{\mathrm{ref}}(\mathrm{id} - P_{\mathrm{ref}})$ are sectorial operators on $\mathrm{R}(P_{\mathrm{ref}})$ and $\mathrm{R}(\mathrm{id} - P_{\mathrm{ref}})$, respectively, and so that their spectrum is contained in the open left half–plane. In particular, the semigroups $\mathrm{e}^{A_{\mathrm{ref}}P_{\mathrm{ref}}\xi}$, defined on $\mathrm{R}(P_{\mathrm{ref}})$ for $\xi \ge 0$, and $\mathrm{e}^{A_{\mathrm{ref}}(\mathrm{id} - P_{\mathrm{ref}})\xi}$, defined on $\mathrm{R}(\mathrm{id} - P_{\mathrm{ref}})$ for $\xi \le 0$, exist and are exponentially decaying in ξ on $\mathbb{R}^+$ and $\mathbb{R}^-$, respectively. Define

$$V(\xi) \;=\; \begin{cases} \mathrm{e}^{A_{\mathrm{ref}}P_{\mathrm{ref}}\xi} P_{\mathrm{ref}} G_0 & \text{for} \quad \xi \;>\; 0\,, \\ -\mathrm{e}^{A_{\mathrm{ref}}(\mathrm{id} - P_{\mathrm{ref}})\xi}(\mathrm{id} - P_{\mathrm{ref}}) G_0 & \text{for} \quad \xi \;<\; 0\,. \end{cases} \tag{5.4}$$

The function V then satisfies $\frac{\mathrm{d}}{\mathrm{d}\xi}V = A_{\mathrm{ref}}V$ for $\xi \neq 0$, and $V \in \mathcal{H}$ with $|V|_{\mathcal{H}} \leq C\,|G_0|_Y$. Suppose that $G_0 \in Y^1$. For any $\psi \in C_0^\infty\big(\mathbb{R}, Y^1\big)$, we have

$$\begin{aligned}
-\int_{-\infty}^{\infty}\big(V,\psi_\xi + A_{\mathrm{ref}}^*\psi\big)_Y\,\mathrm{d}\xi \;&=\; -\int_0^\infty \big(\mathrm{e}^{A_{\mathrm{ref}}P_{\mathrm{ref}}\xi}P_{\mathrm{ref}}G_0,\psi_\xi + A_{\mathrm{ref}}^*\psi\big)_Y\,\mathrm{d}\xi\\
&\quad + \int_{-\infty}^{0}\big(\mathrm{e}^{A_{\mathrm{ref}}(\mathrm{id}-P_{\mathrm{ref}})\xi}(\mathrm{id}-P_{\mathrm{ref}})G_0,\psi_\xi + A_{\mathrm{ref}}^*\psi\big)_Y\,\mathrm{d}\xi\\
&=\; \big(\mathrm{e}^{A_{\mathrm{ref}}P_{\mathrm{ref}}\xi}P_{\mathrm{ref}}G_0,\psi\big)_Y\big|_{\xi=0}\\
&\quad + \big(\mathrm{e}^{A_{\mathrm{ref}}(\mathrm{id}-P_{\mathrm{ref}})\xi}(\mathrm{id}-P_{\mathrm{ref}})G_0,\psi\big)_Y\big|_{\xi=0}\\
&=\; \Big(\lim_{\xi\searrow 0}V(\xi) - \lim_{\xi\nearrow 0}V(\xi),\psi(0)\Big)_Y\\
&=\; (G_0,\psi(0))_Y\\
&=\; \int_{-\infty}^{\infty}(G_0\delta(\xi),\psi(\xi))_Y\,\mathrm{d}\xi\,,
\end{aligned}$$

where we used integration by parts with respect to ξ and the fact that V satisfies $V_\xi - A_{\mathrm{ref}}V = 0$ for $\xi \neq 0$. This proves that V is a solution of $\big(\mathcal{T}_{\mathrm{ref}}^*\big)^{\mathrm{ad}}V = G$ for any $G_0 \in Y^1$. For $G_0 \in Y$, we approximate G_0 by functions $G_0^{(n)} \in Y^1$ so that $G_0^{(n)} \to G_0$ in Y and observe that the associated solutions $V^{(n)}$ converge to the function V defined in (5.4) by strong continuity of the semigroups $\mathrm{e}^{A_{\mathrm{ref}}P_{\mathrm{ref}}\xi}$ and $\mathrm{e}^{A_{\mathrm{ref}}(\mathrm{id}-P_{\mathrm{ref}})\xi}$.

In summary, for any $G_0 \in Y$, the weak solution of $\big(\frac{\mathrm{d}}{\mathrm{d}\xi} - A_{\mathrm{ref}}\big)V = G_0\delta(\xi)$ is a strong solution for $\xi \neq 0$, it belongs to $L^\infty(\mathbb{R}, Y)$ with $|V|_{L^\infty(\mathbb{R},Y)} \leq C\,|G_0|_Y$, and we have that $V|_{[0,\infty)}$ and $V|_{(-\infty,0]}$ are continuous. In fact, these restrictions are differentiable in $(0,\infty)$ and $(-\infty,0)$, respectively. In addition, we have that $V \in \mathcal{H}$ with $|V|_{\mathcal{H}} \leq C\,|G_0|_Y$.

In the next step, we solve

$$(\mathcal{T}^*)^{\mathrm{ad}}\widetilde{U} \;=\; -BV\,, \tag{5.5}$$

where V is the solution defined in (5.4). Note that $BV \in \mathcal{H}$. Therefore, there is a unique solution $\widetilde{U}$ of (5.5) with $\widetilde{U} \in \mathcal{D}$ since $\mathcal{T} : \mathcal{D} \to \mathcal{H}$ is invertible and it coincides with $(\mathcal{T}^*)^{\mathrm{ad}} : \mathcal{H} \to \mathcal{D}^*$ restricted to $\mathcal{D}$. In addition, we have that $\big|\widetilde{U}\big|_{\mathcal{D}} \leq C\,|V|_{\mathcal{H}}$. In particular, we conclude that $\widetilde{U} \in C^0(\mathbb{R}, Y)$. Since the weak derivative of $\widetilde{U}$ is continuous on $(0,\infty)$ and $(-\infty,0)$, we conclude that $\widetilde{U}$ is actually differentiable in the strong sense on these open intervals.

Thus, we have shown that $U = \widetilde{U} + V$ is the weak solution to the original problem $(\mathcal{T}^*)^{\mathrm{ad}}U = G$. Moreover, U is continuous on $\mathbb{R}^+$ and $\mathbb{R}^-$, and we have $|U|_{L^\infty(\mathbb{R},Y)} \leq C\,|G_0|_Y$. The jump of U at $\xi = 0$ is equal to $U_+ - U_- = V_+ - V_- = G_0$ since $\widetilde{U}$ is continuous at $\xi = 0$ and the jump of V is equal to G_0 by construction. □

We define the map

$$\Pi \;:\; Y \;\longrightarrow\; Y \times Y\,, \quad G_0 \;\longmapsto\; (U_+, U_-)\,,$$

where $G_0 = U_+ - U_-$, and U_+, U_- have been defined in Lemma 5.2. The map Π is continuous and injective; injectivity follows since Π has a bounded left–inverse given by $Y \times Y \to Y$, $(U_+, U_-) \mapsto U_+ - U_-$. Finally, we introduce the canonical projections $P_i : Y \times Y \to Y$, $(U_1, U_2) \mapsto U_i$ for $i = 1, 2$. By definition, we then have

$$G \;=\; P_1 \Pi G - P_2 \Pi G$$

for any $G \in Y$.

Lemma 5.3. *Suppose that $\mathcal{T}$ is invertible. The images $\mathrm{R}(P_i\Pi)$ are then closed subspaces of Y. Moreover, we have that*

$$\mathrm{R}(P_1\Pi) \oplus \mathrm{R}(P_2\Pi) \;=\; Y\,.$$

Proof. Closedness of the ranges of $P_i\Pi$ follows once we have proved that $\mathrm{R}(\Pi)$ is closed. This, however, is an immediate consequence of the continuity of the left–inverse of Π given above. Hence, Π is an isomorphism onto its closed range, and the first assertion of the lemma is proved.

Next, we prove that $\mathrm{R}(P_1\Pi) \cap \mathrm{R}(P_2\Pi) = \{0\}$. If $P_1\Pi(G_1) = P_2\Pi(G_2)$, then

$$\Pi G_1 \;=\; (U_*, U_1)\,, \quad \Pi G_2 \;=\; (U_2, U_*)\,.$$

Thus, U_* is an initial value to two bounded weak solutions defined for $\xi \geq 0$ and $\xi \leq 0$, respectively, each of which is equal to the same value U_* at $\xi = 0$. Patching these solutions together gives a weak solution of the homogeneous equation with jump $G_0 = 0$. It is not difficult to see that this solution is in fact a strong solution and therefore belongs to the null space of $\mathcal{T}$; this null space, however, is trivial. Hence, $U_* = 0$, and $\mathrm{R}(P_1\Pi) \cap \mathrm{R}(P_2\Pi) = \{0\}$.

It remains to prove that the sum of the spaces $\mathrm{R}(P_1\Pi)$ and $\mathrm{R}(P_2\Pi)$ is Y. Arguing by contradiction, we suppose that $G^\perp$ is orthogonal to their direct sum. We have

$$G^\perp \;=\; P_1\Pi G^\perp - P_2\Pi G^\perp\,,$$

and therefore $G^\perp \in \mathrm{R}(P_1\Pi) \oplus \mathrm{R}(P_2\Pi)$. Thus, $G^\perp = 0$. This proves the lemma. □

The above construction can be carried out for any initial time $\xi = \xi_0$. We obtain families of operators $\Pi(\xi)$ and subspaces $\mathrm{R}(P_1\Pi(\xi))$ and $\mathrm{R}(P_2\Pi(\xi))$ that depend on ξ. With the families $\mathrm{R}(P_1\Pi(\xi))$ and $\mathrm{R}(P_2\Pi(\xi))$, we can associate a family of projections $P(\xi)$ defined by $\mathrm{R}(P(\xi)) = \mathrm{R}(P_1\Pi(\xi))$ and $\mathrm{N}(P(\xi)) = \mathrm{R}(P_2\Pi(\xi))$.

Remark 5.4. There is a constant $C > 0$ with the following property. For any $U_+ \in \mathrm{R}(P(\xi_0))$, there is a unique strong solution $V^{\mathrm{s}}(\xi)$ of (5.2) for $\xi > \xi_0$ with initial value $V^{\mathrm{s}}(\xi_0) = U_+$. This solution satisfies $|V^{\mathrm{s}}(\xi)| \leq C\,|U_+|$ for $\xi \geq \xi_0$, and it is continuous in $\xi \geq \xi_0$. Similarly, for any $U_- \in \mathrm{N}(P(\xi_0))$, there is a unique strong solution $V^{\mathrm{u}}(\xi)$ defined for $\xi < \xi_0$ with initial value $V^{\mathrm{u}}(\xi_0) = U_-$, we have $|V^{\mathrm{u}}(\xi)| \leq C\,|U_-|$ for $\xi \leq \xi_0$, and the solution is continuous for $\xi \leq \xi_0$.

Lemma 5.5. *Suppose that $\mathcal{T}$ is invertible. The family of projections defined above together with the solutions V^{s} and V^{u} given in Remark 5.4 defines an exponential dichotomy. In particular, the estimates in Definition 2.1 are satisfied on $J = \mathbb{R}$ for appropriate positive constants K and η.*

Proof. We have already constructed the solutions V^{s} and V^{u} above. It remains therefore to prove the uniform exponential decay estimates and invariance.

First, we prove uniform boundedness, i.e. the estimates for $\eta = 0$. Observe that $G_0\delta(\xi - \xi_0)$ is continuous and uniformly bounded in ξ_0 with values in $\mathcal{D}^*$. Therefore, the map Π as well as the subspaces that have been used to define the projections vary continuously with ξ_0. The norm of the map Π is also bounded uniformly in ξ_0: it is not difficult to check that the constant C that appears in Lemma 5.2 does not depend upon the point ξ_0. The angle between the two subspaces is bounded away from zero; otherwise, the norm of the operators Π could not be uniformly bounded since there would be elements $U_\pm$ of unit norm that correspond to arbitrarily small jumps G_0. This establishes uniform bounds on the norms of the projections. The projections are also strongly continuous in ξ; that follows once more from the continuity of Π. Uniform bounds on the solutions in terms of the initial values can then be established as follows: to each initial value U_+, we choose the jump $G_0 = U^+$, and use that the solution is uniformly bounded in terms of its jump.

By Lemma 5.2, the solutions constructed above are continuously differentiable in ξ. Differentiability with respect to the initial time can be seen as follows. We need to show that $\frac{\mathrm{d}}{\mathrm{d}h}U_h(\xi)$ exists for any fixed $\xi > \xi_0$ where $U_h(\xi)$ satisfies $\frac{\mathrm{d}}{\mathrm{d}\xi}U = A(\xi)U$ with initial time ξ_0+h. We define $\widetilde{U}_h(\xi) := U_h(\xi+h)$ so that $\widetilde{U}_h(\xi)$ satisfies $\frac{\mathrm{d}}{\mathrm{d}\xi}U = A(\xi+h)U$ with initial time ξ_0. It then suffices to show that $\frac{\mathrm{d}}{\mathrm{d}h}\widetilde{U}_h(\xi - h)$ exists. Since the operator $\mathcal{T}_h = \frac{\mathrm{d}}{\mathrm{d}\xi} - A(\xi + h)$ depends smoothly upon the parameter h, we see that $\frac{\mathrm{d}}{\mathrm{d}h}\widetilde{U}_h(\xi)$ exists for every fixed $\xi > \xi_0$. This proves the claim since $\widetilde{U}_h(\xi)$ is also differentiable with respect to ξ.

The exponential decay estimates can be derived by applying the aforementioned arguments to the equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi}U \;=\; (A(\xi) + \eta\,\mathrm{sign}(\xi - \xi_0))U \;=:\; A_{\eta,\xi_0}(\xi)U$$

with $\eta > 0$. The operator $\mathcal{T}_{\eta,\xi_0} = \frac{\mathrm{d}}{\mathrm{d}\xi} - A_{\eta,\xi_0}(\xi)$ is continuous in (η, ξ_0) and invertible provided η is small enough. Applying the results established above to the operator $\mathcal{T}_{\eta,\xi_0}$ shows that the modified operator $\frac{\mathrm{d}}{\mathrm{d}\xi} - A_{\eta,\xi_0}(\xi)$ on $\mathbb{R}^\pm$ has dichotomies with uniform bounds. It then suffices to remark that the solutions to the modified equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\widetilde{U} \;=\; A_{\eta,\xi_0}(\xi)\widetilde{U}$$

are given by $\widetilde{U}(\xi) = U(\xi)\mathrm{e}^{\eta\,|\xi-\xi_0|}$ where $U(\xi)$ satisfies the original equation (5.2). Hence, the uniform bounds for the solutions of the modified equation provide exponential bounds for the dichotomies of $\frac{\mathrm{d}}{\mathrm{d}\xi} - A(\xi)$.

It remains to prove invariance of the ranges of the projection. Given a solution $U(\xi)$ for $\xi \geq 0$, we have to show that $U(\xi_0) \in \mathrm{R}(P(\xi_0))$ for any $\xi_0 > 0$. Assume that, for some positive ξ_0, we have

$$U(\xi_0) \;=\; U_+(\xi_0) + U_-(\xi_0) \in \mathrm{R}(P(\xi_0)) \oplus \mathrm{N}(P(\xi_0))\,.$$

By construction, there exists then a bounded solution $U_-(\xi)$ defined for $\xi < \xi_0$ with initial value $U_-(\xi_0)$. The pair of solutions $U_-(\xi)$ with $\xi < \xi_0$ and $U(\xi)$ with $\xi > \xi_0$

is a solution on $\mathbb{R}$ that has the jump $G_0 = U_+(\xi_0)$. Another pair of solutions that have the same jump can be constructed as follows: use $U_-(\xi) = 0$ for $\xi < \xi_0$ and the bounded solution to the initial value $U_+(\xi_0)$ for $\xi > \xi_0$; the latter exists on account of Remark 5.4. Since both these pairs have the same jump, they must coincide, and we conclude that $U_-(\xi_0) = 0$. □

In summary, we have established the existence of exponential dichotomies on $\mathbb{R}$ provided $\mathcal{T}$ is invertible. Next, we address this issue for dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$ in the case where $\mathcal{T}$ is merely Fredholm.

Remark 5.6. Note that the Hypothesis (U1) has not been used in this section.

5.3.2. The operator $\mathcal{T}$ is Fredholm

Let $\phi_1(\xi), \ldots, \phi_k(\xi)$ be a basis of the null space of $\mathcal{T}$. Exploiting regularity properties of solutions to the homogeneous equation (5.2), we see that the null spaces of $(\mathcal{T}^*)^{\mathrm{ad}}$ and $\mathcal{T}$ are equal: weak solutions to the homogeneous elliptic equation (5.2) are weak solutions of the underlying parabolic equation which in turn are strong solutions due to standard parabolic regularity theory; see [40] for similar arguments. Let $\psi_1(\xi), \ldots, \psi_m(\xi)$ be a basis of the null space of $\mathcal{T}^*$. By duality, this space is the orthogonal complement of the ranges of both operators $\mathcal{T}$ and $(\mathcal{T}^*)^{\mathrm{ad}}$. On account of Hypothesis (U1), $\phi_1(0), \ldots, \phi_k(0)$ as well as $\psi_1(0), \ldots, \psi_m(0)$ are linearly independent and we may assume that they are orthonormal bases. We remark that this is the only place where we use Hypothesis (U1). We write

$$\mathcal{V}_0 \;=\; \mathrm{span}\{\phi_1(0), \ldots, \phi_k(0)\}\,, \quad \mathcal{W}_0 \;=\; \mathrm{span}\{\psi_1(0), \ldots, \psi_m(0)\}\,.$$

We first prove that elements in the null space of $(\mathcal{T}^*)^{\mathrm{ad}}$ decay exponentially in ξ.

Lemma 5.7. *There are constants $C > 0$ and $\eta > 0$ such that*

$$|V(\xi)|_Y \;\le\; C\mathrm{e}^{-\eta\,|\xi|}\,|V(0)|_Y$$

for any $V(0) \in \mathcal{V}_0$ and any $\xi \in \mathbb{R}$.

Proof. As before, we consider the operator

$$\mathcal{T}_\eta \;=\; \frac{\mathrm{d}}{\mathrm{d}\xi} - A(\xi) + \eta\,\mathrm{sign}(\xi)\,,$$

and note that the operator $\widetilde{B}$ defined by $\big(\widetilde{B}U\big)(\xi) \;=\; \mathrm{sign}(\xi)U(\xi)$ is contained in $\mathrm{L}(\mathcal{D},\mathcal{H})$. Choose $\phi \in \mathrm{N}(\mathcal{T})$, and let $U = \sum_{n=0}^{\infty} \eta^n U_n$ with $U_0 = \phi$ and $U_n \in \mathrm{N}(\mathcal{T})^\perp$ for $n \ge 1$. We then have $\mathcal{T}_\eta U = 0$ provided the series for U converges and

$$\mathcal{T}U_n \;=\; -\widetilde{B}U_{n-1}$$

for any $n \ge 1$. This equation can be solved for U_n provided $\widetilde{B}U_{n-1} \in \mathrm{R}(\mathcal{T})$. In this case, we have

$$|U_n|_{\mathcal{D}} \;\le\; \big\|\widetilde{B}\big\|\,\big\|\mathcal{T}^{-1}\big\|\,|U_{n-1}|_{\mathcal{D}} \;=\; C\,|U_{n-1}|_{\mathcal{D}}$$

where $\mathcal{T}^{-1}$ is the inverse of the invertible operator $\mathcal{T} : \mathrm{N}(\mathcal{T})^\perp \to \mathrm{R}(\mathcal{T})$. If $\widetilde{B}U_n \in \mathrm{R}(\mathcal{T})$ for all n, then we obtain $|U_n|_\mathcal{D} \leq C^n\,|\phi|_\mathcal{D}$ and the series for U converges provided η is smaller than $1/C$. It therefore remains to prove that $\widetilde{B}U_n \in \mathrm{R}(\mathcal{T})$ for all $n \geq 0$. We proceed by induction and claim that, if $(\psi(\xi), U_{n-1}(\xi))_Y = 0$ for all ξ and any $\psi \in \mathrm{N}(\mathcal{T}^*)$, then we can solve $\mathcal{T}U_n = -\widetilde{B}U_{n-1}$ and $(\psi(\xi), U_n(\xi))_Y = 0$ for all ξ and any $\psi \in \mathrm{N}(\mathcal{T}^*)$. Indeed, if $(\psi(\xi), U_{n-1}(\xi))_Y = 0$, then also $\big(\psi(\xi), \widetilde{B}U_{n-1}(\xi)\big)_Y = 0$, and therefore $\widetilde{B}U_{n-1} \in \mathrm{R}(\mathcal{T})$. Thus, there is a unique $U_n \in \mathrm{N}(\mathcal{T})^\perp$ such that $\mathcal{T}U_n = -\widetilde{B}U_{n-1}$. We shall prove that $(\psi(\xi), U_n(\xi))_Y = 0$ for all ξ. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\xi}(\psi(\xi), U_n(\xi))_Y &= (\psi(\xi), (\mathcal{T}U_n)(\xi))_Y \\ &= -\big(\psi(\xi), \big(\widetilde{B}U_{n-1}\big)(\xi)\big)_Y \\ &= -\operatorname{sign}(\xi)\,(\psi(\xi), U_{n-1}(\xi))_Y \\ &= 0\,.\end{aligned}$$

Since $(\psi(\xi), U_n(\xi))_Y \in L^1(\mathbb{R})$ and $(\psi(\xi), U_n(\xi))_Y$ does not depend upon ξ by the above argument, we conclude that $(\psi(\xi), U_n(\xi))_Y = 0$ which proves our claim. Finally, the same arguments show that $(\psi(\xi), \phi(\xi))_Y = 0$ completing the induction. □

The equation $(\mathcal{T}^*)^{\mathrm{ad}}U = G_0(t)\delta(\xi)$ has a solution if, and only if, $G_0 \perp \mathcal{W}_0$ in Y. The difference of any two such solutions, evaluated at $\xi = 0$, lies in $\mathcal{V}_0$. We have the following regularity result.

Lemma 5.8. *Suppose that $\mathcal{T}$ is Fredholm. There is then a constant $C > 0$ with the following property. Let $G = G_0\delta(\xi)$ for some $G_0 \in Y$ with $G_0 \perp \mathcal{W}_0$. There is then a unique solution $U \in \mathcal{H}$ of $(\mathcal{T}^*)^{\mathrm{ad}}U = G$ that, at the same time, is perpendicular to the null space of $\mathcal{T}$ in $\mathcal{H}$. The restrictions of U to $\mathbb{R}^\pm$ belong to $C^0\big(\mathbb{R}^\pm, Y\big)$; in fact, they are differentiable. In particular, the limits $U_+ = \lim_{\xi\searrow 0} U(\xi)$ and $U_- = \lim_{\xi\nearrow 0} U(\xi)$ exist in Y, and the jump at $\xi = 0$ is given by $U_+ - U_- = G_0$. Finally, we have $|U|_{L^\infty(\mathbb{R},Y)} + |U|_\mathcal{H} \leq C\,|G_0|_Y$.*

Proof. We closely follow the proof of Lemma 5.2. As in Lemma 5.2, we first solve $\big(\mathcal{T}^*_{\mathrm{ref}}\big)^{\mathrm{ad}}V = G$. Using the definition $B = (\mathcal{T}^*)^{\mathrm{ad}} - \big(\mathcal{T}^*_{\mathrm{ref}}\big)^{\mathrm{ad}}$, we obtain

$$(\mathcal{T}^*)^{\mathrm{ad}}V \;=\; \big(\mathcal{T}^*_{\mathrm{ref}}\big)^{\mathrm{ad}}V + BV \;=\; G + BV\,,$$

and we conclude that $BV \in \mathrm{R}\big((\mathcal{T}^*)^{\mathrm{ad}}\big)$ since $G \in \mathrm{R}\big((\mathcal{T}^*)^{\mathrm{ad}}\big)$. We can therefore proceed exactly as in Lemma 5.2, and obtain a unique solution $\widetilde{U}$ of $(\mathcal{T}^*)^{\mathrm{ad}}\widetilde{U} = -BV$ with $\widetilde{U} \in \mathrm{N}\big((\mathcal{T}^*)^{\mathrm{ad}}\big)^\perp$. Finally, we project the function V onto $\mathrm{N}\big((\mathcal{T}^*)^{\mathrm{ad}}\big)$ along the orthogonal complement $\mathrm{N}\big((\mathcal{T}^*)^{\mathrm{ad}}\big)^\perp$. The resulting function in $\mathrm{N}\big((\mathcal{T}^*)^{\mathrm{ad}}\big)$ is denoted by ϕ. The element $U = \widetilde{U} + V - \phi \in \mathcal{H}$ is then the desired solution of $(\mathcal{T}^*)^{\mathrm{ad}}U = G_0\delta$. The estimates for U follow as in Lemma 5.2. □

As before, we define the map

$$\Pi \;:\; \mathcal{W}_0^\perp \;\longrightarrow\; Y \times Y\,, \quad G_0 \;\longmapsto\; (U_+, U_-)\,,$$

where U_+ and U_- have been defined in Lemma 5.8. The map Π is again continuous and injective. Proceeding as in Lemma 5.3, and using that functions in $\mathrm{N}(\mathcal{T})$ decay exponentially in ξ by Lemma 5.7, we see that

$$\mathrm{R}(P_1\Pi) \cap \mathrm{R}(P_2\Pi) \;=\; \mathcal{V}_0\,, \quad \mathrm{R}(P_1\Pi) + \mathrm{R}(P_2\Pi) \;=\; \mathcal{W}_0^{\perp}\,.$$

In other words, initial values of bounded solutions on $\mathbb{R}^+$ are contained in $\mathrm{R}(P_1\Pi)$, initial values of bounded solutions on $\mathbb{R}^-$ lie necessarily in $\mathrm{R}(P_2\Pi)$, and the sum of the subspaces $\mathrm{R}(P_1\Pi)$ and $\mathrm{R}(P_2\Pi)$ is equal to the entire phase space up to the orthogonal complement $\mathcal{W}_0$.

As in the case where $\mathcal{T}$ is invertible, we can carry out the construction described above for any initial time $\xi = \xi_0$, and obtain families of operators $\Pi(\xi)$ and subspaces $\mathrm{R}(P_1\Pi(\xi))$ and $\mathrm{R}(P_2\Pi(\xi))$ that depend on ξ. We emphasize that the constant C appearing in Lemma 5.8 does not depend upon the initial time ξ_0. Define

$$\mathcal{V}_{\xi_0} \;=\; \mathrm{span}\{\phi_1(\xi_0),\,\ldots,\,\phi_k(\xi_0)\}\,, \quad \mathcal{W}_{\xi_0} \;=\; \mathrm{span}\{\psi_1(\xi_0),\,\ldots,\,\psi_m(\xi_0)\}\,.$$

Initial values of bounded solutions on $\xi \geq \xi_0$ are contained in $\mathrm{R}(P_1\Pi(\xi_0))$, initial values of bounded solutions on $\xi \leq \xi_0$ lie necessarily in $\mathrm{R}(P_2\Pi(\xi_0))$. The two subspaces intersect along the space $\mathcal{V}_{\xi_0}$ and their sum is equal to the entire phase space up to the orthogonal complement $\mathcal{W}_{\xi_0}$.

To complete the construction of dichotomies, we have to find forward and backward solutions of (5.2) with initial conditions in appropriate complements of the spaces $\mathrm{R}(P_2\Pi(\xi_0))$ and $\mathrm{R}(P_1\Pi(\xi_0))$, respectively. We concentrate on the case where $\xi_0 < 0$.

Lemma 5.9. *There is a constant $C > 0$ and projections $P(\xi_0)$ in $\mathrm{L}(Y)$ defined for $\xi_0 \leq 0$ with the following properties. First, we have $\|P(\xi_0)\| \leq C$ uniformly in $\xi_0 \leq 0$. Also, for any element $U(\xi_0) \in \mathrm{R}(P(\xi_0))$, there is a unique solution $U(\xi)$ of* (5.2) *defined for $\xi_0 \leq \xi \leq 0$, and we have $|U(\xi)|_Y \leq C\,|U(\xi_0)|_Y$ for all $\xi_0 \leq \xi \leq 0$. Furthermore, for any $U(\xi_0) \in \mathrm{N}(P(\xi_0))$, there is a unique solution $U(\xi)$ of* (5.2) *defined for $\xi < \xi_0$, and $|U(\xi)|_Y \leq C\,|U(\xi_0)|_Y$ for all $\xi \leq \xi_0$. Finally, we have $\mathrm{N}(P(\xi_0)) = \mathrm{R}(P_2\Pi(\xi_0))$ for $\xi_0 \leq 0$.*

In contrast to the situation where $\mathcal{T}$ is invertible, the assertions of the lemma do not follow directly from Lemma 5.8 since the spaces $\mathrm{R}(P_1\Pi(\xi_0))$ and $\mathrm{R}(P_2\Pi(\xi_0))$ have a non–zero intersection.

Proof. We begin by constructing $\mathrm{N}(P(\xi_0))$. On account of Lemma 5.8 and its proof, there is a constant C that does not depend upon ξ_0 with the following properties. For $G \in \mathrm{R}(P_2\Pi(\xi_0))$, we have $\Pi(\xi_0)G = (V_0, G + V_0)$ for some $V_0 \in \mathcal{V}_{\xi_0}$ with $|V_0|_Y \leq C\,|G|_Y$. Furthermore, there is a solution $U(\xi)$ of (5.2) defined for $\xi < \xi_0$ such that $U(\xi_0) = G + V_0$. Also, there is a $V \in \mathrm{N}(\mathcal{T})$ with $V(\xi_0) = V_0$. These solutions satisfy $|U(\xi)|_Y \leq C\,|G|_Y$ for $\xi < \xi_0$ and $|V(\xi)|_Y \leq C\,|G|_Y$ for $\xi > \xi_0$. Since $\xi_0 < 0$, it follows that $|V(0)|_Y \leq C\,|G|_Y$. Due to Lemma 5.7, we therefore conclude that $|V(\xi)|_Y \leq C\,|V_0|_Y \leq C\,|G|_Y$ for any $\xi \in \mathbb{R}$. Thus, the solution $U - V$ of (5.2) defined for $\xi \leq \xi_0$ satisfies $(U - V)(\xi_0) = G$ and $|(U - V)(\xi)|_Y \leq C\,|G|_Y$ for $\xi \leq \xi_0$. Hence, set $\mathrm{N}(P(\xi_0)) = \mathrm{R}(P_2\Pi(\xi_0))$. This proves the part of the lemma related to $\xi < \xi_0$.

We continue by constructing the intersection of $\mathrm{R}(P(\xi_0))$ with $\mathrm{R}(P_1\Pi(\xi_0))$. Let $Y^{\mathrm{s}}_{\xi_0}$ be the unique space that satisfies $Y^{\mathrm{s}}_{\xi_0} \oplus \mathcal{V}_{\xi_0} = \mathrm{R}(P_1\Pi(\xi_0))$ and $Y^{\mathrm{s}}_{\xi_0} \perp \mathcal{V}_{\xi_0}$. If $G \in Y^{\mathrm{s}}_{\xi_0}$, then $\Pi G = (G + V_0, V_0)$ where $\mathcal{V}_0 \in V_{\xi_0}$ with $|V_0|_Y \leq C\,|G|_Y$. We are interested in vectors for which the second component U_- of ΠG is equal to zero. Hence, for any fixed $G \in Y^{\rm s}_{\xi_0}$, we shall find an element $\hat{V}_0 \in \mathcal{V}_{\xi_0}$ such that $\Pi\big(G + \hat{V}_0\big) = \big(G + \hat{V}_0, 0\big)$. This is equivalent to solving

$$\Pi\hat{V}_0 \;=\; \big(\hat{V}_0 - V_0, -V_0\big)\,,$$

where we used $\Pi G = (G + V_0, V_0)$. Using the definition of Π, we see that this equation is satisfied provided we define $\hat{V}_0 = \beta V_0$ where

$$\beta \;=\; \sqrt{\frac{\int_{-\infty}^{\infty} |V(\xi)|_Y^2\,\mathrm{d}\xi}{\int_{\xi_0}^{\infty} |V(\xi)|_Y^2\,\mathrm{d}\xi}}$$

and $V \in \mathrm{N}(\mathcal{T})$ with $V(\xi_0) = V_0$. On account of Lemma 5.7, we have $|\beta| \leq C$ uniformly in $\xi_0 \leq 0$. Therefore, for any $G \in Y^{\rm s}_{\xi_0}$, there is a unique $\hat{V}_0 \in \mathcal{V}_{\xi_0}$ such that $\Pi\big(G + \hat{V}_0\big) = \big(G + \hat{V}_0, 0\big)$, and we have $\big|\hat{V}_0\big|_Y \leq C\,|G|_Y$ since $|V_0|_Y \leq C\,|G|_Y$. Let

$$\widetilde{Y}^{\rm s}_{\xi_0} \;:=\; \big\{G + \hat{V}_0;\; G \in Y^{\rm s}_{\xi_0}\big\}\,,$$

then $\widetilde{Y}^{\rm s}_{\xi_0} \oplus \mathcal{V}_{\xi_0} = \mathrm{R}(P_1\Pi(\xi_0))$ and the associated projection is bounded uniformly in ξ_0 since $\big|\hat{V}_0\big|_Y \leq C\,|G|_Y$. Moreover, by construction, for any $U_0 \in \widetilde{Y}^{\rm s}_{\xi_0}$, there is a unique strong solution of (5.2) defined for $\xi > \xi_0$ such that $U(\xi_0) = U_0$. It also follows from Lemma 5.8 that $|U(\xi)|_Y \leq C\,|U_0|_Y$ for any $\xi \geq \xi_0$.

It remains to construct a subspace that is transverse to $\mathrm{R}(P_1\Pi(\xi_0)) + \mathrm{R}(P_2\Pi(\xi_0))$ such that, for elements U_0 in that subspace, we can solve the equation for $0 \geq \xi \geq \xi_0$, and the associated solution $U(\xi)$ satisfies $U(\xi_0) = U_0$ and $|U(\xi)|_Y \leq C\,|U_0|_Y$ for $0 \geq \xi \geq \xi_0$. Let $G_0 \in \mathcal{W}_{\xi_0}$, and choose $\psi_0 \in \mathcal{W}_0$ so that $\psi_0 = \sum_{j=1}^m (\psi_j(\xi_0), G_0)_Y \psi_j(0)$. Note that $|\psi_0|_Y \leq C\,|G_0|_Y$ on account of Lemma 5.7 applied to the adjoint operator. Finally, choose the right–hand side of the equation $(\mathcal{T}^*)^{\rm ad} U = G$ according to

$$G \;=\; G_0\delta(\xi - \xi_0) - \psi_0\delta(\xi)\,.$$

By construction, the function G is orthogonal to $\psi_j(\xi)$ in $\mathcal{H}$ for any j, and G is therefore contained in the range of $(\mathcal{T}^*)^{\rm ad}$. Thus, we can construct a unique solution U of $(\mathcal{T}^*)^{\rm ad} U = G$ with $U \in \mathrm{N}\big((\mathcal{T}^*)^{\rm ad}\big)$ as in Lemma 5.8: first, we solve the reference equation with right–hand side $G_0\delta(\xi - \xi_0)$, and then again for the right–hand side $-\psi_0\delta(\xi)$. The sum of these two solutions to the reference equation has the jumps G_0 at $\xi = \xi_0$ and $-\psi_0$ at $\xi = 0$. Afterwards, we proceed as in the proof of Lemma 5.8 and obtain, as claimed, a solution U to $(\mathcal{T}^*)^{\rm ad} U = G$ with $|U|_{L^\infty(\mathbb{R},Y)} + |U|_{\mathcal{H}} \leq C\,|G_0|_Y$. By adding suitable elements in $\mathrm{N}(P(\xi_0)) + \mathrm{N}\big((\mathcal{T}^*)^{\rm ad}\big)$ as above in the case of the space $Y^{\rm s}_{\xi_0}$, we can then construct an appropriate complement of $\mathrm{R}(P_1\Pi(\xi_0)) + \mathrm{R}(P_2\Pi(\xi_0))$ on which we can solve the equation for $0 \geq \xi \geq \xi_0$. We omit the details as they are similar to the arguments given before. □

The complements for $\xi \geq 0$ can be constructed in an analogous fashion.

The uniform exponential decay estimates of solutions can be established as in the case where $\mathcal{T}$ is invertible. Indeed, Lemma 5.7 shows that the dimensions of the null spaces of $\big(\mathcal{T}^*_\eta\big)^{\rm ad}$ and $\mathcal{T}^*_\eta$ defined in Lemma 5.7 do not depend upon η. Moreover, the dimensions of the spaces $\mathcal{V}_{\eta,\xi_0}$ and $\mathcal{W}_{\eta,\xi_0}$ for the operators $\left(\mathcal{T}_\eta^*\right)^{\mathrm{ad}}$ and $\mathcal{T}_\eta^*$ are independent of ξ_0 since the functions in the null spaces $\mathcal{V}_\eta$ and $\mathcal{W}_\eta$ are given by the functions in $\mathcal{V}$ and $\mathcal{W}$ multiplied by $\mathrm{e}^{\eta\,|\xi-\xi_0|}$.

The projections that we constructed above do not necessarily satisfy the invariance property. Using the exponential decay estimates, it is, however, not difficult to redefine these projections so that their ranges and null spaces are invariant; see [9] for the relevant arguments that apply also in our case.

6. Proof of Theorem 2.8

We can assume that $\lambda = 1$ so that $\alpha = 0$; otherwise, we replace $a(\xi,t)$ by $a(\xi,t)-\alpha$.

In Section 6.1, we prove that $\mathcal{T}$ is invertible if, and only if, $\Phi - \mathrm{id}$ is invertible. In Section 6.3, we demonstrate that $\Phi - \mathrm{id}$ is Fredholm with index zero if, and only if, $\mathcal{T}$ enjoys the same property. In addition, we show that the null spaces of $\mathcal{T}$ and $\Phi - \mathrm{id}$ have the same dimension. Once we have established these results, it follows from Theorem 2.6 that $\lambda = 1$ is in the essential spectrum of Φ if, and only if, either $\mathcal{T}$ does not have dichotomies on $\mathbb{R}^+$ or $\mathbb{R}^-$, or else $\mathcal{T}$ has exponential dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$ but their indices are different at $+\infty$ and $-\infty$.

Note that the last statement of Theorem 2.8 is a consequence of the above. Let λ vary in a connected component of $\mathbb{C}\backslash\Sigma_{\mathrm{ess}}$ and consider the set Λ of values of λ for which $\Phi - \lambda$ has a non–trivial null space. Since the family $\Phi - \lambda$ is analytic in λ, it follows from [18, Theorem VII.1.9] that the set Λ is either the entire component or else consists of isolated elements. Therefore, if there is a point λ_0 in the aforementioned connected component for which $\Phi - \lambda_0$ is invertible, only the latter case can occur; consequently, any element in the point spectrum that is contained in the fixed connected component of $\mathbb{C} \setminus \Sigma_{\mathrm{ess}}$ belongs necessarily to the pure point spectrum.

6.1. $\mathcal{T}$ is invertible if, and only if, $\Phi - \mathrm{id}$ is invertible

We begin by demonstrating that $\Phi - \mathrm{id}$ is invertible whenever $\mathcal{T}$ is invertible. The operator $\Phi - \mathrm{id}$ is invertible if there is a constant C such that, for any $g \in L^2(\mathbb{R})$, there is a unique solution v of

$$\begin{aligned} v_t &= Dv_{\xi\xi} + a(\xi,t)v + cv_\xi\,, \\ v(\xi,T) &= v(\xi,0) + g(\xi) \end{aligned} \tag{6.1}$$

and $|v(\,\cdot\,,T)|_{L^2(\mathbb{R})} \le C\,|g|_{L^2(\mathbb{R})}$. Following [40], we first solve the equation

$$G_t = DG_{\xi\xi} + a(\xi,t)G + cG_\xi - mG \tag{6.2}$$

with $G(\xi,T) = G(\xi,0) + g(\xi)$ for some large $m > 0$. If m is sufficiently large, the time–T map of the parabolic equation (6.2) is a contraction. Therefore, for any g, there is a unique solution to (6.2) such that $G(\xi,T) = G(\xi,0) + g(\xi)$. If we set $v = G + \widetilde{w}$, then v satisfies (6.1) if, and only if, $\widetilde{w}$ is a solution to

$$\begin{aligned} \widetilde{w}_t &= D\widetilde{w}_{\xi\xi} + a(\xi,t)\widetilde{w} + c\widetilde{w} + mG\,, \\ \widetilde{w}(\xi,0) &= \widetilde{w}(\xi,T)\,. \end{aligned} \tag{6.3}$$

If $\mathcal{T}$ admits an exponential dichotomy, then $\mathcal{T}$ is invertible due to Theorem 2.6. In particular, for any $\widetilde{G} \in L^2\big(\mathbb{R}, L^2(\mathbb{R}/T\mathbb{Z})\big)$, there exists a unique solution to

$$v_\xi \;=\; w\,, \quad w_\xi \;=\; D^{-1}(v_t - cw - a(\,\cdot\,,\xi)v) + \widetilde{G}(\xi)$$

in $H^1(\mathbb{R}, Y) \cap L^2\big(\mathbb{R}, Y^1\big)$. Hence, upon setting $\widetilde{G} = -mG$, we can solve (6.3) in a unique fashion. Under the given smoothness assumptions, it is straightforward to verify that G as well as $\widetilde{w}$ are bounded in terms of g; the details can be found in [40, Lemma 4.1]. We conclude that the inverse of the operator $\Phi - \mathrm{id}$ exists and is bounded.

It remains to prove the converse. Hence, assume that $\Phi - \mathrm{id}$ is invertible; we have to prove that, for any $G \in L^2(\mathbb{R}, Y)$, there is a unique solution $V \in H^1(\mathbb{R}, Y) \cap L^2\big(\mathbb{R}, Y^1\big)$ of

$$V_\xi \;=\; A(\xi)V + G(\xi) \tag{6.4}$$

and that the norm of V can be estimated by the norm of G. We first solve the equation

$$\widetilde{V}_\xi \;=\; A_{\mathrm{ref}}\widetilde{V} + G(\xi)\,. \tag{6.5}$$

Using Fourier series, it is straightforward to prove that (6.5) has a unique solution $\widetilde{V} = (\tilde{v}, \widetilde{w}) \in H^1(\mathbb{R}, Y) \cap L^2\big(\mathbb{R}, Y^1\big)$. We set $V = \widetilde{V} + W$, then V satisfies (6.4) provided W is a solution of

$$W_\xi \;=\; A(\xi)W + (A(\xi) - A_{\mathrm{ref}})\widetilde{V}(\xi) \;=\; A(\xi)W + \begin{pmatrix} 0 \\ D^{-1}a\tilde{v} \end{pmatrix} \tag{6.6}$$

where $\tilde{v} \in H^1\big(\mathbb{R}, H^{\frac{1}{2}}(\mathbb{R}/T\mathbb{Z})\big) \cap L^2\big(\mathbb{R}, H^1(\mathbb{R}/T\mathbb{Z})\big)$. In particular, we can rewrite (6.6) according to

$$w_t \;=\; Dw_{\xi\xi} + cw_\xi + aw - a\tilde{v} \tag{6.7}$$

with $w(\xi, T) = w(\xi, 0)$. Using the regularity properties of $\tilde{v}$, we see that

$$\tilde{v} \in C^0\big([0,T], L^2(\mathbb{R})\big)\,.$$

Hence, we can solve (6.7) using the variation–of–constant formula and get

$$w(\,\cdot\,,t) \;=\; \Phi_{t,0}w(\,\cdot\,,0) - \int_0^t \Phi_{t,s}\, a(\,\cdot\,,s)\tilde{v}(\,\cdot\,,s)\,\mathrm{d}s\,, \tag{6.8}$$

where $\Phi_{t,s}$ is the evolution associated with the homogeneous part of (6.7); see also Section 2.1. The integral on the right–hand side of (6.8) is well defined and belongs to $L^2(\mathbb{R})$ due to the regularity properties of $\tilde{v}$. We shall solve $w(\xi, T) = w(\xi, 0)$, i.e.,

$$w(\,\cdot\,,0) \;=\; \Phi w(\,\cdot\,,0) - \int_0^T \Phi_{T,s}\, a(\,\cdot\,,s)\tilde{v}(\,\cdot\,,s)\,\mathrm{d}s\,,$$

which can be written as

$$(\Phi - \mathrm{id})w(\,\cdot\,,0) \;=\; \int_0^T \Phi_{T,s}\, a(\,\cdot\,,s)\tilde{v}(\,\cdot\,,s)\,\mathrm{d}s\,.$$

Since, by assumption, $\Phi - \mathrm{id}$ is invertible, we can solve this equation for $w(\xi, 0)$. Substituting the expression for $w(\xi, 0)$ into (6.8), we see that $W = (w, w_\xi)$ satisfies (6.6). Exploiting the various regularity results obtained above, it is straightforward to establish the necessary estimate of V in terms of G.

6.2. The adjoint equation

As a preparation for the next section, we consider the adjoint parabolic equation

$$-v_t \;=\; Dv_{\xi\xi} - cv_\xi + a(\xi, t)^* v\,.$$

The evolution problem for this equation is well–posed in backward time. The corresponding time–$(-T)$ map, which maps profiles $v(\,\cdot\,, T)$ to profiles $v(\,\cdot\,, 0)$, is equal to the adjoint operator Φ^* of Φ. We may then associate with Φ^* the elliptic operator

$$\begin{aligned} \widetilde{\mathcal{T}} : H^1(\mathbb{R}, Y) \cap L^2\big(\mathbb{R}, Y^1\big) \quad &\longrightarrow \quad L^2(\mathbb{R}, Y)\,, \\ (v, w) \quad &\longmapsto \quad \big(v_\xi - w, w_\xi - D^{-1}(-v_t + cw - a(\xi, t)^* v)\big)\,. \end{aligned}$$

Next, we calculate the adjoint operator $\mathcal{T}^*$ of $\mathcal{T}$ considered as a closed operator in $L^2(\mathbb{R}, Y)$. Let $\mathcal{T} = \frac{\mathrm{d}}{\mathrm{d}\xi} - A$, where

$$A \;=\; \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\partial_t - a(\xi,\,\cdot\,)) & -D^{-1}c \end{pmatrix}.$$

We then have $\mathcal{T}^* = -\frac{\mathrm{d}}{\mathrm{d}\xi} - A^*$. Define the linear isomorphism

$$\mathcal{J} \;:\; H^1(\mathbb{R}/T\mathbb{Z}) \;\longrightarrow\; L^2(\mathbb{R}/T\mathbb{Z})\,, \quad v \;=\; \sum_{k\in\mathbb{Z}} v_k \mathrm{e}^{\mathrm{i}kt/T} \;\longmapsto\; \mathcal{J}v \;=\; \sum_{k\in\mathbb{Z}} (1+|k|) v_k \mathrm{e}^{\mathrm{i}kt/T}\,.$$

Using the explicit scalar product

$$\left\langle \begin{pmatrix} v \\ w \end{pmatrix}, \begin{pmatrix} \tilde v \\ \widetilde w \end{pmatrix} \right\rangle_Y \;:=\; \langle v, \mathcal{J}\tilde v\rangle_{L^2} + \langle w, \widetilde w\rangle_{L^2}$$

on Y, we calculate A^* as follows:

$$\begin{aligned} \left\langle A\begin{pmatrix} v \\ w \end{pmatrix}, \begin{pmatrix} \tilde v \\ \widetilde w \end{pmatrix} \right\rangle_Y \;&=\; \left\langle \begin{pmatrix} w \\ D^{-1}(\partial_t - a(\xi,\,\cdot\,))v - D^{-1}cw \end{pmatrix}, \begin{pmatrix} \tilde v \\ \widetilde w \end{pmatrix} \right\rangle_Y \\ &=\; \langle w, \mathcal{J}\tilde v\rangle_{L^2} + \big\langle D^{-1}(\partial_t - a(\xi,\,\cdot\,))v - D^{-1}cw, \widetilde w\big\rangle_{L^2} \\ &=\; \big\langle v, (\partial_t - a(\xi,\,\cdot\,))^* D^{-1}\widetilde w\big\rangle_{L^2} + \big\langle w, \mathcal{J}\tilde v - D^{-1}c\widetilde w\big\rangle_{L^2} \\ &=\; \left\langle \begin{pmatrix} v \\ w \end{pmatrix}, \begin{pmatrix} \mathcal{J}^{-1}(\partial_t - a(\xi,\,\cdot\,))^* D^{-1}\widetilde w \\ \mathcal{J}\tilde v - D^{-1}c\widetilde w \end{pmatrix} \right\rangle_Y \\ &=\; \left\langle \begin{pmatrix} v \\ w \end{pmatrix}, A^*\begin{pmatrix} \tilde v \\ \tilde w \end{pmatrix} \right\rangle_Y \end{aligned}$$

and therefore

$$A^* \;=\; \begin{pmatrix} 0 & \mathcal{J}^{-1}(\partial_t - a(\xi,\,\cdot\,))^* D^{-1} \\ \mathcal{J} & -D^{-1}c \end{pmatrix}$$

with $D(A^*) = H^1(\mathbb{R}/T\mathbb{Z}) \times H^{\frac{1}{2}}(\mathbb{R}/T\mathbb{Z})$.

In summary, we have

$$\mathcal{T}^* \;=\; -\frac{\mathrm{d}}{\mathrm{d}\xi} - \begin{pmatrix} 0 & -\mathcal{J}^{-1}(\partial_t + a(\xi,\,\cdot\,)^*)D^{-1} \\ \mathcal{J} & -D^{-1}c \end{pmatrix}$$

and

$$\widetilde{\mathcal{T}} \;=\; \frac{\mathrm{d}}{\mathrm{d}\xi} - \begin{pmatrix} 0 & \mathrm{id} \\ -D^{-1}(\partial_t + a(\xi,\,\cdot\,)^*) & -D^{-1}c \end{pmatrix}.$$

Lemma 6.1. *We have that* $\mathrm{N}(\mathcal{T}^*) \cong \mathrm{N}(\widetilde{\mathcal{T}})$.

Proof. The functions (v,w) and $(\tilde{v},\widetilde{w})$ are in the null spaces of $\mathcal{T}^*$ and $\widetilde{\mathcal{T}}$, respectively, if

$$v_\xi \;=\; \mathcal{J}^{-1}(\partial_t + a^*)D^{-1}w\,, \quad w_\xi \;=\; -\mathcal{J}v + cD^{-1}w$$

and

$$\tilde{v}_\xi \;=\; \widetilde{w}\,, \quad \widetilde{w}_\xi \;=\; -D^{-1}(\partial_t + a^*)\tilde{v} - cD^{-1}\widetilde{w}\,.$$

Exploiting the regularity properties of these equations and using the fact that $\mathcal{J} : H^1(\mathbb{R}/T\mathbb{Z}) \to L^2(\mathbb{R}/T\mathbb{Z})$ is an isomorphism, we see that the maps

$$\begin{aligned} (v(\xi), w(\xi)) &\longmapsto \left(D^{-1}w(-\xi), -D^{-1}w_\xi(-\xi)\right), \\ (\tilde{v}(\xi), \widetilde{w}(\xi)) &\longmapsto \left(\mathcal{J}^{-1}(c\tilde{v}(-\xi) - D\tilde{v}_\xi(-\xi)), D\tilde{v}(-\xi)\right) \end{aligned}$$

are the desired isomorphisms between $\mathrm{N}(\mathcal{T}^*)$ and $\mathrm{N}(\widetilde{\mathcal{T}})$ and its inverse. □

6.3. $\mathcal{T}$ is Fredholm with index zero if, and only if, $\Phi - \mathrm{id}$ is Fredholm with index zero

First, we construct an isomorphism between $\mathrm{N}(\mathcal{T})$ and $\mathrm{N}(\Phi - \mathrm{id})$. Given an element $V(\xi,t) = (v,w)(\xi,t)$ in $\mathrm{N}(\mathcal{T})$, we see that $v(\,\cdot\,,0)$ is in $\mathrm{N}(\Phi - \mathrm{id})$. The inverse of this map can be constructed as follows. Given an element $v_0(\xi)$ in $\mathrm{N}(\Phi - \mathrm{id})$, we solve the parabolic equation forward in time with initial condition $v_0(\xi)$ at $t = 0$. Let $v(\xi,t)$ be the resulting unique solution. Setting $V = (v, v_\xi)$, we see that V belongs to $\mathrm{N}(\mathcal{T})$. If either $\mathcal{T}$ or $\Phi - \mathrm{id}$ is Fredholm, the one of the two corresponding null spaces is finite–dimensional which proves that $\dim \mathrm{N}(\mathcal{T}) = \dim \mathrm{N}(\Phi - \mathrm{id}) < \infty$.

Using the results obtained in Section 6.1, it is not hard to prove that the range of $\mathcal{T}$ is closed provided $\Phi - \mathrm{id}$ has closed range. Analogously, if $\mathcal{T}$ has closed range, we see that the range of $\Phi - \mathrm{id}$ is closed.

Exploiting the results obtained above, we have that

$$\mathrm{N}(\mathcal{T}) \;\cong\; \mathrm{N}(\Phi - \mathrm{id})\,, \quad \mathrm{N}(\widetilde{\mathcal{T}}) \;\cong\; \mathrm{N}(\Phi^* - \mathrm{id})\,; \tag{6.9}$$

we also know that

$$\mathrm{N}(\Phi^* - \mathrm{id}) \;=\; \mathrm{R}(\Phi - \mathrm{id})^\perp\,, \quad \mathrm{N}(\mathcal{T}^*) \;=\; \mathrm{R}(\mathcal{T})^\perp\,. \tag{6.10}$$

In the last section, we have proved that $\mathrm{N}(\mathcal{T}^*) \cong \mathrm{N}(\widetilde{\mathcal{T}})$; see Lemma 6.1. Hence, using (6.9) and (6.10), we conclude that

$$\mathrm{R}(\Phi - \mathrm{id})^\perp \;\cong\; \mathrm{N}(\widetilde{\mathcal{T}}) \;\cong\; \mathrm{N}(\mathcal{T}^*) \;=\; \mathrm{R}(\mathcal{T})^\perp\,, \quad \mathrm{N}(\Phi - \mathrm{id}) \;\cong\; \mathrm{N}(\mathcal{T})\,.$$

This proves that $\mathcal{T}$ is Fredholm with index zero if, and only if, $\Phi - \mathrm{id}$ enjoys this property.

7. Proof of Proposition 2.10 and Theorem 3.1

In this section, we prove Proposition 2.10, Corollary 2.12 and Theorem 3.1.

7.1. Proof of Proposition 2.10

In this section, we prove Proposition 2.10. Thus, we have to demonstrate that the equation

$$\begin{pmatrix} u_\xi \\ v_\xi \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\partial_t + \alpha - a(\xi,t)) & -cD^{-1} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{7.1}$$

has an exponential dichotomy on $\mathbb{R}^+$ if, and only if, the equation

$$\begin{pmatrix} u_\xi \\ v_\xi \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\partial_t + \alpha - a_+(\xi,t)) & -cD^{-1} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{7.2}$$

does not have a purely imaginary spatial Floquet exponent.

If (7.2) does not have a purely imaginary Floquet exponent, then the operator $\mathcal{T}_+$ associated with (7.2) is invertible due to [27, Theorem 2.3]. Thus, on account of Remark 5.6 and Theorem 2.6, (7.2) has an exponential dichotomy on $\mathbb{R}$. Due to (U2) and [32, Theorem 1], we know that (7.1) then has an exponential dichotomy on $\mathbb{R}^+$.

Next, assume that (7.2) has a purely imaginary Floquet exponent. Thus, there exists a non–zero solution $U_{\mathrm{bd}}(\xi)$ of (7.2) and a real number β such that $U_{\mathrm{bd}}(\xi + np_+) = \mathrm{e}^{\mathrm{i}n\beta p_+} U_{\mathrm{bd}}(\xi)$ for any ξ and any integer n. We argue by contradiction. Hence, suppose that (7.1) has an exponential dichotomy on $\mathbb{R}^+$. By assumption, we know that U_{bd} satisfies

$$\begin{pmatrix} u_\xi \\ v_\xi \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\partial_t + \alpha - a(\xi,t)) & -cD^{-1} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ D^{-1}(a(\xi,t) - a_+(\xi,t))u \end{pmatrix}.$$

For any $\xi > \xi_0 \geq 0$, and with $G(\xi) := \big(0, D^{-1}(a(\xi,t) - a_+(\xi,t))u_{\mathrm{bd}}(\xi)\big)$, we therefore have

$$U_{\mathrm{bd}}(\xi) = \varphi_+^{\mathrm{s}}(\xi;\xi_0)U_{\mathrm{bd}}(\xi_0) + \int_{\xi_0}^{\xi} \varphi_+^{\mathrm{s}}(\xi;\zeta)G(\zeta)\,\mathrm{d}\zeta + \int_{\infty}^{\xi} \varphi_+^{\mathrm{u}}(\xi;\zeta)G(\zeta)\,\mathrm{d}\zeta\,,$$

since we assumed that (7.1) has an exponential dichotomy on $\mathbb{R}^+$. Using the estimates

$$\begin{aligned} |\varphi^{\mathrm{s}}(\xi;\zeta)U|_Y &\leq K\mathrm{e}^{-\eta\,|\xi-\zeta|}\,|U|_Y \quad \text{for} \quad \xi \geq \zeta \geq 0\,, \\ |\varphi^{\mathrm{u}}(\xi;\zeta)U|_Y &\leq K\mathrm{e}^{-\eta\,|\xi-\zeta|}\,|U|_Y \quad \text{for} \quad \zeta \geq \xi \geq 0\,, \end{aligned}$$

we therefore get

$$|U_{\mathrm{bd}}(\xi)|_Y \leq K\mathrm{e}^{-\eta\,|\xi-\xi_0|}\,|U_{\mathrm{bd}}(\xi_0)|_Y + \frac{2K}{\eta}|D^{-1}| \sup_{\xi\geq\xi_0} |a(\xi,t) - a_+(\xi,t)|\,|U_{\mathrm{bd}}(\xi)|_Y\,.$$

Evaluating at $\xi_0 = np_+$ and $\xi = (n+m)p_+$ for large integers n and m, and using that $|U_{\rm bd}(\xi + np_+)|_Y = |U_{\rm bd}(\xi)|_Y$ for all ξ and any integer n, we get

$$\begin{aligned} |U_{\rm bd}(0)|_Y &\leq K\mathrm{e}^{-\eta m p_+}\,|U_{\rm bd}(0)|_Y \\ &\quad + \frac{2K}{\eta}\left|D^{-1}\right|\left(\sup_{\zeta\geq np_+}|a(\zeta,t)-a_+(\zeta,t)|\right)\left(\sup_{0\leq\zeta\leq p_+}|U_{\rm bd}(\zeta)|_Y\right). \end{aligned}$$

Since $|a(\zeta,t) - a_+(\zeta,t)| \to 0$ as $\zeta \to \infty$, we see that $\sup_{\zeta\geq np_+}|a(\zeta,t)-a_+(\zeta,t)|$ is as small as we wish after choosing n sufficiently large. Therefore, $U_{\rm bd}(0) = 0$ which contradicts either Hypothesis (U2) or the assumption that $U_{\rm bd}(\xi)$ is a non–trivial bounded solution on $\mathbb{R}^+$.

7.2. Proof of Corollary 2.12

We use Theorem 2.8 to prove the assertion of Corollary 2.12. Let Σ_2 and Σ_∞ denote the spectra of Φ as an operator on $L^2(\mathbb{R},\mathbb{C}^n)$ and $C^0_{\rm unif}(\mathbb{R},\mathbb{C}^n)$, respectively. If $\lambda = \mathrm{e}^{\alpha T} \notin \Sigma_2$, then the spatial dynamical system

$$v_\xi = w, \quad w_\xi = D^{-1}(v_t + \alpha v - cw - a(\xi,t)v) \tag{7.3}$$

with $(u,v)(\xi) \in Y$ has an exponential dichotomy on $\mathbb{R}$ by Theorem 2.8. It then follows from [40, Lemma 4.1] that the map $\Phi - \lambda$ is invertible on the space $C^0_{\rm unif}(\mathbb{R},\mathbb{C}^n)$. Hence, $\lambda \notin \Sigma_\infty$. It remains to show that, if $\lambda \in \Sigma_2$, then $\lambda \in \Sigma_\infty$. Thus, assume that $\lambda = \mathrm{e}^{\alpha T} \in \Sigma_2$. If (7.3) has exponential dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$, then λ is also in Σ_∞; this follows again from [40, Lemma 4.1].

The remaining case is that (7.3) does not have exponential dichotomies on $\mathbb{R}^+$ or on $\mathbb{R}^-$. Let us assume that (7.3) does not have an exponential dichotomy on $\mathbb{R}^+$; the other case is analogous. In this case, we know that the asymptotic equation

$$v_\xi = w, \quad w_\xi = D^{-1}(v_t + \alpha v - cw - a_+(\xi,t)v)$$

has a bounded solution on $\mathbb{R}$ by Proposition 2.10. It then follows from the proof of [40, Lemma 4.1] that λ is in Σ_∞; see also [45, Lemma 6.4].

7.3. Proof of Theorem 3.1

The theorem follows using the arguments presented in Sections 4 and 5. Most of the arguments apply verbatim to (3.2) and (3.3). The only change is that we employ Galerkin projections as in [24] instead of Fourier series to approximate the infinite–dimensional system by ODEs.

8. Applications: spectra of long–wavelength periodic waves

We illustrate our results by investigating the existence and stability of spatially–periodic modulated waves that accompany modulated pulses.

To motivate our interest in this issue, suppose that a certain reaction–diffusion system exhibits a pulse that travels with non–zero wave speed c_∞. Typically, this pulse is then accompanied by a one–parameter family of spatially–periodic wave trains that consist of an infinite number of copies of the pulse, spaced periodically, i. e. equidistantly, along the domain $\mathbb{R}$. The wave trains are parametrized by their spatial period $2L$, with $L \to \infty$, and they have wave speeds c_L close to c_∞. Considered as waves on the finite interval $(-L, L)$ with periodic boundary conditions, these spatially–periodic waves are stable provided the pulse is stable. Considered as wave trains on the entire real line, however, the spectrum of the linearization about these spatially periodic waves consists entirely of essential spectrum: near each eigenvalue of the linearization about the pulse, there is a small circle of eigenvalues in the spectrum of the linearization about each of the wave trains; the diameter of this circle shrinks to zero as the spatial period $2L$ tends to infinity [11]. In particular, there is a critical circle of eigenvalues near $\lambda = 0$, which is always an eigenvalue of the pulse due to translation symmetry. Using exponential dichotomies for the spatial dynamics, we recently demonstrated [42] how the exact location of this circle can be determined from spectral information about the single pulse and geometric information about the homoclinic orbit that corresponds to the pulse in the spatial dynamics; see also Section 8.1 below.

Suppose next that the pulse destabilizes in a Hopf bifurcation, say. As a consequence, modulated pulses bifurcate. The issue that interests us is the existence of modulated spatially–periodic waves, with large spatial period, that accompany the modulated pulse. These additional waves arise through Hopf bifurcations from the wave trains that accompany the pulse; alternatively, they can be viewed as bifurcating from the modulated pulse in very much the same fashion as the wave trains bifurcate from the pulse. In this section, we investigate the existence and the stability of such spatially–periodic modulated waves that bifurcate from modulated pulses. We also discuss two mechanisms that generate modulated pulses: temporally–periodic forcing and Hopf bifurcations.

It is known that pulses for ill–posed spatial dynamical systems are often accompanied by spatially periodic waves; see [24]. As mentioned above, the stability of these wave trains has been studied in [42] using again spatial dynamics. The results that we derived in the previous sections relate the spectrum of the temporal period map, linearized about a modulated wave, to exponential dichotomies of the spatial dynamical system. Therefore, these results allow us to apply the theorems in [24, 42] to modulated travelling waves. We will constantly refer to these two articles for further details.

Our results imply in particular that modulated pulses can be computed safely by truncating the real line to a finite but large interval and imposing periodic boundary conditions; the modulated spatially periodic waves are then close to the modulated pulse and share its stability properties.

8.1. Modulated wave trains with long wavelength in non–autonomous systems

Consider the non–autonomous equation

$$u_t = Du_{\xi\xi} + cu_\xi + f(t, u), \quad u \in \mathbb{R}^n, \tag{8.1}$$

where the nonlinearity f is periodic in t with period $T > 0$ so that $f(t, u) = f(t+T, u)$. Throughout this section, we assume for the sake of clarity that the nonlinearity f is analytic; this assumption guarantees that the Hypotheses (U1) and (U2) are satisfied.

Notation 8.1. For every modulated wave $u(\xi, t)$ of (8.1) that satisfies $u(\xi, t) = u(\xi, t + T)$ for all (ξ, t), we consider the linearization

$$(8.2) \qquad v_t \;=\; Dv_{\xi\xi} + cv_\xi + f_u(t, u(\xi, t))v$$

of (8.1) about the wave $u(\xi, t)$ on the space $X = L^2(\mathbb{R}, \mathbb{C}^n)$. The temporal period map

$$\Phi(u) \;:\; X \;\longrightarrow\; X\,, \quad v(\,\cdot\,, 0) \;\longmapsto\; \Phi(u)v(\,\cdot\,, 0) \;=\; v(\,\cdot\,, T)$$

of (8.2) maps an initial value $v(\,\cdot\,, 0) \in X$ to the associated solution of (8.2) at time T.

Remark 8.2. Throughout Section 8, we consider the spectra of the linearization (8.2) about modulated pulses and spatially–periodic modulated waves. In both cases, the spectrum on $X = L^2(\mathbb{R}, \mathbb{C}^n)$ coincides with the spectrum on $C^0_{\mathrm{unif}}(\mathbb{R}, \mathbb{C}^n)$ by Corollary 2.12.

As a standing assumption, we assume that $q(\xi, t)$ is a smooth solution of (8.1) so that $q(\xi, t) = q(\xi, t + T)$ for all (ξ, t) and $q(\xi, t) \to p(t)$ for $|\xi| \to \infty$, uniformly in t. In other words, $q(\xi, t)$ is a modulated pulse with temporal period T that converges to the time–periodic spatially homogeneous solution $p(t)$ as $|\xi| \to \infty$.

First, we consider the existence of spatially–periodic modulated waves with large spatial period that bifurcate from $q(\xi, t)$. We begin by formulating the assumptions on the asymptotic state p and the pulse q. The next section contains an application of the forthcoming results to weakly–forced fast pulses of the FitzHugh–Nagumo equation.

Consider the linearization

$$(8.3) \qquad v_t \;=\; Dv_{\xi\xi} + cv_\xi + f_u(t, p(t))v$$

of (8.1) about $p(t)$. Of interest is then the spectrum of the temporal period map $\Phi(p)$.

Hypothesis (H). We assume that $\lambda = 1$ is not in the spectrum of the period map $\Phi(p)$ associated with (8.3).

Alternatively, we could require hyperbolicity of the equilibrium $p(t)$ for the spatial ξ–dynamics: as before, the equation

$$u_\xi \;=\; v\,, \quad v_\xi \;=\; D^{-1}(u_t - cv - f(t, u))\,,$$

with periodic boundary conditions in t describes the spatial dynamics of time–periodic solutions to (8.1). The linearization of this equation about $p(t)$ is given by

$$v_\xi \;=\; w\,, \quad w_\xi \;=\; D^{-1}(v_t - cw - f_u(t, p(t))v)\,,$$

which we also write as

$$(8.4) \qquad \frac{\mathrm{d}}{\mathrm{d}\xi} V \;=\; A(p)V\,, \quad V \;=\; (v, w)$$

on the space $Y = H^{\frac{1}{2}}(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^n) \times L^2(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^n)$ of T–periodic functions of t.

Remark 8.3. By Theorem 2.8, Hypothesis (H) is equivalent to the assumption that (8.4) has an exponential dichotomy on $\mathbb{R}$. In either case, there is a number $\eta > 0$ such that the spectrum of $A(p) : Y \to Y$ does not intersect the strip $\{z \in \mathbb{C};\ |\operatorname{Im} z| \le \eta\}$. In fact, a possible choice for η is the rate of exponential decay in Definition 2.1 for an exponential dichotomy.

Next, consider the linearization

$$v_t \;=\; Dv_{\xi\xi} + cv_\xi + f_u(t, q(\xi, t))v \tag{8.5}$$

of (8.1) about the modulated pulse $q(\xi, t)$. We are again interested in the spectrum of the temporal period map $\Phi(q) : X \to X$ associated with (8.5). Since the nonlinearity f does not explicitly depend upon ξ, we have that $\lambda = 1$ is in the spectrum of $\Phi(q)$ with eigenfunction $q_\xi(\xi, 0)$. On the other hand, $\lambda = 1$ is not contained in the essential spectrum due to Hypothesis (H) and the fact that $f_u(t, q(\xi, t))$ is a relatively compact perturbation of $f_u(t, p(t))$.

Hypothesis (T). We assume that $\lambda = 1$ has geometric and algebraic multiplicity one as an eigenvalue of the period map $\Phi(q)$ to (8.5).

We emphasize that Hypothesis (T) is never satisfied in the situation where the nonlinearity does not depend upon t since then the time–derivative $q_t(\xi, 0)$ of the modulated pulse supplies another linearly independent eigenfunction of $\lambda = 1$.

The spatial dynamical system associated with (8.5) is given by

$$v_\xi \;=\; w\,, \quad w_\xi \;=\; D^{-1}(v_t - cw - f_u(t, q(\xi, t))v)$$

or, in shorter notation, by

$$\frac{\mathrm{d}}{\mathrm{d}\xi} V \;=\; A(q)V\,, \quad V \;=\; (v, w) \in Y\,.$$

We also introduce the adjoint variational equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi} V \;=\; -A^*(q)V\,, \tag{8.6}$$

where $A^*(q) = A(q(\xi, \cdot\,))^*$ is the adjoint, taken pointwise for every fixed ξ, of $A(q(\xi, \cdot\,))$ in the Hilbert space Y. Due to Hypothesis (T) and Lemma 6.1, there exists a unique, up to constant scalar multiples, non–zero bounded solution to (8.6), which we denote by $\psi(\xi, t)$.

The following theorem guarantees the existence of time–periodic waves that are spatially periodic with large wavelength. Similar results were proved in [4, 23] for ODEs and evolutionary PDEs, and in [24] for elliptic PDEs.

Theorem 8.4. *Assume that the hyperbolicity assumption* (H) *and the transversality assumption* (T) *are both met. There are then positive numbers L_* and C so that the following is true. For every $L > L_*$, there exists a modulated travelling wave $q_L(\xi, t)$*

for some wave speed c_L so that $q_L(\xi,t) = q_L(\xi,t+T)$ and $q_L(\xi,t) = q_L(\xi+2L,t)$ as well as

$$|c_L - c_\infty| + \sup_{|\xi|\le L,\, 0\le t\le T} |q_L(\xi,t) - q(\xi,t)| \;\le\; C\mathrm{e}^{-\eta L}$$

are satisfied, where c_∞ is the wave speed of the modulated pulse $q(\xi,t)$ and $\eta > 0$ is the constant appearing in Remark 8.3.

Proof. The proof is a consequence of [24, Corollary 3]. The assumptions [24, (A1) – (A3), (H1) – (H2)] are immediate consequences of our set–up together with the Hypotheses (H) and (T) that we imposed. It remains to check the Melnikov condition [24, (H3)] which, in our notation, reads

$$M \;=\; \int_{-\infty}^{\infty} \big\langle \psi(\xi,\,\cdot\,), \big(0, D^{-1}q_\xi(\xi,\,\cdot\,)\big)\big\rangle_Y \,\mathrm{d}\xi \;\ne\; 0\,. \tag{8.7}$$

The brackets $\langle\,\cdot\,,\,\cdot\,\rangle_Y$ denote the scalar product in Y. The integrand in the above integral is the scalar product of the non–zero function $\psi(\xi,\,\cdot\,)$, which lies in the null space of the adjoint operator, and the derivative of the right–hand side of (8.1) with respect to the wave speed c, evaluated at the modulated pulse $(q, q_\xi) \in Y$. Following [36, Lemma 5.5], we argue that $M = 0$ implies that $\lambda = 1$ has algebraic multiplicity of at least two as an eigenvalue of the temporal period map $\Phi(q)$ of (8.5); this contradicts Hypothesis (T).

Hence, we argue by contradiction, and suppose that $M = 0$ is zero. The function $\big(0, D^{-1}q_\xi(\xi,\,\cdot\,)\big)$ would then belong to the range of $\mathcal{T}$ as defined in (2.6) with $a(\xi,t) = f_u(t, q(\xi,t))$. Choose (v,w) so that $\mathcal{T}(v,w) = -\big(0, D^{-1}q_\xi(\xi,\,\cdot\,)\big)$, then v satisfies

$$Dv_{\xi\xi} \;=\; v_t - cv_\xi - f_u(t, q(\xi,t))v - q_\xi(\xi,t)\,.$$

Therefore, $\tilde v(\xi,t) := v(\xi,t) + tq_\xi(\xi,t)$ satisfies

$$D\tilde v_{\xi\xi} \;=\; \tilde v_t - c\tilde v_\xi - f_u(t, q(\xi,t))\tilde v$$

and $\tilde v(\xi,T) = \tilde v(\xi,0) + Tq_\xi(\xi,0)$. Thus, in contradiction to Hypothesis (T), $\tilde v$ is a principal eigenfunction to the non–zero eigenfunction Tq_ξ associated with the eigenvalue $\lambda = 1$ of the period map $\Phi(q)$ to the linearized parabolic equation (8.3). □

Next, we address the issue of the stability of the spatially–periodic modulated waves $q_L(\xi,t)$ under the assumption that the modulated pulse $q(\xi,t)$ is stable.

Hypothesis (S). Assume that the modulated pulse $q(\xi,t)$ is stable, that is,

$$\Sigma \cap \{\lambda \in \mathbb{C};\ |\lambda| \ge 1\} \;=\; \{1\}$$

where Σ is the spectrum of the period map $\Phi(q) : X \to X$ of the linearized equation (8.5) about $q(\xi,t)$.

Let $\lambda = \mathrm{e}^{\alpha T}$, and consider the eigenvalue problem

$$v_\xi \;=\; w\,, \quad w_\xi \;=\; D^{-1}(v_t + \alpha v - c_L w - f_u(t, q_L(\xi,t))v)\,. \tag{8.8}$$

By Theorem 2.8 and Proposition 2.10, λ is in the spectrum Σ_L of the temporal period map $\Phi(q_L) : X \to X$ if, and only if, there exists a solution $(v,w)(\xi,t)$ to (8.8) for $|\xi| < L$ so that

$$(v,w)(L,t) \;=\; \mathrm{e}^{\mathrm{i}\gamma}(v,w)(-L,t) \tag{8.9}$$

for some $\gamma \in \mathbb{R}$; recall that the spatial period of $q_L(\xi,t)$ is equal to $2L$. We call γ the spatial Floquet exponent. We emphasize that it is here where we apply the results obtained in the earlier sections of this article. In the remaining part of this section, we consider the eigenvalue problem (8.8) with the boundary condition (8.9).

The following lemma states that the spectrum Σ_L of $\Phi(q_L)$ is close to the spectrum Σ of $\Phi(q)$. It therefore suffices to locate the spectrum Σ_L of the temporal period map $\Phi(q_L) : X \to X$ near $\lambda = 1$ to prove spectral stability of the spatially–periodic wave trains $q_L(\xi,t)$. Recall that, due to the Hypotheses (H) and (S), $\lambda = 1$ is a simple isolated eigenvalue of $\Phi(q)$.

Lemma 8.5. *Assume that the Hypotheses* (H), (T) and (S) *are satisfied. There exists a number* $0 < r < 1$ *so that the following is true. For every neighborhood* U *of* $\lambda = 1$ *in* $\mathbb{C}$*, there is a positive number* L_0 *so that* $\Sigma_L \subset U \cup \{|\lambda| < r\}$ *for all* $L > L_0$*.*

Proof. Consider a complex number $\lambda = \mathrm{e}^{\alpha T}$ that is not contained in the spectrum of the temporal period map $\Phi(q)$ associated with the modulated pulse q. For such values of α, the eigenvalue problem

$$v_\xi \;=\; w\,, \quad w_\xi \;=\; D^{-1}(v_t + \alpha v - cw - f_u(t,q(\xi,t))v) \tag{8.10}$$

has an exponential dichotomy on $\mathbb{R}$ by Theorem 2.8. Denote the associated projections by $P(\xi)$; we suppress the dependence on α. Due to [32, Corollary 2], the projections $P(\xi)$ converge exponentially, as $\xi \to \pm\infty$, to the spectral projection P_∞ of the asymptotic equation

$$v_\xi \;=\; w\,, \quad w_\xi \;=\; D^{-1}(v_t + \alpha v - cw - f_u(t,p(t))v)\,.$$

The boundary conditions (8.9) are transverse to the asymptotic stable and unstable eigenspaces given by $\mathrm{R}(P_\infty)$ and $\mathrm{N}(P_\infty)$ in the sense of [24, (T1)(ii)] since periodic boundary conditions satisfy [24, (T1)(ii)] by [24, Corollary 3]. Arguing as in [24, Section 5] demonstrates that the eigenvalue problem (8.9 – 8.10) has only the trivial solution $(v,w) = 0$. Since the waves q_L are close to q on the spatial interval $[-L,L]$, the eigenvalue problem (8.8 – 8.9) for q_L does not have a non–trivial solution provided L is sufficiently large. This proves that $\mathrm{e}^{\alpha T}$ is not in the spectrum of $\Phi(q_L)$.

The argument given above is uniform in compact subsets of the resolvent set of $\Phi(q)$. It therefore remains to show that it suffices to consider compact subsets in the resolvent set.

Consider the linearization

$$v_t \;=\; Dv_{\xi\xi} + cv_\xi + f_u(t,u(\xi,t))v\,, \quad v(\,\cdot\,,t) \in X \;\text{ for every }\; t \;\geq\; 0$$

of (8.1) about any modulated wave $u(\xi,t)$. The temporal period map $\Phi(u)$ associated with the above equation is a bounded linear operator on X. In particular, the spectrum of $\Phi(u)$ is bounded in $\mathbb{C}$ by a constant R that depends only on the sup–norm of the modulated wave u. Thus, the spectrum of $\Phi(q_L)$ is contained in a ball with radius R that is independent of L. Also, by Hypothesis (S), there is a number r with $0 < r < 1$ such that the entire spectrum of $\Phi(q)$, with the exception of the simple eigenvalue at $\lambda = 1$, is contained in the ball with radius r, centered at zero, in the complex plane. Hence, it suffices to exclude eigenvalues that are contained in the compact annulus $\{r \leq |\lambda| \leq R\}$ but that do not lie in the chosen neighborhood U of $\lambda = 1$. □

It remains to locate the spectrum of $\Phi(q_L)$ near $\lambda = 1$ to determine the stability properties of the spatially–periodic wave trains $q_L(\xi, t)$ of period $2L$ given in Theorem 8.4. Due to the discussion right before Lemma 8.5, it suffices to find all α close to zero and $\gamma \in \mathbb{R}$ for which the spatial eigenvalue problem (8.8 – 8.9) has a solution (v, w). We show that there is an analytic, complex–valued function $E(\alpha, \gamma)$ so that (8.8 – 8.9) has a solution for α close to zero precisely when $E(\alpha, \gamma) = 0$. The function E is commonly referred to as an Evans function: its zeros correspond to eigenvalues of spatial eigenvalue problems; see, for instance, [1, 10, 11] for more background. We also give an expansion of the Evans function in terms of geometric data of the modulated pulse $q(\xi, t)$ that allows us to actually determine its zeros, i.e. the spectrum of $\Phi(q_L)$ near $\lambda = 1$.

Theorem 8.6. *Assume that the Hypotheses* (H), (T) *and* (S) *are met. There are positive numbers δ, C and L_* such that the following is true for any $L > L_*$. A complex number $\lambda = \mathrm{e}^{\alpha T}$ with $|\alpha| < \delta$ is in the spectrum of $\Phi(q_L) : X \to X$ if, and only if, there is a $\gamma \in \mathbb{R}$ so that*

$$E(\alpha, \gamma) \;=\; 0 \tag{8.11}$$

where $E : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ is an analytic function that also depends on L (though we omit explicit mention of the dependence of E on L). Furthermore, we have the expansion

$$\begin{aligned} E(\alpha, \gamma) \;=\; & \big(\mathrm{e}^{\mathrm{i}\gamma} - 1\big)\langle \psi(L, \,\cdot\,), (q_\xi, q_{\xi\xi})(-L, \,\cdot\,)\rangle_Y \\ & + \big(1 - \mathrm{e}^{-\mathrm{i}\gamma}\big)\langle \psi(-L, \,\cdot\,), (q_\xi, q_{\xi\xi})(L, \,\cdot\,)\rangle_Y - \alpha M \\ & + \big(\mathrm{e}^{\mathrm{i}\gamma} - 1\big) R(\alpha, \gamma) + \alpha\, \widetilde{R}(\alpha, \gamma) \end{aligned} \tag{8.12}$$

where $\psi(\xi, t)$ is a non–zero bounded solution of the adjoint variational equation (8.6), *$q_\xi(\xi, t)$ is the spatial derivative of the modulated pulse, the spatial Floquet exponent γ appears as a parameter in the boundary–value problem* (8.8 – 8.9), *and the constant M is the Melnikov integral*

$$M \;=\; \int_{-\infty}^{\infty} \big\langle \psi(\xi, \,\cdot\,), \big(0, D^{-1} q_\xi(\xi, \,\cdot\,)\big)\big\rangle_Y \,\mathrm{d}\xi$$

given in (8.7). *The remainder terms $R(\alpha, \gamma)$ and $\widetilde{R}(\alpha, \gamma)$ are analytic in (α, γ) and satisfy*

$$\big|\partial_\alpha^{\jmath} \partial_\gamma^{\ell} R(\alpha, \gamma)\big| \;\leq\; C\mathrm{e}^{-3\eta L}\,, \quad \big|\partial_\gamma^{\ell} \widetilde{R}(\alpha, \gamma)\big| \;\leq\; C\mathrm{e}^{-\eta L} \tag{8.13}$$

for $\jmath, \ell \geq 0$, uniformly in L, where η is the rate of hyperbolicity that appears in Remark 8.3. *Both $R(\alpha, \gamma)$ and $\widetilde{R}(\alpha, \gamma)$ are real whenever $\big(\alpha, \mathrm{e}^{\mathrm{i}\gamma}\big)$ is real.*

We refer to [42, Theorem 2.1] for a more detailed expansion and a somewhat stronger statement, though only for ODEs.

Proof. The proof given in [42] for ODEs carries over almost verbatim to the elliptic setting as mentioned in [42, Remark 3.1]. The estimates given in [42, Lemma 3.1] for ODEs are also true in the elliptic case; see [32, Theorem 1 and Corollary 2]. □

Finally, we summarize the main conclusions obtained in [42]. Since we assumed in Hypothesis (T) that M is nonzero, the temporal Floquet exponent α lies on a circle close to zero: for each fixed L, we can solve (8.11) for α as a function of γ and obtain a small circle of critical eigenvalues that are parametrized by γ. Thus, the stability of the spatially–periodic modulated waves q_L depends on the precise location of the circle of critical eigenvalues. Typically, we expect that one of the first two terms in the expansion (8.12) for the bifurcation function is dominant (an exception are standing pulses in symmetric systems where the two terms are of the same order; see [42, Theorem 5.4]). There are then two different cases that we need to consider: denote by ν the leading eigenvalue of the operator $A(p)$, i.e. the eigenvalue with the smallest real part in modulus. This eigenvalue is then either real or not.

Case I: monotone tails. Suppose that there is a simple real eigenvalue ν^{u} in the spectrum of $A(p) : Y \to Y$ such that $|\mathrm{Re}\,\nu| > \nu^{\mathrm{u}} > 0$ for all other eigenvalue ν of $A(p)$. Assume also that, for some $\delta > 0$, we have

$$
\begin{aligned}
(q_\xi, q_{\xi\xi})(\xi,\,\cdot\,) &= \mathrm{e}^{\nu^{\mathrm{u}}\xi}\big(\rho + \mathrm{O}(\mathrm{e}^{\delta\xi})\big) \qquad \text{as} \quad \xi \to -\infty\,,\\
\psi(\xi,\,\cdot\,) &= \mathrm{e}^{-\nu^{\mathrm{u}}\xi}\big(\rho_* + \mathrm{O}(\mathrm{e}^{-\delta\xi})\big) \quad \text{for} \quad \xi \to \infty
\end{aligned}
$$

in Y for appropriate function ρ and ρ_* in Y with $\langle \rho, \rho_*\rangle_Y \neq 0$. The spatially–periodic modulated waves q_L are then stable if, and only if,

$$
M\langle \rho, \rho_*\rangle_Y \;>\; 0\,. \tag{8.14}
$$

Case II: oscillatory tails. Suppose that there is a pair of simple complex–conjugate eigenvalues ν^{u} and $\overline{\nu^{\mathrm{u}}}$ in the spectrum of $A(p)$ with $\mathrm{Im}\,\nu^{\mathrm{u}} \neq 0$ such that $|\mathrm{Re}\,\nu| > \mathrm{Re}\,\nu^{\mathrm{u}} > 0$ for all other eigenvalue ν in the spectrum of $A(p)$. We then have

$$
\big\langle \psi(L,\,\cdot\,), \big(q_\xi, q_{\xi\xi}\big)(-L,\,\cdot\,)\big\rangle_Y \;=\; a\sin(2\,\mathrm{Im}\,\nu^{\mathrm{u}}L + b)\mathrm{e}^{-2\,\mathrm{Re}\,\nu^{\mathrm{u}}L} + \mathrm{O}\Big(\mathrm{e}^{-(2\,\mathrm{Re}\,\nu^{\mathrm{u}}+\delta)L}\Big)
$$

for certain constants a, b. Assume that $a \neq 0$. The spatially–periodic modulated waves q_L then stabilize and destabilize periodically as the wavelength $2L$ tends to infinity. The frequency of the exchange of stability is equal to $\mathrm{Im}\,\nu^{\mathrm{u}}$, i.e. to the wavenumber of the spatial oscillations in the dominant tail of the modulated pulse q.

A proof of these statements can be found in [42, Section 5.1].

8.2. Application I: periodically forced pulses

Modulated pulses can be created by adding a small time–periodic perturbation to an autonomous reaction–diffusion equation that exhibits travelling pulses. We demonstrate that the forcing term may change the decay properties at the tails of the pulse;

as a consequence, the individual modulated pulses in a spatially–periodic modulated wave train interact with each other in a qualitatively different fashion.

Consider

$$u_t \;=\; Du_{\xi\xi} + cu_\xi + f(u) + \mu g(t,u;\mu) \tag{8.15}$$

where μ is small and, for some temporal period $T>0$, we have $g(t,u;\mu) = g(t{+}T,u;\mu)$ for all (t,u,μ). We assume that, for $\mu = 0$ and an appropriate value of c, equation (8.15) exhibits a stationary pulse $q(\xi)$ such that $q(\xi) \to 0$ as $|\xi| \to \infty$. Suppose that this pulse is stable so that the spectrum of the linearization $L = D\partial_{\xi\xi} + c\partial_\xi + f_u(q(\xi))$ is contained in the open left half–plane with the exception of a simple eigenvalue at $\lambda = 0$ that is inevitable due to translation symmetry.

Applying center–manifold theory to the family $q(\,\cdot + \tau)$ of equilibria of the parabolic equation (8.15), we see that there is a family $q(\xi,t;\mu)$ of modulated pulses with wave speed $c = c(\mu)$ to (8.15) for every μ close to zero. We are interested in spatially–periodic modulated waves that accompany this family of modulated pulses. Thus, we consider the spatial dynamics

$$u_\xi \;=\; v\,, \quad v_\xi \;=\; D^{-1}(u_t - cv - f(u) + \mu g(t,u;\mu)) \tag{8.16}$$

on the space Y of time–periodic functions with period T in t. The linearization of this equation about $q(\xi)$ at $\mu = 0$ is given by

$$v_\xi \;=\; w\,, \quad w_\xi \;=\; D^{-1}(v_t - cw - f_u(q(\xi))v)\,,$$

where $(v,w)(\xi,\,\cdot\,)$ has period T in t. For $\mu = 0$, the pulse $q(\xi)$ corresponds to a homoclinic orbit $(q,q_\xi)(\xi)$ for the spatial dynamics (8.16) on Y. Theorem 2.8 implies that $q(\xi)$ satisfies Hypotheses (H) and (T); in other words, the equilibrium $(u,v) = 0$ of (8.16) is hyperbolic, and the linearization about $q(\xi)$ admits a unique bounded solution. Theorem 8.4 then guarantees the existence of spatially–periodic modulated waves $q_L(\xi,t;\mu)$ with large wavelength $2L$ to (8.15) for every sufficiently small μ.

As mentioned above, we are interested in the stability of the bifurcating spatially–periodic modulated waves. Note that these waves can be interpreted as trains of infinitely many equidistantly spaced copies of the modulated pulse $q(\xi,t;\mu)$. The stability of the wave trains depends then on the interaction properties of the individual modulated pulses that make up the wave train. The most interesting situation occurs if the periodic travelling waves $q_L(\xi)$ that accompany the original pulse $q(\xi)$ for $\mu = 0$ are all stable. The issue is then whether the modulated wave trains inherit stability.

Before we address this issue, we comment on equation (8.16)

$$u_\xi \;=\; v\,, \quad v_\xi \;=\; D^{-1}(u_t - cv - f(u)) \tag{8.17}$$

with $\mu = 0$ posed on the space Y of time–periodic functions with period T. Note that this equation does not depend explicitly on t, and is therefore equivariant with respect to shifts in time. The perturbation $g(t,u;\mu)$ breaks this S^1–symmetry for $\mu \neq 0$. The fixed–point space $Y_{\rm fix}$ of the S^1–symmetry consists of all time–independent functions; it is therefore isomorphic to $\mathbb{R}^{2n}$. The dynamics on $Y_{\rm fix}$ is given by the travelling–wave ODE. In particular, the homoclinic orbit $(q,q_\xi)(\xi)$ is contained in $Y_{\rm fix}$ as are the periodic orbits associated with $q_L(\xi)$. As we shall see below, the different stability properties of the travelling and modulated wave trains are related to the different dynamical properties of (8.17) regarded as an equation on either Y_{fix} or Y.

The periodic travelling waves $q_L(\xi)$ are stable provided the following assumptions are met. First, we assume that the leading eigenvalue ν^{u} of the matrix

$$\begin{pmatrix} 0 & 1 \\ -D^{-1}f_u(0) & -cD^{-1} \end{pmatrix}$$

on the space $Y_{\text{fix}} \cong \mathbb{R}^{2n}$ is real, positive and simple. Note that this matrix represents the linearization of (8.16) about $(u,v) = 0$ at $\mu = 0$ restricted to the fixed–point space of the symmetry group. In addition, we assume that $M\langle\rho,\rho_*\rangle_{\mathbb{R}^{2n}} > 0$, where M is the Melnikov integral

$$M \;=\; \int_{-\infty}^{\infty} \big\langle \psi(\xi), \big(0, D^{-1}q_\xi(\xi)\big)\big\rangle_{\mathbb{R}^{2n}}\,\mathrm{d}\xi$$

with respect to the parameter c, and ρ, ρ_* are defined by

$$\rho \;=\; \lim_{\xi\to-\infty}(q_\xi, q_{\xi\xi})(\xi)\mathrm{e}^{-\nu^{\text{u}}\xi}\,, \qquad \rho_* \;=\; \lim_{\xi\to\infty}\psi(\xi)\mathrm{e}^{\nu^{\text{u}}\xi}\,.$$

Note that the unique bounded solution $\psi(\xi)$ to the adjoint variational equation about $q(\xi)$ is independent of t. Under these hypotheses, [42, Corollary 5.1] asserts that the periodic travelling waves $q_L(\xi)$ are stable.

After these preparations, we return to the issue of whether the modulated wave trains inherit the stability from the travelling wave trains. We shall see below that either all modulated wave trains are stable or else locking phenomena occur that render unstable wave trains for a range of spatial wavelengths. Which of these two cases occurs depends on the linearization

$$v_\xi \;=\; w\,, \quad w_\xi \;=\; D^{-1}(v_t - cw - f_u(0)v) \tag{8.18}$$

of (8.17) about $(u,v) = 0$ posed on the entire space Y. We denote by ν_* the leading eigenvalue of the right–hand side of (8.18) posed on Y. Recall that ν^{u} is the leading eigenvalue of (8.18) restricted to the fixed–point space $Y_{\text{fix}} \cong \mathbb{R}^{2n}$.

Case I: monotone tails. If $\nu^{\text{u}} = \nu_*$ is also the leading eigenvalue of (8.18) on Y, then the modulated wave trains $q_L(\xi,t;\mu)$ are stable for all sufficiently large L. In other words, the modulated wave trains inherit the stability from the travelling wave trains. Indeed, the tails of the homoclinic orbit that corresponds to the modulated pulse $q(\xi,t;\mu)$ decay with rate ν^{u} and are therefore monotone. This follows immediately from the continuity of the expression $M\langle\rho,\rho_*\rangle_Y$ in μ. As an example for this case, we discuss the FitzHugh–Nagumo equation; see below.

Case II: oscillatory tails. This second case occurs if $\nu^{\text{u}} \neq \nu_*$. Hence, the eigenfunction to the leading eigenvalue ν_* is then time–dependent since it is not contained in Y_{fix}. Therefore, ν_* is typically non–real due to the non–trivial S^1–symmetry on the eigenspace. We assume that the leading eigenvalues ν_* and $\overline{\nu_*}$ are simple and complex conjugate, and that there are no other eigenvalues with real part equal to $\pm|\mathrm{Re}\,\nu_*|$. It is convenient to assume that $\mathrm{Re}\,\nu_* > 0$; we emphasize that the results outlined below are also true if $\mathrm{Re}\,\nu_* < 0$.

For $\mu = 0$, the homoclinic orbit for the spatial dynamics on Y lies in $Y_{\rm fix}$. Therefore, it is forced to approach the equilibrium $(u, v) = 0$ along the strong unstable manifold. This situation is robust in the class of S^1–equivariant perturbations. The parameter μ, however, breaks the S^1–symmetry since the perturbation $\mu g(t, u; \mu)$ depends explicitly on t. Thus, for generic nonlinearities g, the homoclinic orbits that correspond to the modulated pulses $q(\xi, t; \mu)$ approach the equilibrium $(u, v) = 0$ along the direction with the smallest possible exponential decay rate $\mathrm{Re}\,\nu_*$ as soon as $\mu \neq 0$. Since the eigenvalue ν_* in the weakest direction has non–zero imaginary part, the tails of the modulated pulse become spatially oscillatory under periodic forcing, and the accompanying spatially–periodic modulated waves are alternately stable and unstable for sufficiently large wavelengths L.

As mentioned above, for $\mu \neq 0$, the modulated pulses typically approach the asymptotic state $(u, v) = 0$ along the weak unstable direction as $\xi \to -\infty$. The resulting bifurcation is called an orbit–flip since the homoclinic orbit typically flips from one side of the strong unstable manifold to the other as μ crosses through zero. Generically, many multi–hump pulses are created in such an orbit–flip bifurcation as has been proved in [35]; we also refer to [38] where orbit–flips were studied in equivariant systems with symmetry–breaking perturbations.

8.3. The FitzHugh–Nagumo equation

In this section, we apply the results above to the FitzHugh–Nagumo equation with a small diffusion term in the second variable. Consider

$$u_t \;=\; u_{xx} + f(u) - w\,, \quad w_t \;=\; \delta^2 w_{xx} + \epsilon(u - \gamma w)\,, \tag{8.19}$$

where $f(u) = u(1 - u)(a - u)$, $0 < a < 1/2$ and the positive parameters δ, ϵ and γ are small. Travelling waves $q(x - ct)$ are described by a differential equation in $\mathbb{R}^4$ that involves (u, u_ξ, w, w_ξ) and the wave speed c. It has been proved in [1] that, for an appropriate wave speed c, equation (8.19) has a stable pulse $q(x - ct)$ that is asymptotic to $u = w = 0$.

We argue that the spatially–periodic wave trains that accompany this pulse are also stable: the construction in [1] shows that, for δ small enough, the spatial dynamics can be reduced to a three–dimensional slow manifold. On the slow manifold, the flow is close to the spatial dynamics with $\delta = 0$. For this flow, we proved in [42] that the condition $M\langle \rho, \rho_* \rangle > 0$ which implies stability of the spatially–periodic travelling waves is satisfied. Since the slow manifold is normally hyperbolic, and the solutions ψ and $(q_\xi, q_{\xi\xi})$ lie in it, we conclude that the inequality $M\langle \rho, \rho_* \rangle > 0$ is also true for the full four–dimensional flow. Thus, the spatially–periodic wave trains with large wavelength are also stable.

Next, we consider the FitzHugh–Nagumo equation (8.19) subject to small external time–periodic forcing. We argue that, for any period of the forcing, we always encounter Case I which implies that the spatially–periodic modulated wave trains with large wavelengths are also stable regardless of their spatial period. Indeed, consider the linearization of the FitzHugh–Nagumo system about the origin, written as a dynamical system in the spatial variable ξ on the space Y of time–periodic functions. For ϵ sufficiently small, the leading eigenvalue $\nu^{\rm u}$ in the fixed–point space $Y_{\rm fix}$ is real, positive and of order ϵ. On the other hand, it is easy to verify that the other eigenvalues in the entire space Y keep an O(1)–distance from the imaginary axis as ϵ tends to zero. The reason is that an imaginary eigenvalue for $\epsilon = 0$ would imply that the origin undergoes a Hopf bifurcation at $\epsilon = 0$; it is known, however, that such a bifurcation does not occur. This proves that the eigenvalue ν^{u} is the leading eigenvalue for spatial dynamics on Y.

More generally, the above considerations suggest that, in singularly perturbed systems, the stability of long–wavelength spatially–periodic wave trains is typically robust under temporally periodic forcing.

8.4. Modulated wave trains with long wavelength in autonomous systems

In this section, we concentrate on the autonomous reaction–diffusion system

$$u_t \;=\; Du_{\xi\xi} + cu_\xi + f(u)\,, \quad u \in \mathbb{R}^n\,. \tag{8.20}$$

We assume that $q(\xi, t)$ is a modulated pulse to (8.20) with temporal period $T > 0$ so that $q(\xi, t)$ converges to some homogeneous time–independent rest state $p \in \mathbb{R}^n$ of (8.20) as $|\xi| \to \infty$. We use the same notation as in Section 8.1.

We begin by stating the assumptions on the asymptotic rest state p and the modulated pulse $q(\xi, t)$. As above, we assume hyperbolicity of the asymptotic rest state p.

Hypothesis $\big(\widetilde{\mathbf{H}}\big)$. We assume that $\lambda = 1$ is not in the spectrum of the temporal period map $\Phi(p) : X \to X,\ v(\,\cdot\,, 0) \mapsto v(\,\cdot\,, T)$ associated with the linearization $v_t = Dv_{\xi\xi} + cv_\xi + f_u(p)v$ of (8.20) about p.

Note that this assumption implies that p cannot depend upon t; otherwise, $p_t(t)$ would be an eigenfunction of $\Phi(p)$ associated with the eigenvalue $\lambda = 1$. We also remark that $\big(\widetilde{\mathrm{H}}\big)$ is equivalent to the condition

$$\det\big(- D\nu^2 + \mathrm{i}c\nu + f_u(p) - \mathrm{i}\,nT\big) \;\neq\; 0$$

for all $\nu \in \mathbb{R}$ and $n \in \mathbb{Z}$.

Equation (8.20) is invariant under shifts in space and time. Thus, the space and time translates $q(\xi + \xi_0, t + t_0)$ are also modulated pulses. Therefore, $q_\xi(\xi, t)$ and $q_t(\xi, t)$ are eigenfunctions associated with the eigenvalue $\lambda = 1$ of the temporal period map $\Phi(q)$ of the linearization

$$v_t \;=\; Dv_{\xi\xi} + cv_\xi + f_u(q(\xi, t))v \tag{8.21}$$

about the modulated pulse. We assume that the modulated pulse q is not a travelling pulse so that q_ξ and q_t are linearly independent.

Hypothesis $\big(\widetilde{\mathbf{T}}\big)$. We assume that $\lambda = 1$ is an eigenvalue of the temporal period map $\Phi(q)$ of (8.21) with geometric and algebraic multiplicity two and associated linearly independent eigenfunctions q_ξ and q_t.

Equation (8.21), written in the spatial dynamics, reads

$$\begin{pmatrix} v_\xi \\ w_\xi \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\partial_t - f_u(q(\xi,t))) & -cD^{-1} \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = A(q) \begin{pmatrix} v \\ w \end{pmatrix}.$$

Its adjoint $V_\xi = -A(q)^* V$ then has two linearly independent solutions that we denote by $\psi_1(\xi,t)$ and $\psi_2(\xi,t)$.

We shall investigate the existence and stability of spatially–periodic modulated waves that bifurcate from the modulated pulse in the spatial dynamics. Since the eigenspace, associated with the eigenvalue $\lambda = 1$, of the temporal period map $\Phi(q)$ is two–dimensional, we need two parameters for a proper unfolding. The first parameter is the wave speed c, while the temporal period T supplies the second parameter. We begin by calculating the relevant Melnikov integrals with respect to the parameters (c,T).

Lemma 8.7. *Assume that Hypotheses* $(\widetilde{\mathrm{H}})$ *and* $(\widetilde{\mathrm{T}})$ *are met. Define*

$$M_1 = \left(\int_{-\infty}^{\infty} \langle \psi_1(\xi,\cdot), (0, D^{-1} q_\xi(\xi,\cdot)) \rangle_Y \,\mathrm{d}\xi, \int_{-\infty}^{\infty} \langle \psi_2(\xi,\cdot), (0, D^{-1} q_\xi(\xi,\cdot)) \rangle_Y \,\mathrm{d}\xi \right),$$
$$M_2 = \left(\int_{-\infty}^{\infty} \langle \psi_1(\xi,\cdot), (0, D^{-1} q_t(\xi,\cdot)) \rangle_Y \,\mathrm{d}\xi, \int_{-\infty}^{\infty} \langle \psi_2(\xi,\cdot), (0, D^{-1} q_t(\xi,\cdot)) \rangle_Y \,\mathrm{d}\xi \right),$$

then M_1 *and* M_2 *are linearly independent vectors in* $\mathbb{R}^2$.

P r o o f. Suppose that the two vectors are linearly dependent. We argue that the algebraic multiplicity of the eigenvalue $\lambda = 1$ of the linearized temporal period map $\Phi(q)$ is then at least three, contradicting Hypothesis $(\widetilde{\mathrm{T}})$; see the proof of Theorem 8.4 for a similar argument. If $rM_1 + sM_2 = 0$ for some constants r and s with $rs \neq 0$, then $(0, rq_\xi(\xi,\cdot) + sq_t(\xi,\cdot))$ belongs to the range of the operator $\mathcal{T}$, defined in (2.6), with $a(\xi,t) = f_u(q(\xi,t))$. Hence, there is a function $v(\xi,t)$ with $v(\xi,T) = v(\xi,0)$ such that

$$-(rq_\xi(\xi,t) + sq_t(\xi,t)) = Dv_{\xi\xi} - v_t + cv_\xi + f_u(q(\xi,t))v,$$

and $w = v + t(rq_\xi + sq_t)$ satisfies

$$w_t = Dw_{\xi\xi} + cw_\xi + f_u(q(\xi,t))w, \quad w(\xi,T) = w(\xi,0) + rTq_\xi(\xi,0) + sTq_t(\xi,0).$$

The function $w(\xi,0)$ is therefore a generalized eigenfunction associated with the eigenvalue $\lambda = 1$ of $\Phi(q)$. This contradicts $(\widetilde{\mathrm{T}})$. □

A consequence of this observation is that modulated pulses are accompanied by spatially–periodic modulated waves with large spatial wavelengths.

Theorem 8.8. *Assume that* $(\widetilde{\mathrm{H}})$ *and* $(\widetilde{\mathrm{T}})$ *are met. There are then positive numbers* L_*, C *and* $\eta > 0$ *so that the following is true. For every* $L > L_*$, *there exists a modulated travelling wave* $q_L(\xi,t)$ *of* (8.20) *for some wave speed* c_L *and some temporal period* T_L *so that* $q_L(\xi,t) = q_L(\xi,t+T_L)$ *and* $q_L(\xi,t) = q_L(\xi+2L,t)$ *as well as*

$$|c_L - c_\infty| + |T_L - T_\infty| + \sup_{|\xi| \le L,\, 0 \le t \le T_\infty} |q_L(\xi, tT_L/T_\infty) - q(\xi,t)| \le C\,\mathrm{e}^{-\eta L}$$

are satisfied, where c_∞ *and* T_∞ *are the wave speed and the temporal period, respectively, of the modulated pulse* $q(\xi,t)$.

The exponential rate $\eta > 0$ is determined by the linearization of (8.20) about the homogeneous rest state p; see Remark 8.3.

Proof. Modulated waves with temporal period $\hat{T}$ are solutions to (8.20) written as the dynamical system

$$u_\xi \;=\; v\,, \quad v_\xi \;=\; D^{-1}(u_t - cv - f(u))$$

in the spatial variable ξ posed on the space $(u,v) \in H^{\frac{1}{2}}\big(\mathbb{R}/\hat{T}\mathbb{Z}\big) \times L^2\big(\mathbb{R}/\hat{T}\mathbb{Z}\big)$. Since the parameter $\hat{T}$ appears in the above function spaces, it is convenient to rescale time $t \to \omega t$ where $\omega = \hat{T}/T_\infty$. We then obtain the new system

$$u_\xi \;=\; v\,, \quad v_\xi \;=\; D^{-1}(\omega u_t - cv - f(u))\,, \tag{8.22}$$

with $(u,v) \in H^{\frac{1}{2}}(\mathbb{R}/T_\infty\mathbb{Z}) \times L^2(\mathbb{R}/T_\infty\mathbb{Z})$. Any bounded solution to (8.22) with parameter ω is a modulated wave to (8.20) with temporal period $\hat{T} = \omega T_\infty$. In [42, Theorem 4.1] we derived the following bifurcation equations for the existence of periodic orbits (i. e. modulated spatially–periodic waves) with period $2L$ near a homoclinic orbit (i. e. a modulated pulse) of the equivariant spatial dynamics:

$$M_1(c-c_\infty) + M_2(\omega - 1) + R(c,\omega) \;=\; 0\,, \tag{8.23}$$

where we omit the dependence of the remainder term $R : \mathbb{R}^2 \to \mathbb{R}^2$ on L. The remainder term is differentiable in (c,ω), and we have

$$\begin{aligned} |R(c,\omega)| \;&\le\; C\big(\mathrm{e}^{-\eta L} + |c-c_\infty| + |\omega - 1|\big)^2\,, \\ \big|\partial_{(c,\omega)}R(c,\omega)\big| \;&\le\; C\big(\mathrm{e}^{-\eta L} + |c-c_\infty| + |\omega - 1|\big) \end{aligned}$$

for (c,ω) near $(c_\infty,1)$, uniformly in L. Here, η is the exponential decay rate that appears in Remark 8.3. More precisely, there exists a spatially–periodic solution with period $2L$, wave speed c and temporal period ωT_∞ if, and only if, (8.23) is satisfied. The proof given in [42] for ODEs carries over immediately to the spatial dynamics considered here. Exploiting that M_1 and M_2 are linearly independent by Lemma 8.7, we can solve (8.23) for (c,ω) as a function of L for all sufficiently large L by the implicit function theorem. □

Finally, we consider the stability of the spatially–periodic wave trains $q_L(\xi,t)$. We have the following result that reduces the issue of locating the spectrum of the temporal period map $\Phi(q_L)$ near $\lambda = 1$ to solving a nonlinear equation of the form $\det E(\alpha,\gamma) = 0$ where α and γ are temporal and spatial Floquet exponents. The analytic function $E(\alpha,\gamma)$ is a matrix–valued Evans function; see also Theorem 8.6 above.

We define

$$Q_1(\xi,t) \;:=\; (q_\xi, q_{\xi\xi})(\xi,t) \quad \text{and} \quad Q_2(\xi,t) \;:=\; (q_t, q_{t\xi})(\xi,t)\,.$$

Theorem 8.9. *Assume that the Hypotheses* $(\widetilde{\mathrm{H}})$, $(\widetilde{\mathrm{T}})$ *and* (S) *are met. There are positive numbers* δ, C *and* L_* *such that the following is true for any* $L > L_*$. *A* *complex number* $\lambda = \mathrm{e}^{\alpha T}$ *with* $|\alpha| < \delta$ *is in the spectrum of* $\Phi(q_L) : X \to X$ *if, and only if, there is a* $\gamma \in \mathbb{R}$ *so that*

$$\det E(\alpha,\gamma) \;=\; 0$$

where $E : \mathbb{C}\times\mathbb{R} \to \mathbb{C}^{2\times 2}$ *is an analytic matrix–valued function that also depends on* L *(though we shall omit explicit mention of the dependence of* E *on* L*). The entries* E_{jk} *of* E *are given by*

$$\begin{aligned} E_{jk}(\alpha,\gamma) \;=\; & \left(\mathrm{e}^{\mathrm{i}\gamma}-1\right)\big\langle \psi_j(L,\,\cdot\,), Q_k(-L,\,\cdot\,)\big\rangle_Y \\ & + \left(1-\mathrm{e}^{-\mathrm{i}\gamma}\right)\big\langle \psi_j(-L,\,\cdot\,), Q_k(L,\,\cdot\,)\big\rangle_Y - \alpha M_{j,k} \\ & + \left(\mathrm{e}^{\mathrm{i}\gamma}-1\right) R_{jk}(\alpha,\gamma) + \alpha\, \widetilde{R}_{jk}(\alpha,\gamma) \end{aligned}$$

where $j,k = 1,2$*. The functions* $\psi_j(\xi,t)$ *are two linearly independent, non–zero bounded solutions of the adjoint variational equation, see Lemma* 8.7*,* Q_1 *and* Q_2 *have been introduced right before the theorem, the spatial Floquet exponent* γ *appears as a parameter in the boundary–value problem* (8.8 – 8.9)*, and the constant* $M_{j,k}$ *is the* j*th component of the vector* M_k *introduced in Lemma* 8.7*. Finally, the remainder terms* R *and* $\widetilde{R}$ *satisfy the estimates* (8.13) *in Theorem* 8.6*.*

The theorem is a consequence of [42, Theorem 2.2] and the discussion in the proof of Theorem 8.6 above.

In general, it is still difficult to use Theorem 8.9 to locate the spectrum near $\lambda = 1$ since the double eigenvalue $\lambda = 1$ typically splits into two small circles for finite values of L. A similar problem arises for travelling pulses to complex Ginzburg–Landau equations (CGL). For these equations, we have an additional S^1–symmetry that is induced by multiplication by $\mathrm{e}^{\mathrm{i}\beta}$. This gauge symmetry actually models the time–shift symmetry that we investigated here; it is therefore no surprise that the stability analysis for travelling pulses to the CGL is very similar to the one for modulated pulses to reaction–diffusion equations. We refer to a forthcoming work [37] of one of the authors for a stability analysis for the CGL that can then also be used for modulated pulses.

Note that Theorem 8.9 is related to the PDE stability of the modulated wave trains $q_L(\xi,t)$ considered as solutions of

$$u_t \;=\; Du_{\xi\xi} + cu_\xi + f(u)\,, \quad \xi \in \mathbb{R}$$

posed on the real line. It is often natural to regard the waves $q_L(\xi,t)$ as solutions to

$$\begin{aligned} u_t \;&=\; Du_{\xi\xi} + cu_\xi + f(u)\,, \\ u(-kL,t) \;&=\; u(kL,t)\,, \quad u_\xi(-kL,t) \;=\; u_\xi(kL,t) \end{aligned} \tag{8.24}$$

on the interval $(-kL,kL)$ for some positive integer k. Recall that the spatial period of $q_L(\xi,t)$ is $2L$. We then have the following stability result for q_L considered as a solution to (8.24).

Corollary 8.10. *Fix an integer* $k > 0$*. Under the assumptions, and the setting, of Theorem* 8.9*, the following is true. A complex number* $\lambda = \mathrm{e}^{\alpha T}$ *with* $|\alpha| < \delta$ *is in the spectrum of the linearization of* (8.24) *about* $q_L(\xi,t)$ *if, and only if,*

$$\det E(\alpha, 2\pi m/k) \;=\; 0 \tag{8.25}$$

for some integer m with $0 \le m < k$. In particular, the spatially–periodic waves $q_L(\xi,t)$ are stable on the minimal interval $(-L,L)$, and the spectrum near $\lambda = 1$ consists of a double eigenvalue at $\lambda = 1$.

Proof. Due to Theorem 8.9, we have that solutions to (8.25) correspond to solutions of the eigenvalue problem (8.8)

$$v_\xi \;=\; w\,, \quad w_\xi \;=\; D^{-1}(v_t + \alpha v - c_L w - f_u(q_L(\xi,t))v)$$

on $(-L,L)$ with boundary conditions (8.9)

$$(v,w)(L,t) \;=\; \mathrm{e}^{\mathrm{i}\gamma}(v,w)(-L,t)\,.$$

Such solutions fit into the interval $(-kL,kL)$ with periodic boundary conditions if, and only if, $\gamma = 2\pi m/k$ for some m with $0 \le m < k$. Thus, the first assertion of the corollary follows. For $k = 1$, we necessarily have $m = 0$ so that $\mathrm{e}^{\mathrm{i}\gamma} = 1$. The entries of the matrix $E(\alpha,0)$ are given by

$$E_{jk}(\alpha,\gamma) \;=\; -\alpha M_{j,k} + \alpha\,\widetilde{R}_{jk}(\alpha,\gamma)\,,$$

and we have $\det E(\alpha,0) = 0$ if, and only if, $\alpha = 0$. In fact, since the matrix M is invertible, we have

$$\det E(\alpha,0) \;=\; \alpha^2 \det M + \mathrm{O}\big(|\alpha|^3\big)\,.$$

Thus, α is a zero of $E(\alpha,0)$ of order two. It is a consequence of [36, Lemma 4.1] and the construction of $E(\alpha,\gamma)$ in [42] that the order of α as a zero of $E(\alpha,0)$ is equal to the algebraic multiplicity of $\lambda = \mathrm{e}^{\alpha T}$ as an eigenvalue of the linearization about q_L; see also [10] for a proof of this assertion for periodic travelling waves. Thus, $\lambda = 1$ is a double eigenvalue. □

In numerical simulations, modulated pulses are often calculated by restricting the real line to an interval and imposing periodic boundary conditions; see, for instance, [21]. Theorem 8.8 shows that the modulated pulse can indeed be approximated by spatially–periodic modulated waves provided the interval is sufficiently large. The corollary then shows that the resulting modulated wave train is also stable. This justifies the numerical procedure.

8.5. Application II: Hopf bifurcations from pulses

We briefly comment on Hopf bifurcations from travelling to modulated pulses. Suppose that $q(\xi)$ is a pulse to the reaction–diffusion equation

$$u_t \;=\; Du_{\xi\xi} + cu_\xi + f(u;\mu) \tag{8.26}$$

with $\xi \in \mathbb{R}$ so that q experiences a Hopf instability upon varying μ near zero: the spectrum Σ of the operator $D\partial_{\xi\xi} + c\partial_\xi + f_u(q(\xi);0) : X \to X$ satisfies

$$\Sigma \cap \mathrm{i}\mathbb{R} \;=\; \{0, \pm\mathrm{i}\omega\}\,.$$

We assume that these three eigenvalues are algebraically simple and isolated in the spectrum, that the imaginary eigenvalues at $\pm\mathrm{i}\omega$ cross the imaginary axis transversely as μ crosses through zero, and that the Hopf bifurcation is super–critical. It follows then from center–manifold theory [14, 44] that there is a unique stable modulated pulse $q(\xi, t)$ that bifurcates from the pulse $q(\xi)$. The eigenvalue $\lambda = 1$ of the linearized period map about $q(\xi, t)$ has geometric and algebraic multiplicity two, and Hypotheses $(\widetilde{\mathrm{H}})$, $(\widetilde{\mathrm{T}})$ and (S) are satisfied.

Therefore, due to Theorem 8.8, the modulated pulse $q(\xi, t)$ is accompanied by long–wavelength spatially–periodic modulated waves $q_L(\xi, t)$. Corollary 8.10 implies that these waves are stable when we regard them as solutions to (8.26) on the interval $(-L, L)$ with periodic boundary conditions. Regarded as solutions on the entire real line, however, they could well be unstable due to interaction instabilities between adjacent pulses in the wave train.

9. Discussion

In this section, we summarize our results and comment on open problems.

For dynamical systems or differential equations on $\mathbb{R}^n$, exponential dichotomies are a useful tool when investigating the linearization about trajectories [34]. On a linear level, exponential dichotomies separate exponentially decaying from exponentially growing components. We focused on partial differential equations (PDE) that are posed on the real line. Instead of viewing the PDE as a dynamical system in the time variable t, we interpreted the PDE as a dynamical system in the spatial variable ξ while imposing periodicity or growth conditions in the time variable t. Trajectories of the resulting dynamical system correspond to travelling or modulated waves. The linearized equation provides us with information about the spatial growth or decay of perturbations with a prescribed temporal evolution. This point of view allows us to characterize the spectrum of the PDE linearization about a modulated wave in terms of properties of the exponential dichotomies. In particular, the boundary of the essential spectrum consist of those points for which the linear eigenvalue problem, considered as a spatial dynamical system, does not have an exponential dichotomy. This viewpoint has been exploited for travelling waves on the real line where the linear eigenvalue problem is an ODE; see, for instance, [31, 13, 16, 49]. Spatial dynamics has been used in [6, 28] for the stability analysis of stationary spatially–periodic patterns of small amplitude. Our contribution is the extension of these ideas to travelling waves in unbounded cylinders and to modulated waves. We would also like to point out that the approach via spatial dynamics has been used first in [15] to establish the existence of small modulated waves.

Another advantage of this approach is that it allows us to investigate patterns that exhibit the same temporal behavior as a given primary wave but that are not feasible to a regular perturbation analysis. For instance, we might be interested in patterns that are not close, in a uniform sense, to the original wave with respect to the spatial variable. Or we may want to investigate patterns that have essential spectrum up to the imaginary axis so that we cannot apply an implicit function theorem. Below, we comment on these two situations.

Certain patterns that are not close to the modulated pulse in a uniform sense were studied in Section 8, where we concentrated on the existence and stability of spatially and temporally periodic waves with large wavelength. These wave trains can be thought of as consisting of infinitely many, well separated and equidistant copies of the original modulated pulse. The stability properties of such wave trains depends on the interaction between adjacent pulses that were, on a linear level and to leading order, described by a transmission coefficient; see (8.14).

Certain bifurcations that involve the essential spectrum have been studied in [39, 40] where we gave an existence and stability proof of pulses that travel through a Turing pattern near a Turing instability. The construction of these pulses was facilitated by exponential dichotomies, as well as associated invariant manifolds and foliations, that we constructed in an ad–hoc fashion [39]. In that article, we made extensive use of the fact that the time–periodic modulations that are created by the Turing patterns ahead and behind the pulse are of small amplitude. We also refer to [41] where the interaction of Turing patterns and fronts is studied.

One drawback of viewing the PDE as a dynamical system in the spatial variable is that the associated initial–value problem is ill–posed. The construction of exponential dichotomies is then not as standard as for ODEs; for elliptic equations, it has been carried out in [32]. Similar problems arise for travelling waves on lattices where the linearized equations are ill–posed forward–backward delay equations. We refer, for instance, to [25, 8] for results on Fredholm properties in the context of equations on lattices. We would also like to mention work by Robbin and Salamon [33], see also [3], on infinite–dimensional variational problems. In [33, 3], Fredholm properties were established for abstract self–adjoint equations. In contrast to our results, compactness of the lower–order terms is not required; instead, the unbounded operator has to be self–adjoint.

Finally, we comment on open problems. Given the equivalence of Fredholm properties and the existence of dichotomies, one might argue that the aforementioned results should somehow follow directly from Fredholm properties without deriving and using exponential dichotomies. Even for the existence of periodic orbits that accompany homoclinic orbits in ODEs, however, such a direct proof does not seem to exist.

Exponential dichotomies also exist for systems of elliptic equations posed on the entire space $\mathbb{R}^N$, at least for asymptotically constant coefficients; see [32]. Again, the equation can be written as an ill–posed dynamical system in one of the unbounded directions. Note, however, that the choice of this direction is quite arbitrary, and the approach does not seem to cover all possible phenomena. In the case where the translational symmetry is broken so that the linearization about a stationary pulse is invertible, Angenent [2] proved the existence of multi–pulses and even of chaotic spatial patterns without rewriting the elliptic equation as a spatial dynamical system. An interesting issue is the existence of patterns near a travelling pulse upon breaking the translational invariance; this occurs naturally in weakly inhomogeneous media that have a spatially periodic structure. We may, for instance, add a small spatially periodic function to the diffusion coefficients. In this setting, and in a moving coordinate frame, the travelling pulse experiences both temporal and spatial forcing. It is then not clear what patterns are created by such an interaction between temporal and spatial effects.

The results presented in Sections 2 and 3 apply to reaction–diffusion equations on unbounded cylinders. In particular, we exploited that the equation is semilinear, and we used compactness properties of the non–autonomous terms as well as uniqueness of solutions to certain initial–value problems. We do not know whether one or all of these assumptions can be dropped. The question of dropping these assumptions is not just one of finding the most general setting:

There are many interesting applied problems that involve quasilinear rather than semilinear equations. For instance, water waves and problems in elasticity are typically described by quasilinear equation [26]. There are also various interesting equations that involve operators which are not parabolic. Examples are the FitzHugh–Nagumo equation without diffusion in the second variable or dispersive equations such as the nonlinear Schrödinger equation. Dispersive equations, for instance, lead to spatial dynamical systems that have an infinite–dimensional neutral part on which solutions neither decay nor grow.

Similar difficulties arise for quasi–periodic modulated waves. The approach we introduced here for time–periodic waves fails for temporally quasi–periodic patterns. Again, the linearization of the spatial dynamical system is of mixed hyperbolic–elliptic type.

We believe that the view point of spatial dynamics that consists of replacing Fredholm's alternative by intersection properties of suitable subspaces, pointwise in the spatial coordinate ξ, can facilitate the analysis of many other problems. We hope that the abstract framework of relative Morse indices and exponential dichotomies that we presented here can be generalized further and that some of the aforementioned obstacles will be removed eventually.

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