Gluing unstable fronts and backs together can produce stable pulses

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Abstract. We investigate the stability of pulses that are created at T-points in reaction-diffusion systems on the real line. The pulses are formed by gluing unstable fronts and backs together. We demonstrate that the bifurcating pulses can nevertheless be stable, and establish necessary and sufficient conditions that involve only the front and the back for the stability of the bifurcating pulses.

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1. Introduction

We consider travelling waves of reaction-diffusion equations posed on the entire real line. Suppose that the ordinary differential equation (ODE) that describes travelling waves admits a heteroclinic cycle so that the first connection is transversely constructed while the other connection is of codimension two; see figure 1. This situation is often called a T-point [10]. The interpretation for the partial differential equation (PDE) is as follows. There are two homogeneous rest states so that one of them, say p_0, is stable while the other one, p_1, is unstable. There is also a front that connects p_0 to p_1, and a back that connects p_1 to p_0. Furthermore, the front and back have the same wave speed. It is known [3, 12] that, for certain nearby parameter values, the PDE exhibits pulses that connect the stable rest state p_0 to itself; see figure 2. These pulses are created by gluing the front and the back together near p_1. The bifurcating pulses are characterized uniquely by the length $2L$ of the plateau where the pulse is close to the unstable rest state p_1; see again figure 2. An interesting issue is the stability of these pulses. Since the pulses resemble concatenated copies of the front and the back, one might expect that the spectrum of the pulses is close to the union of the spectra of the front and the back. Thus, the pulses should then always be unstable as the front and the back are both unstable since they connect to an unstable rest state. Nevertheless, in direct numerical simulations, stable bifurcating pulses have sometimes been observed, see [23, 25]. It is the goal of this article to shed some light on this phenomenon.

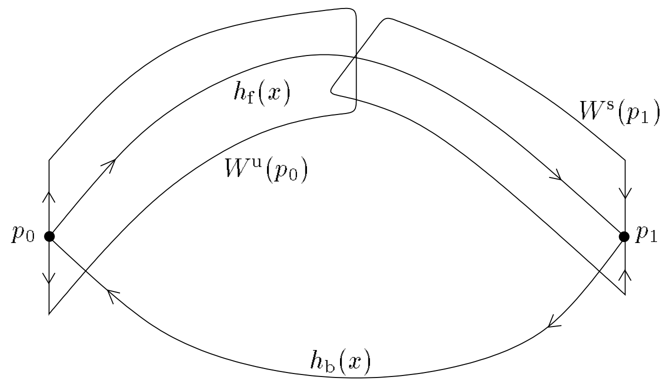


Figure 1. The geometric configuration of stable and unstable manifolds at a T-point.

Hence, we shall investigate the spectrum of the bifurcating pulses. Recall that the pulses are parametrized by the characteristic length L where the limit $L \to \infty$ corresponds to the bifurcation point. The idea is to consider the limiting spectral set that is obtained as the limit (so it exists) of the spectra about the pulses as L tends to infinity. If this limiting set exists, then the spectrum of the pulses is close to it for all sufficiently large L. We demonstrate that the limiting spectral set indeed exists, at least typically, and that it is the union of the following three sets: the spectrum Σ^0_{ess} of the stable rest state p_0, the absolute spectrum Σ^1_{abs} of the unstable rest state p_1, and a finite number of uniformly isolated eigenvalues. The spectral sets Σ^0_{ess} and Σ^1_{abs} consist of curve segments and can be calculated using only information about the asymptotic rest states. In fact, the spectrum of the pulse contains the essential spectrum Σ^0_{ess} about the stable rest state. Each point in the absolute spectrum, however, is approached by infinitely many different discrete eigenvalues in the spectrum of the pulse as $L \to \infty$. In other words, more and more eigenvalues of the pulse accumulate onto the limiting absolute spectrum. We emphasize that the absolute spectrum of the unstable rest state differs, in general, from the rest state's essential spectrum; in fact, the absolute spectrum Σ^1_{abs} is to the left of the essential spectrum Σ^0_{ess}. In particular, the bifurcating pulses can be stable. We remark that the part of the absolute spectrum Σ^1_{abs} of p_1 that lies to the right of the essential spectrum Σ^0_{ess} of p_0 does not depend upon p_0. We call it the *absolute* spectrum as it is related to absolute instabilities that are visible on the entire domain (in contrast to convective instabilities); we refer to [21] for references.

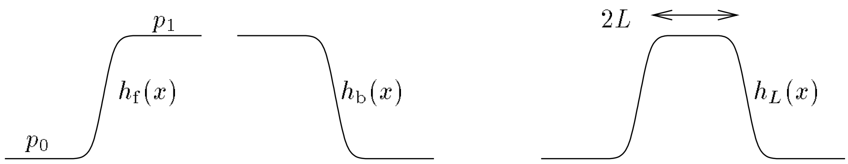


Figure 2. A schematic picture of the front $h_{\mathrm{f}}(x)$, the back $h_{\mathrm{b}}(x)$, and the bifurcating pulses $h_L(x)$.

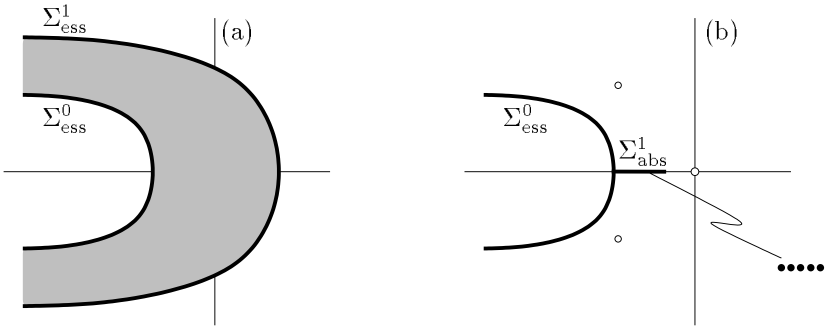


Figure 3. A schematic picture of the spectrum of the front $h_{\rm f}(x)$ or the back $h_{\rm b}(x)$ in (a) and the spectrum of the pulse $h_L(x)$ in (b). Note that the pulse has a single eigenvalue near $\lambda = 0$. Additional eigenvalues of the pulse, indicated by circles in (b), may arise inside the spectrum of the front or back. The absolute spectrum breaks up into a large number of eigenvalues as indicated in (b). Observe that the spectra of both the front and the back in (a) contain open subsets of the complex plane.

The part of the absolute spectrum $\Sigma^1_{\rm abs}$ that lies to the left of the essential spectrum $\Sigma^0_{\rm ess}$ will depend on p_0; with an abuse of notation we still refer to it as the absolute spectrum of p_1; see section 3 for more details. Finally, the remaining finitely many eigenvalues are isolated uniformly in L. They are created by eigenvalues of the front and the back, computed in an exponentially weighted norm. In other words, they arise as zeros of the Evans functions of the front and the back, computed for the linearization in a function space with exponential weights. Such eigenvalues are often referred to as resonance poles; they do not necessarily correspond to eigenvalues of the front or the back on the original C^0 or L^2 space since the associated eigenfunctions may increase exponentially. Our results demonstrate in particular that the pulses have generically only one eigenvalue near the origin, namely $\lambda = 0$. This is in sharp contrast to pulses that are constructed from fronts and backs that connect two stable rest states: in this situation, it is known that the pulses have two eigenvalues near the origin; see [1]. We refer to figure 3 for an illustration of the spectra of the front (or the back) and the pulse.

To prove the aforementioned results, we employ the ideas and methods that we used in [21] where we proved that the spectrum of PDE operators on large bounded intervals is a perturbation not of the essential spectrum computed on the real line but of the operator's absolute spectrum. In particular, we use exponential dichotomies for the linearization in certain exponentially weighted spaces. Exponential weights have been used, for the first time, by Sattinger [22]. Since then, they have been applied to a variety of different problems; see, for instance, [16, 5, 6] for applications.

Matching or gluing the pulses from fronts and backs is similar to imposing a boundary condition in the middle of the domain. Thus, given the results in [21], we expected that the stability properties of the pulse are not determined by the essential spectrum of the unstable rest state but rather by its absolute spectrum which can be stable even though the essential spectrum is unstable.

Simultaneously and independently, Nii [15] obtained results that are related to those presented here. He proved that the bifurcating pulses are unstable whenever the dispersion relation of the unstable rest state has a double root in the right half-plane. His result is a consequence of ours as the absolute spectrum of the unstable rest state is to the right of the imaginary axis whenever its dispersion relation has a double root that lies in the right half-plane (but not vice versa). Again simultaneously and independently, Jones and Romeo [11] constructed an explicit example where the bifurcating pulses are indeed stable.

This paper is organized as follows. We begin in section 2 by reviewing the necessary existence theory near T-points. The essential and absolute spectra of the homogeneous rest states are studied in section 3. In section 4, we consider the PDE linearization about the front and the back, while section 5 contains the set-up for the linearization about the pulse. The main results are theorems 2 and 3 in sections 6 and 7 where we compute isolated and non-isolated eigenvalues, respectively. In section 8, we apply our results to a reaction-diffusion model of FitzHugh-Nagumo type.

2. T-points arising in reaction-diffusion equations

Consider the reaction-diffusion system

$$U_t = DU_{xx} + F(U, \epsilon) \qquad U \in \mathbb{R}^m \qquad x \in \mathbb{R} \tag{2.1}$$

where $\epsilon \in \mathbb{R}$ is a parameter and $D = \mathrm{diag}(d_j)$ is a diagonal diffusion matrix with non-negative coefficients $d_j \geq 0$. We order the components of U so that $d_j > 0$ for $j = 1, \ldots, k$ and $d_j = 0$ for $j = k+1, \ldots, m$. We are interested in travelling-wave solutions to (2.1) that satisfy $U(x,t) = U_*(x - ct)$ for some non-zero wave speed c. It is then convenient to introduce the moving-frame coordinate $\xi = x - ct$. We obtain

$$U_t = DU_{\xi\xi} + cU_\xi + F(U, \epsilon) \qquad U \in \mathbb{R}^m \qquad \xi \in \mathbb{R}$$

or, upon replacing ξ by x,

$$U_t = DU_{xx} + cU_x + F(U, \epsilon) \qquad U \in \mathbb{R}^m \qquad x \in \mathbb{R}. \tag{2.2}$$

Travelling waves with wave speed c satisfy the ODE

$$DU_{xx} + cU_x + F(U, \epsilon) = 0$$

which, for non-zero speeds c, can be rewritten as the first-order system

$$u' = f(u, \epsilon, c) \qquad u \in \mathbb{R}^n \tag{2.3}$$

where $u = (U_1, \ldots, U_k, \partial_x U_1, \ldots, \partial_x U_k, U_{k+1}, \ldots, U_m)$ so that $n = m + k$, while $f_j(u, \epsilon, c) = u_{k+j}$ and $f_{k+j}(u, \epsilon, c) = -(cu_{k+j} + F_j(U, \epsilon))/d_j$ for $j = 1, \ldots, k$ and $f_{k+j}(u, \epsilon, c) = -F_j(U, \epsilon)/c$ for $j = k+1, \ldots, m$.

We begin by discussing (2.3). We assume that there are parameter values (ϵ_*, c_*) with $c_* \neq 0$ such that (2.3) has two hyperbolic equilibria p_0 and p_1 with

$$\dim W^{\mathrm{u}}(p_0) = \dim W^{\mathrm{u}}(p_1) + 1; \tag{2.4}$$

in other words, the equilibria have different Morse (or saddle) indices. We also assume that there are heteroclinic connections $h_{\rm f}(x)$ and $h_{\rm b}(x)$ such that

$$h_{\rm f}(x) \in W^{\rm u}(p_0) \cap W^{\rm s}(p_1) \qquad h_{\rm b}(x) \in W^{\rm u}(p_1) \cap W^{\rm s}(p_0) \tag{2.5}$$

at (ϵ_*, c_*); see figure 1. Note that we typically need two parameters to obtain these connections. Using (2.4), we expect that the first intersection in (2.5) is transverse. On the other hand, we see that the dimensions of the manifolds $W^{\rm u}(p_1)$ and $W^{\rm s}(p_0)$ in the second intersection add up to $n-1$ so that we need two parameters to make them intersect along a curve. We assume that the intersections appearing in (2.5) are as transverse as possible.

Hypothesis 1 *We assume that*

$$\begin{aligned}
\operatorname{span}\{h_{\rm f}'(0)\} &= T_{h_{\rm f}(0)} W^{\rm u}(p_0) \mathbin{\overline{\pitchfork}} T_{h_{\rm f}(0)} W^{\rm s}(p_1) \\
\operatorname{span}\{h_{\rm b}'(0)\} &= T_{h_{\rm b}(0)} W^{\rm u}(p_1) \cap T_{h_{\rm b}(0)} W^{\rm s}(p_0).
\end{aligned}$$

The front that connects p_0 with p_1 is then transversely constructed. We assume that the two parameters (ϵ, c) unfold the back in a generic fashion.

Hypothesis 2 *The center-unstable and center-stable manifolds $W^{\rm cu}(p_1, \epsilon_*, c_*)$ and $W^{\rm cs}(p_0, \epsilon_*, c_*)$ of the equation $(u, \epsilon, c)' = (f(u, \epsilon, c), 0, 0)$ intersect transversely along the back $(h_{\rm b}(x), \epsilon_*, c_*)$, i.e.*

$$T_{(h_{\rm b}(0),0,0)} W^{\rm cu}(p_1, \epsilon_*, c_*) \mathbin{\overline{\pitchfork}} T_{(h_{\rm b}(0),0,0)} W^{\rm cs}(p_0, \epsilon_*, c_*).$$

Since the equilibria p_0 and p_1 are hyperbolic, they persist upon varying (ϵ, c) near (ϵ_*, c_*). Possibly after changing the coordinates, we can assume that p_0 and p_1 do not depend upon (ϵ_*, c_*). We then have the following theorem.

Theorem 1 ([3, 12, 13]) *Assume that the hypotheses 1 and 2 are met. There are then positive constants C, θ and L_* so that (2.3) has a pulse $h_L(x)$ with $\lim_{|x|\to\infty} h_L(x) = p_0$ for parameter values (ϵ_L, c_L) and*

$$\begin{aligned}
|\epsilon_* - \epsilon_L| + |c_* - c_L| &+ \sup_{-\infty < x \le 0} |h_{\rm f}(x+L) - h_L(x)| \\
&+ \sup_{0 \le x < \infty} |h_{\rm b}(x-L) - h_L(x)| \le C{\rm e}^{-\theta L}
\end{aligned} \tag{2.6}$$

uniformly in $L \ge L_*$. Besides these pulses, there are no other pulses to the equilibrium p_0 for parameters (ϵ, c) close to (ϵ_*, c_*).

Proof. The theorem has been proved in [3, 12]. In these references, additional assumptions on the eigenvalues were imposed to make the dependence of (ϵ_L, c_L) on L more explicit. It is a consequence of the results in [13, 24] that these assumptions are not needed for the statement of the theorem as we have formulated it. The exponential estimates are also a consequence of [13]. □

In other words, the pulses h_L are glued together from the front $h_{\rm f}$ and the back $h_{\rm b}$ so that $h_L(x)$ is close to p_1 for x in an interval of length approximately equal to $2L$.

3. PDE-spectra of the homogeneous rest states

We return to the PDE (2.2) and begin by discussing the stability of the rest state P_0 that correspond to the equilibrium p_0 to (2.3). The stability properties of the homogeneous rest state P_0 to (2.2) are determined as follows. Upon linearizing (2.2) about P_0,

$$V_t = DV_{xx} + cV_x + \partial_U F(P_0, \epsilon)V,$$

we see that $V(x,t) = \mathrm{e}^{\lambda t + \mathrm{i}kx}V_0$ satisfies the linearized equation if, and only if,

$$\det[-k^2 D + \mathrm{i}kc + \partial_U F(P_0, \epsilon) - \lambda] = 0. \tag{3.1}$$

This is equivalent to solving

$$\det[\partial_u f(p_0, \epsilon, c) + \lambda B - \mathrm{i}k] = 0,$$

where the matrix B is given in block structure with three blocks of size k, k and $m-k$, respectively, by

$$B = \begin{pmatrix} 0 & 0 & 0 \\ D_k^{-1} & 0 & 0 \\ 0 & 0 & c^{-1} \end{pmatrix}$$

where $D_k = \mathrm{diag}_{j=1,\ldots,k}(d_j)$. In other words, the PDE spectrum of the homogeneous state P_0 can be computed by locating those values of λ for which the matrix

$$\partial_u f(p_0, \epsilon, c) + \lambda B$$

has a purely imaginary spatial eigenvalue $\nu = \mathrm{i}k$. We assume that the homogeneous rest state p_0 is stable.

Hypothesis 3 *The spectrum*

$$\Sigma^0_{\mathrm{ess}} = \{\lambda \in \mathbb{C};\ \ \textit{(3.1) has a solution } k \textit{ for some } k \in \mathbb{R}\}$$

of the rest state P_0 at (ϵ_*, c_*) is contained in $\{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda < -2\delta\}$ for some $\delta > 0$.

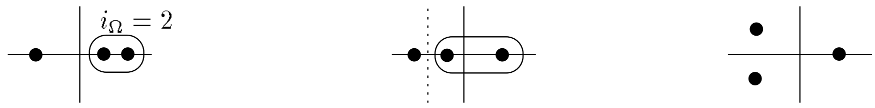
Define

$$A_0(\lambda) := \partial_u f(p_0, \epsilon_*, c_*) + \lambda B \qquad A_1(\lambda) := \partial_u f(p_1, \epsilon_*, c_*) + \lambda B.$$

It is a consequence of hypothesis 3 that the number of unstable eigenvalues of the matrix $A_0(\lambda)$ does not depend upon λ for λ in a fixed connected component of $\mathbb{C} \setminus \Sigma^0_{\mathrm{ess}}$. Furthermore, the matrix $A_0(\lambda)$ is hyperbolic for $\lambda \in \mathbb{C} \setminus \Sigma^0_{\mathrm{ess}}$.

Thus, we choose, and fix, an open, bounded and connected subset $\Omega \subset \mathbb{C} \setminus \Sigma^0_{\mathrm{ess}}$. Throughout the remainder of this article, we consider $\lambda \in \Omega$.

As shown above, the matrix $A_0(\lambda)$ is hyperbolic for $\lambda \in \Omega$. We denote its stable and unstable eigenspaces by $E_0^{\mathrm{s,u}}(\lambda)$. Define the Morse index $i_\Omega = \dim E_0^{\mathrm{u}}(\lambda)$ so that $A_0(\lambda)$ has i_Ω eigenvalues with positive real part and $(n - i_\Omega)$ eigenvalues with negative real part, counted with multiplicity. Note that i_Ω is independent of $\lambda \in \Omega$. We emphasize that i_Ω may change once we choose a different connected component Ω of $\mathbb{C} \setminus \Sigma^0_{\mathrm{ess}}$.



(a) $\lambda \notin \Sigma^0_{\rm ess}$ (b) $\lambda \notin \Sigma^{1,\Omega}_{\rm abs}$ (c) $\lambda \in \Sigma^{1,\Omega}_{\rm abs}$

Figure 4. We fix $\lambda \notin \Sigma^0_{\rm ess}$. The eigenvalues of the matrix $A_0(\lambda)$ are plotted in the leftmost picture while the two pictures to the right contain two possible eigenvalue configurations of $A_1(\lambda)$. We have $\lambda \in \Sigma^{1,\Omega}_{\rm abs}$ if, and only if, the spectrum of $A_1(\lambda)$ cannot be divided by a line $\mathrm{Re}\,\nu = -\eta$ so that i_Ω eigenvalues are to the right of this line and $(n - i_\Omega)$ eigenvalues to the left of it.

Next, we consider the matrix $A_1(\lambda)$. We order its eigenvalues $\nu_j(\lambda)$, repeated with their algebraic multiplicity, according to their real part so that

$$\mathrm{Re}\,\nu_1(\lambda) \geq \mathrm{Re}\,\nu_2(\lambda) \geq \ldots \geq \mathrm{Re}\,\nu_{n-1}(\lambda) \geq \mathrm{Re}\,\nu_n(\lambda).$$

We then define the absolute spectrum of p_1 in Ω by

$$\Sigma^{1,\Omega}_{\rm abs} := \{\lambda \in \Omega;\ \mathrm{Re}\,\nu_{i_\Omega}(\lambda) = \mathrm{Re}\,\nu_{i_\Omega+1}(\lambda)\}; \tag{3.2}$$

see also figure 4. Hence, $\lambda \in \Omega$ is in the absolute spectrum of p_1 if we cannot find a line $\mathrm{Re}\,\nu = \eta$ so that $A_1(\lambda)$ has exactly i_Ω eigenvalues strictly to the right of this line and $(n - i_\Omega)$ eigenvalues strictly to the left of it. Note that the absolute spectrum of p_1 depends crucially on the Morse index i_Ω that is obtained from the rest state p_0. With some abuse of notation, we nevertheless refer to $\Sigma^{1,\Omega}_{\rm abs}$ as the absolute spectrum of p_1 and omit its dependence on p_0. We emphasize that the Morse index i_{Ω_∞} that belongs to the connected component Ω_∞ of the resolvent set that contains the positive real axis depends only on the PDE but not on p_0. Thus, the part of the absolute spectrum that lies to the right of the essential spectrum $\Sigma^0_{\rm ess}$ of p_0 depends only on p_1 and not on p_0.

Observe that we have $\mathrm{Re}\,\nu_{i_\Omega}(\lambda) > \mathrm{Re}\,\nu_{i_\Omega+1}(\lambda)$ for $\lambda \notin \Sigma^{1,\Omega}_{\rm abs}$; we then define the subspaces $\tilde{E}^{\rm s}_1(\lambda)$ and $\tilde{E}^{\rm u}_1(\lambda)$ as the generalized eigenspaces of $A_1(\lambda)$ associated with eigenvalues ν with $\mathrm{Re}\,\nu \leq \mathrm{Re}\,\nu_{i_\Omega+1}(\lambda)$ and with $\mathrm{Re}\,\nu \geq \mathrm{Re}\,\nu_{i_\Omega}(\lambda)$, respectively. Note that $\dim \tilde{E}^{\rm u}_1(\lambda) = i_\Omega$ for $\lambda \notin \Sigma^{1,\Omega}_{\rm abs}$.

4. The PDE linearizations about front and back

Recall that Ω is a fixed open, bounded and connected subset of $\mathbb{C} \setminus \Sigma^0_{\rm ess}$.

4.1. Exponential dichotomies for front and back

First, let $\lambda \in \Omega$ be arbitrary. Define

$$A_{\rm f}(x;\lambda) := \partial_u f(h_{\rm f}(x), \epsilon_*, c_*) + \lambda B.$$

The PDE eigenvalue problem about the front can be written as

$$v' = A_{\rm f}(x;\lambda)v \qquad v \in \mathbb{R}^n. \tag{4.1}$$

We denote by $E_{\mathrm{f}}^{\mathrm{u}}(x;\lambda)$ the subspace of those initial conditions for which the associated solutions decay exponentially as $x \to -\infty$. Note that $E_{\mathrm{f}}^{\mathrm{u}}(x;\lambda)$ converges to $E_0^{\mathrm{u}}(\lambda)$ as $x \to -\infty$.

Next, we restrict to $\lambda \in \Omega \setminus \Sigma_{\mathrm{abs}}^{1,\Omega}$ and consider (4.1) for $x \geq 0$. Since $\lambda \notin \Sigma_{\mathrm{abs}}^{1,\Omega}$, there are numbers η and $\kappa^{\mathrm{s,u}}$, which possibly depend on λ, such that

$$\mathrm{Re}\, \nu_{i_\Omega+1}(\lambda) < \kappa^{\mathrm{s}} < -\eta < \kappa^{\mathrm{u}} < \mathrm{Re}\, \nu_{i_\Omega}(\lambda).$$

Hence, if the number of unstable eigenvalues of $A_0(\lambda)$ is equal to i_Ω, then the first i_Ω eigenvalues of $A_1(\lambda)$ have larger real part than the remaining $(n - i_\Omega)$ eigenvalues of A_1; see figure 4(b) in section 3. The evolution $\varphi_{\mathrm{f}}(x,y;\lambda)$ of (4.1) can then be written as

$$\varphi_{\mathrm{f}}(x,y;\lambda) = \varphi_{\mathrm{f}}^{\mathrm{s}}(x,y;\lambda) + \varphi_{\mathrm{f}}^{\mathrm{u}}(x,y;\lambda) \qquad x, y \geq 0$$

so that $\varphi_{\mathrm{f}}^{\mathrm{s}}(x,x;\lambda)$ is a projection and

$$\begin{aligned} |\varphi_{\mathrm{f}}^{\mathrm{s}}(x,y;\lambda)| &\leq C\mathrm{e}^{\kappa^s|x-y|} \qquad & x \geq y \geq 0 \\ |\varphi_{\mathrm{f}}^{\mathrm{u}}(x,y;\lambda)| &\leq C\mathrm{e}^{-\kappa^{\mathrm{u}}|x-y|} \qquad & y \geq x \geq 0. \end{aligned}$$

To prove this claim, we argue as follows. Consider the equation

$$w' = (A_{\mathrm{f}}(x;\lambda) + \eta)w \tag{4.2}$$

and observe that solutions to (4.1) and (4.2) are related via

$$v(x) = \mathrm{e}^{-\eta x} w(x).$$

Note that the asymptotic matrix $A_1(\lambda) + \eta$ of (4.2) is hyperbolic and has precisely i_Ω unstable eigenvalues due to our choice of η. Thus, (4.2) has an exponential dichotomy on $\mathbb{R}^+$, see [7, 17], which proves the claim. We define

$$\tilde{E}_{\mathrm{f}}^{\mathrm{s,u}}(x;\lambda) := \mathrm{R}(\varphi_{\mathrm{f}}^{\mathrm{s,u}}(x,x;\lambda))$$

for $x \geq 0$.

Finally, we apply the same arguments to the linearization

$$v' = A_{\mathrm{b}}(x;\lambda)v \tag{4.3}$$

about the back where

$$A_{\mathrm{b}}(x;\lambda) := \partial_u f(h_{\mathrm{b}}(x), \epsilon_*, c_*) + \lambda B.$$

For $\lambda \in \Omega$, we denote by $E_{\mathrm{b}}^{\mathrm{s}}(x;\lambda)$ the stable subspace of (4.3) for $x \geq 0$. These subspaces converge to the stable subspace $E_0^{\mathrm{s}}(\lambda)$ as $x \to \infty$. In addition, for $\lambda \in \Omega \setminus \Sigma_{\mathrm{abs}}^{1,\Omega}$, we define the stable and unstable subspaces $\tilde{E}_{\mathrm{b}}^{\mathrm{s,u}}(x;\lambda)$ of (4.3) for $x \leq 0$.

4.2. The Evans functions of the front and the back

Let $E_\pm(\lambda)$ be two subspaces of $\mathbb{C}^n$ that depend analytically on λ such that $n_- + n_+ = n$ where $n_\pm := \dim E_\pm(\lambda)$ is independent of λ. Choose vectors $v_1^\pm(\lambda), \ldots, v_{n_\pm}^\pm(\lambda)$ such that

$$E_\pm(\lambda) = \mathrm{span}\{v_1^\pm(\lambda), \ldots, v_{n_\pm}^\pm(\lambda)\}$$

and $v_j^{\pm}(\lambda)$ is analytic in λ for all j. We then define

$$E_-(\lambda) \wedge E_+(\lambda) := \det[v_1^-(\lambda), \ldots, v_{n_-}^-(\lambda), v_1^+(\lambda), \ldots, v_{n_+}^+(\lambda)] \in \mathbb{C}.$$

Note that this function is analytic in λ. In addition, the order of any of its zeros does not depend on the choice of the bases; in fact, any two such functions differ by a product with a non-zero analytic complex-valued function.

We define the Evans functions $D_{\rm f}(\lambda)$ and $D_{\rm b}(\lambda)$ of the front and the back, respectively, by

$$D_{\rm f}(\lambda) = E_{\rm f}^{\rm u}(0;\lambda) \wedge \tilde{E}_{\rm f}^{\rm s}(0;\lambda) \qquad D_{\rm b}(\lambda) = \tilde{E}_{\rm b}^{\rm u}(0;\lambda) \wedge E_{\rm b}^{\rm s}(0;\lambda). \tag{4.4}$$

These functions are defined and analytic for $\lambda \in \Omega \setminus \Sigma_{\rm abs}^{1,\Omega}$. The front generates a zero of $D_{\rm f}$ if the linearization about the front connects the i_Ω-dimensional unstable eigenspace of A_0 at $-\infty$ with the eigenspace of A_1 at $+\infty$ that is generated by the $(n-i_\Omega)$ eigenvalues of A_1 that have the smallest real part. Similarly, the back generates a zero of $D_{\rm b}$ if it connects the eigenspace associated with the i_Ω eigenvalues of A_1 with largest real part at $-\infty$ with the $(n-i_\Omega)$-dimensional stable subspace of A_0 at $+\infty$. Note that the stable and unstable eigenspaces of A_1 might not be of dimension i_Ω and $(n-i_\Omega)$, respectively. Therefore, the aforementioned connections may not be related at all to eigenvalues of the front or the back. In fact, the functions $D_{\rm f}$ and $D_{\rm b}$ count eigenvalues of the front and the back, respectively, precisely when $\mathrm{Re}\,\nu_{i_\Omega}(\lambda) > 0 > \mathrm{Re}\,\nu_{i_\Omega+1}(\lambda)$; otherwise, they count resonance poles, i.e. eigenvalues of the PDE operator cast in an exponentially weighted function space.

5. The PDE linearization about the pulse

We are interested in the eigenvalue problem

$$v' = (\partial_u f(h_L(x), a_L, c_L) + \lambda B)v$$

about the pulse $h_L(x)$. The spectrum Σ of the linearization about the pulse h_L is the disjoint union of the essential spectrum and the point spectrum

$$\Sigma = \Sigma_{\rm ess} \cup \Sigma_{\rm pt}$$

where the point spectrum consists of all isolated eigenvalues with finite multiplicity, and the essential spectrum is the complement in Σ of the point spectrum. Since $h_L(x) \to p_0$ as $|x| \to \infty$, the essential spectrum of the pulse is bounded by the essential spectrum of the rest state p_0 at the parameter values (ϵ_L, c_L). Due to the estimate (2.6) and the hypothesis 3, the essential spectrum is therefore to the left of the line $\mathrm{Re}\,\lambda = -\delta$ for all L sufficiently large. It remains to investigate isolated eigenvalues.

5.1. Exponential dichotomies for the pulse

We shall compare the evolution operators of the front with the evolution operator for the linearization about the pulse. Recall that the pulse satisfies the estimate (2.6)

$$|\epsilon_* - \epsilon_L| + |c_* - c_L| + \sup_{-\infty < x \le L} |h_{\rm f}(x) - h_L(x-L)|$$

$$+ \sup_{-L\leq x<\infty} |h_\mathrm{b}(x) - h_L(x+L)| \leq C\mathrm{e}^{-\theta L}.$$

In other words, the coefficients of the linearization

$$v' = (\partial_u f(h_L(x-L), \epsilon_L, c_L) + \lambda B)v \tag{5.1}$$

about the pulse are $\mathrm{e}^{-\theta L}$-close to the coefficients of the linearization

$$v' = (\partial_u f(h_\mathrm{f}(x), \epsilon_*, c_*) + \lambda B)v$$

about the front, uniformly in x for $-\infty < x \leq L$. We denote the evolution operator of (5.1) on the interval $(-\infty, L]$ by $\varphi_{\mathrm{f},L}(x, y; \lambda)$.

Lemma 1 *For $\lambda \in \Omega$, the space $E^\mathrm{u}_{\mathrm{f},L}(0;\lambda)$ of initial conditions at $x = 0$ that correspond to solutions of (5.1) that decay exponentially as $x \to -\infty$ is $\mathrm{e}^{-\theta L}$-close to $E^\mathrm{u}_\mathrm{f}(0;\lambda)$. For $\lambda \in \Omega \setminus \Sigma^{1,\Omega}_\mathrm{abs}$, there exist evolution matrices $\varphi^\mathrm{s,u}_{\mathrm{f},L}(x, y; \lambda)$ defined for $x, y \in [0, L]$ such that*

$$\varphi_{\mathrm{f},L}(x, y; \lambda) = \varphi^\mathrm{s}_{\mathrm{f},L}(x, y; \lambda) + \varphi^\mathrm{u}_{\mathrm{f},L}(x, y; \lambda) \qquad x, y \in [0, L],$$

so that $\varphi^\mathrm{s,u}_{\mathrm{f},L}(x, x; \lambda)$ are complementary projections and

$$|\varphi^\mathrm{s}_\mathrm{f}(x, y; \lambda) - \varphi^\mathrm{s}_{\mathrm{f},L}(x, y; \lambda)| \leq C\mathrm{e}^{-\theta L}\,\mathrm{e}^{\kappa^\mathrm{s}|x-y|} \qquad x \geq y \geq 0$$
$$|\varphi^\mathrm{u}_\mathrm{f}(x, y; \lambda) - \varphi^\mathrm{u}_{\mathrm{f},L}(x, y; \lambda)| \leq C\mathrm{e}^{-\theta L}\,\mathrm{e}^{-\kappa^\mathrm{u}|x-y|} \qquad y \geq x \geq 0$$

for some constant C that does not depend upon L.

Proof. The statement of the lemma is a consequence of the estimate (2.6) and the roughness theorem for exponential dichotomies [18, 17]. □

We define

$$\tilde{E}^\mathrm{s,u}_{\mathrm{f},L}(0;\lambda) := \mathrm{R}(\varphi^\mathrm{s,u}_{\mathrm{f},L}(x, x; \lambda))$$

to be the range of the projection $\varphi^\mathrm{s,u}_{\mathrm{f},L}(x, x; \lambda)$. Lemma 1 is also true for the equation

$$v' = (\partial_u f(h_L(x+L), \epsilon_L, c_L) + \lambda B)v \tag{5.2}$$

and the linearization

$$v' = (\partial_u f(h_\mathrm{b}(x), \epsilon_*, c_*) + \lambda B)v$$

about the back, both considered on the interval $[-L, \infty)$. For $\lambda \in \Omega$, we denote by $E^\mathrm{s}_{\mathrm{b},L}(0;\lambda)$ the space of initial conditions at $x = 0$ that lead to solutions of (5.2) which decay exponentially as $x \to \infty$. This space is exponentially close to the stable space $E^\mathrm{s}_\mathrm{b}(0;\lambda)$ associated with the back. Furthermore, for $\lambda \in \Omega \setminus \Sigma^{1,\Omega}_\mathrm{abs}$, we denote by

$$\varphi_{\mathrm{b},L}(x, y; \lambda) = \varphi^\mathrm{s}_{\mathrm{b},L}(x, y; \lambda) + \varphi^\mathrm{u}_{\mathrm{b},L}(x, y; \lambda) \qquad x, y \in [-L, 0]$$

the evolution operators of (5.2) that are then exponentially close, uniformly in L, to the evolution operators $\varphi^\mathrm{s,u}_\mathrm{b}(x, y; \lambda)$ of the back. As before, we define

$$\tilde{E}^\mathrm{s,u}_{\mathrm{b},L}(0;\lambda) := \mathrm{R}(\varphi^\mathrm{s,u}_{\mathrm{b},L}(x, x; \lambda)).$$

5.2. The Evans function of the pulse

For $\lambda \in \Omega$, we define

$$D_L(\lambda) = \varphi_{\mathrm{f},L}(L,0;\lambda)E^{\mathrm{u}}_{\mathrm{f},L}(0;\lambda) \wedge \varphi_{\mathrm{b},L}(-L,0;\lambda)E^{\mathrm{s}}_{\mathrm{b},L}(0;\lambda);$$

see section 4.2. This is the ordinary Evans function for the pulse. In particular, zeros of $D_L(\lambda)$, counted with their order, are in one-to-one correspondence with eigenvalues of the PDE linearization about the pulse, counted with their algebraic multiplicity; see [1, 9]. It therefore suffices to seek zeros of $D_L(\lambda)$.

For any analytic function $D(\lambda)$, we denote by $\mathrm{ord}(\lambda_*, D)$ the order of λ_* as a zero of $D(\lambda)$. If the order is finite, then it is equal to the winding number of $D(\lambda)$ about any sufficiently small circle in $\mathbb{C}$ that is centered at λ_*.

6. Persistence of eigenvalues

In this section, we consider exclusively $\lambda_* \in \Omega \setminus \Sigma^{1,\Omega}_{\mathrm{abs}}$. We shall demonstrate that D_L has ℓ zeros near λ_* whenever the combined order of λ_* as a zero of D_f and D_b is equal to ℓ. In other words, zeros of D_f and D_b persist with their combined order as zeros of D_L. In particular, if neither D_f nor D_b vanish at $\lambda = \lambda_*$, then λ is not in the spectrum of the pulse for any λ close to λ_* uniformly in $L \geq L_*$ for some L_*.

Lemma 2 *Let $\lambda_* \in \Omega$ with $\lambda_* \notin \Sigma^{1,\Omega}_{\mathrm{abs}}$ so that $D_\mathrm{f}(\lambda_*) \neq 0$ and $\mathrm{ord}(\lambda_*, D_\mathrm{b}) = \ell$. For every small $\delta > 0$, there is then an $L_* > 0$ so that D_L has precisely ℓ eigenvalues (counted with multiplicity) in $U_\delta(\lambda_*)$ for every $L \geq L_*$.*

Proof. Since $D_\mathrm{f}(\lambda_*)$ is not equal to zero, we have

$$E^{\mathrm{u}}_{\mathrm{f}}(0;\lambda_*) \oplus \tilde{E}^{\mathrm{s}}_{\mathrm{f}}(0;\lambda_*) = \mathbb{C}^n.$$

Therefore, lemma 1 implies that

$$E^{\mathrm{u}}_{\mathrm{f},L}(0;\lambda) \oplus \tilde{E}^{\mathrm{s}}_{\mathrm{f},L}(0;\lambda) = \mathbb{C}^n$$

for all λ close to λ_* and all $L \geq L_*$ for some $L_* > 0$. Hence, solving forward in x, we conclude that $\varphi_{\mathrm{f},L}(L,0;\lambda)E^{\mathrm{u}}_{\mathrm{f},L}(0;\lambda)$ is $\mathrm{e}^{-\alpha L}$-close to $\tilde{E}^{\mathrm{u}}_1(\lambda)$, uniformly in λ and L, where $\alpha = \min\{\theta, \kappa^{\mathrm{u}} - \kappa^{\mathrm{s}}\}$; see [21]. Continuing to solve forward in x, and again employing lemma 1, we obtain that

$$\varphi_{\mathrm{b},L}(0,-L;\lambda)\varphi_{\mathrm{f},L}(L,0;\lambda)E^{\mathrm{u}}_{\mathrm{f},L}(0;\lambda)$$

is $\mathrm{e}^{-\alpha L}$-close to $\tilde{E}^{\mathrm{u}}_{\mathrm{b}}(0;\lambda)$, uniformly in λ and L. Therefore,

$$\begin{aligned} D_L(\lambda) &= \frac{1}{\det \varphi_{\mathrm{b},L}(0,-L;\lambda)}\Big([\varphi_{\mathrm{b},L}(0,-L;\lambda)\varphi_{\mathrm{f},L}(L,0;\lambda)E^{\mathrm{u}}_{\mathrm{f},L}(0;\lambda)] \wedge E^{\mathrm{s}}_{\mathrm{b},L}(0;\lambda)\Big) \\ &= \frac{1}{\det \varphi_{\mathrm{b},L}(0,-L;\lambda)}(D_\mathrm{b}(\lambda) + \mathrm{O}(\mathrm{e}^{-\alpha L})) \end{aligned}$$

is exponentially close to the Evans function $D_\mathrm{b}(\lambda)$ of the front up to the non-zero factor $\det \varphi_{\mathrm{b},L}(-L,0;\lambda)$. This proves the statement of the lemma; we refer to [21] for similar arguments. $\square$

Remark 1 *Obviously, the conclusion of lemma 2 remains true if* $\lambda_* \in \Omega \setminus \Sigma_{\rm abs}^{1,\Omega}$ *so that* $\mathrm{ord}(\lambda_*, D_{\rm f}) = \ell$ *and* $D_{\rm b}(\lambda_*) \neq 0$.

Before we discuss the general case when both Evans functions vanish, we comment on the situation near $\lambda = 0$.

Lemma 3 *If* $\lambda = 0$ *is not contained in* $\Sigma_{\rm abs}^{1,\Omega}$*, then generically we have* $D_{\rm f}(0) \neq 0$ *and* $\mathrm{ord}(0, D_{\rm b}) = 1$.

In particular, if $0 \notin \Sigma_{\rm abs}^{1,\Omega}$, and under further generic conditions that are specified explicitly in the proof below, the linearization about the pulse has a simple eigenvalue at $\lambda = 0$, and there are positive numbers L_* and δ such that $\lambda = 0$ is the only eigenvalue in $U_\delta(0)$ for $L \geq L_*$. This is in contrast to the situation for pulses that bifurcate from fronts and backs that connect stable rest states: such pulses always have two eigenvalues near $\lambda = 0$, see [1], and it requires some further analysis to track the non-trivial eigenvalue; see [14, 19, 20].

Proof. Recall that we assumed that

$$\dim W^{\rm u}(p_0) = \dim W^{\rm u}(p_1) + 1;$$

see (2.4). In addition, we have $\dim W^{\rm u}(p_0) = i_\Omega$. Combined with the assumption that $\lambda = 0$ is not contained in $\Sigma_{\rm abs}^{1,\Omega}$, this gives

$$\mathrm{Re}\,\nu_i(0) > 0 > \mathrm{Re}\,\nu_{i_\Omega}(0) > \mathrm{Re}\,\nu_j(0) \tag{6.1}$$

for $i < i_\Omega < j$; see section 3. In particular, we conclude that $\tilde{E}_1^{\rm s}(0)$ is equal to the tangent space of the strong stable manifold $W^{\rm ss}(p_1)$ of the equilibrium p_1. In hypothesis 1, we assumed that

$$\mathrm{span}\{h_{\rm f}'(0)\} = E_{\rm f}^{\rm u}(0;0) \cap T_{h_{\rm f}(0)} W^{\rm s}(p_1).$$

Thus, if $h_{\rm f}(0)$ is not contained in the strong stable manifold of p_1,

$$h_{\rm f}(0) \notin W^{\rm ss}(p_1), \tag{6.2}$$

then we have that $E_{\rm f}^{\rm u}(0;0) \cap \tilde{E}_1^{\rm s}(0;0) = \{0\}$, and therefore $D_{\rm f}(0) \neq 0$.

Next, we consider the back. Denote by $W_{\rm ext}^{\rm u}(p_1)$ the invariant i_Ω-dimensional extended unstable manifold of p_1 that has as its tangent space at p_1 the eigenspace associated with eigenvalues ν that satisfy $\mathrm{Re}\,\nu \geq \mathrm{Re}\,\nu_{i_\Omega}(0)$; see (6.1). While this manifold itself is not unique, its tangent space along $h_{\rm b}(x)$ is unique. If we assume that

$$\mathrm{span}\{h_{\rm b}'(0)\} = T_{h_{\rm b}(0)} W_{\rm ext}^{\rm u}(p_1) \cap T_{h_{\rm b}(0)} W^{\rm s}(p_0) \tag{6.3}$$

and that the intersection between $W_{\rm ext}^{\rm u}(p_1)$ and $W^{\rm s}(p_0)$ along $h_{\rm b}(x)$ unfolds generically as c is varied near c_*, then it is straightforward to see that $\mathrm{ord}(0, D_{\rm b}) = 1$.

Finally, we observe that the conditions (6.2) and (6.3) as well as the transversal unfolding mentioned right above are satisfied for generic two-parameter families. □

It remains to consider the case when both Evans functions vanish.

Theorem 2 *Let* $\lambda_* \in \Omega$ *with* $\lambda_* \notin \Sigma_{\mathrm{abs}}^{1,\Omega}$ *so that* $\mathrm{ord}(\lambda_*, D_\mathrm{f}) = \ell_1$ *and* $\mathrm{ord}(\lambda_*, D_\mathrm{b}) = \ell_2$. *For every small* $\delta > 0$, *there is then an* $L_* > 0$ *such that* D_L *has precisely* $\ell_1 + \ell_2$ *eigenvalues (counted with multiplicity) in* $U_\delta(\lambda_*)$ *for every* $L \geq L_*$.

Proof. Save for notation, the proof is identical to the proof of Theorem 2 in [21], and we shall omit it. The idea is to use that the Evans functions are analytic in λ. We can therefore slightly perturb the equation for λ near λ_* without missing, or gaining, any eigenvalues. In particular, if we change the linearized equation only along the front in an appropriate fashion, we can arrange that λ_* is no longer a zero of D_f; of course, as mentioned above, D_f still has ℓ_1 eigenvalues arbitrarily close to λ_*. The perturbed equation satisfies the assumptions of lemma 2 and remark 1 for any λ near λ_*, and the statement of the theorem follows. □

In summary, zeros of D_f and D_b in $\Omega \setminus \Sigma_{\mathrm{abs}}^{1,\Omega}$ persist with their multiplicity as eigenvalues of the pulse. In particular, if $\Sigma_{\mathrm{abs}}^{1,\Omega}$ is contained in the open left half-plane, if D_f has no zeros in the closed right half-plane, and if D_b has no zeros in the closed right half-plane except a simple zero at $\lambda = 0$, then the pulse has no eigenvalues in Ω that are in the closed right half-plane except a simple eigenvalue at zero.

7. Eigenvalues that accumulate near the absolute spectrum

It remains to investigate the spectrum of the pulse near the absolute spectrum $\Sigma_{\mathrm{abs}}^{1,\Omega}$ of the equilibrium p_1. We shall demonstrate that the number of eigenvalues of the pulse h_L near each fixed element in $\Sigma_{\mathrm{abs}}^{1,\Omega}$ is not bounded uniformly in L. Roughly speaking, as L increases, an unbounded number of eigenvalues of the pulse accumulate at each element of the absolute spectrum $\Sigma_{\mathrm{abs}}^{1,\Omega}$.

Recall that the open set $\Omega \subset \mathbb{C}$ has been chosen such that $\Sigma_{\mathrm{ess}}^{0} \cap \Omega = \emptyset$.

Definition 1 *We say that* $\lambda_* \in \Omega$ *is regular if there is an open neighborhood* $U(\lambda_*)$ *of* λ_* *in* Ω, *an integer* ℓ_* *and a positive number* L_* *such that* D_L *has at most* ℓ_* *zeros in* $U(\lambda_*)$ *for all* $L \geq L_*$. *Recall that zeros are always counted with their multiplicity. Furthermore, we define the extrapolated (essential) spectral set*

$$\Sigma_{\mathrm{ext}}^{\mathrm{e},\Omega} = \{\lambda \in \Omega;\ \lambda \textit{ is not regular}\}.$$

Hence, the extrapolated spectral set $\Sigma_{\mathrm{ext}}^{\mathrm{e},\Omega}$ consists of those points where infinitely many eigenvalues of the linearization about the pulse h_L accumulate as $L \to \infty$. Note that $\Sigma_{\mathrm{ext}}^{\mathrm{e},\Omega}$ is closed since its complement is open by definition.

The next hypothesis excludes the situation that D_f or D_b vanish identically in a connected component of $\Omega \setminus \Sigma_{\mathrm{abs}}^{1,\Omega}$. In other words, we exclude the situation that the entire open connected component consists of eigenvalues.

Hypothesis 4 *Neither* D_f *nor* D_b *vanish identically on any connected component of* $\Omega \setminus \Sigma_{\mathrm{abs}}^{1,\Omega}$.

This hypothesis is met for reaction-diffusion equations if Ω is contained in the connected component of $\mathbb{C} \setminus \Sigma^0_{\rm ess}$ that contains the positive real axis.

Lemma 4 *If hypothesis 4 is met, then* $\Sigma^{\rm e,\Omega}_{\rm ext} \subset \Sigma^{1,\Omega}_{\rm abs}$.

Proof. This is an immediate consequence of theorem 2 and the definition of $\Sigma^{\rm e,\Omega}_{\rm ext}$. $\square$

In fact, as we shall see below, the extrapolated spectral set is actually equal to the absolute spectrum of p_1 provided the following assumption is met.

Hypothesis 5 (Reducible absolute spectrum) *The subset* $\mathcal{S}^{1,\Omega}_{\rm abs}$*, defined below, of the absolute spectrum* $\Sigma^{1,\Omega}_{\rm abs}$ *is dense in* $\Sigma^{1,\Omega}_{\rm abs}$*. Here,* $\lambda_* \in \mathcal{S}^{1,\Omega}_{\rm abs}$ *if* $D_{\rm f}(\lambda_*) \neq 0$*,* $D_{\rm b}(\lambda_*) \neq 0$ *and, in addition,*

$$\mathrm{Re}\,\nu_{i_\Omega-1}(\lambda_*) > \mathrm{Re}\,\nu_{i_\Omega}(\lambda_*) = \mathrm{Re}\,\nu_{i_\Omega+1}(\lambda_*) > \mathrm{Re}\,\nu_{i_\Omega+2}(\lambda_*)$$

with $\nu_{i_\Omega}(\lambda_*) \neq \nu_{i_\Omega+1}(\lambda_*)$ *and* $\frac{\mathrm{d}}{\mathrm{d}\lambda}(\nu_{i_\Omega} - \nu_{i_\Omega+1})|_{\lambda_*} \neq 0$.

Note that the set $\mathcal{S}^{1,\Omega}_{\rm abs}$ consists of curve segments.

Theorem 3 *If hypotheses 4 and 5 are met, then* $\Sigma^{\rm e,\Omega}_{\rm ext} = \Sigma^{1,\Omega}_{\rm abs}$.

Proof. The proof is similar to the proof of [21, theorem 5].

Since $\Sigma^{\rm e,\Omega}_{\rm ext}$ is closed, and due to lemma 4 and hypothesis 5, it suffices to show that $\lambda_* \in \Sigma^{\rm e,\Omega}_{\rm ext}$ whenever $\lambda_* \in \mathcal{S}^{1,\Omega}_{\rm abs}$. Thus, we fix $\lambda_* \in \mathcal{S}^{1,\Omega}_{\rm abs}$ and consider λ close to λ_*.

Throughout the proof, let $\hat{E}^{\rm u}_1(\lambda)$, $\hat{E}^{\rm c}_1(\lambda)$ and $\hat{E}^{\rm s}_1(\lambda)$ be the generalized eigenspaces of $A_1(\lambda)$ associated with the spectral sets $\{\nu_j(\lambda)\}_{j=1,\dots,i_\Omega-1}$, $\{\nu_{i_\Omega}(\lambda), \nu_{i_\Omega+1}(\lambda)\}$ and $\{\nu_j(\lambda)\}_{j=i_\Omega+2,\dots,n}$, respectively. In other words, we single out the two eigenvalues ν_{i_Ω} and $\nu_{i_\Omega+1}$ that prevent the spectral separation at $\lambda = \lambda_*$. Due to hypothesis 5, these three spectral sets are separated by gaps between the real part of their elements.

First, consider the space $E^{\rm u}_{{\rm f},L}(L;\lambda)$. We claim that

$$E^{\rm u}_{{\rm f},L}(L;\lambda) = \mathrm{span}\{u^{\rm c}_{{\rm f},L}(L;\lambda)\} \oplus (\hat{E}^{\rm s}_1(\lambda) + \mathrm{O}(\mathrm{e}^{-\alpha L})) \tag{7.1}$$

for $L \geq L_*$ and some $\alpha > 0$ that does not depend upon L, where

$$u^{\rm c}_{{\rm f},L}(x;\lambda) = a^{\rm f}_{i_\Omega}(\lambda)\mathrm{e}^{\nu_\infty(\lambda)x} + a^{\rm f}_{i_\Omega+1}(\lambda)\mathrm{e}^{\nu_{\infty+1}(\lambda)x} + \mathrm{O}(\mathrm{e}^{-\alpha x}) \qquad x \geq 0 \tag{7.2}$$

for some non-zero vectors $a^{\rm f}_{i_\Omega}(\lambda)$ and $a^{\rm f}_{i_\Omega+1}(\lambda)$ that are contained in $\hat{E}^{\rm c}_1(\lambda)$. Otherwise, we reach a contradiction to hypothesis 5; see [21, proof of theorem 5] for details.

By the same token, we obtain that

$$E^{\rm s}_{{\rm b},L}(-L;\lambda) = \mathrm{span}\{u^{\rm c}_{{\rm b},L}(-L;\lambda)\} \oplus (\hat{E}^{\rm u}_1(\lambda) + \mathrm{O}(\mathrm{e}^{-\alpha L})) \tag{7.3}$$

for $L \geq L_*$, where

$$u^{\rm c}_{{\rm b},L}(x;\lambda) = a^{\rm b}_{i_\Omega}(\lambda)\mathrm{e}^{\nu_\infty(\lambda)x} + a^{\rm b}_{i_\Omega+1}(\lambda)\mathrm{e}^{\nu_{\infty+1}(\lambda)x} + \mathrm{O}(\mathrm{e}^{\alpha x}) \qquad x \leq 0 \tag{7.4}$$

for some non-zero vectors $a^{\rm b}_{i_\Omega}(\lambda)$ and $a^{\rm b}_{i_\Omega+1}(\lambda)$ that are contained in $\hat{E}^{\rm c}_1(\lambda)$.

Eigenvalues of the pulse h_L are given as intersections of $E^{\rm u}_{{\rm f},L}(L;\lambda)$ and $E^{\rm s}_{{\rm b},L}(-L;\lambda)$. The idea is to apply Lyapunov-Schmidt reduction using the characterizations (7.1) and

(7.3) of the stable and unstable subspaces $E^{\mathrm{u}}_{\mathrm{f},L}(L;\lambda)$ and $E^{\mathrm{s}}_{\mathrm{b},L}(-L;\lambda)$. The reduced equation then lives on the center space $\hat{E}^{\mathrm{c}}_{1}(\lambda)$; it is given by

$$u^{\mathrm{c}}_{\mathrm{f},L}(L;\lambda) = u^{\mathrm{c}}_{\mathrm{b},L}(-L;\lambda) + \mathrm{O}(\mathrm{e}^{-\alpha L}). \tag{7.5}$$

Upon substituting the expressions (7.2) and (7.4), and exploiting that $\mathrm{Re}\,\nu_{i_\Omega}(\lambda_*) = \mathrm{Re}\,\nu_{i_\Omega+1}(\lambda_*)$ and $\frac{\mathrm{d}}{\mathrm{d}\lambda}(\nu_{i_\Omega} - \nu_{i_\Omega+1})|_{\lambda_*} \neq 0$ by hypothesis 5, it is then not difficult to prove that the reduced equation (7.5) has $\mathrm{O}(L)$ different solutions for λ close to λ_* so that $\lambda_* \in \Sigma^{\mathrm{e},\Omega}_{\mathrm{ext}}$. The details of the aforementioned arguments are identical to those given in [21, proof of theorem 5]; thus, we omit them. □

As an example, consider a travelling-wave ODE in three space dimensions with $i_{\Omega_\infty} = 2$: a number λ is then in the absolute spectrum if, and only if, the two eigenvalues of $A_1(\lambda)$ with smallest real part have, in fact, the same real part; see figure 4(c). In particular, in the situation shown in figure 1, $\lambda = 0$ is certainly in the absolute spectrum of p_1 if the two stable eigenvalues at p_1 correspond to two complex conjugate eigenvalues, i.e. if p_1 is a saddle-focus rather than a saddle. Thus, it is necessary for stability of the pulses that the equilibrium p_1 is a saddle and not a saddle-focus. If the rest state p_1 is a saddle-focus, then the pulses experience infinitely many saddle-nodes as $L \to \infty$ which are caused by eigenvalues that cross the imaginary axis from left to right and accumulate onto the unstable absolute spectrum.

8. The FitzHugh-Nagumo equation

As an application, we consider a modified FitzHugh-Nagumo equation that partly motivated this article. Zimmermann *et al* [25] found a T-point in this equation and observed that the bifurcating pulses appear to be stable. In a moving coordinate frame, the modified FitzHugh-Nagumo equation is given by

$$\begin{aligned} u_t &= u_{xx} - cu_x - \frac{1}{\epsilon}u(u-1)(u - \frac{w+b}{a}) \\ w_t &= -\,cw_x + f(u) - w \end{aligned} \tag{8.1}$$

where the nonlinearity $f(u)$ is defined by

$$f(u) = \begin{cases} 0 & 0 \le u \le \frac{1}{3} \\ 1 - 6.75\,u(u-1)^2 & \frac{1}{3} \le u \le 1 \\ 1 & 1 \le u \end{cases}$$

and the parameters a and b are given by

$$a = 0.84 \qquad b = 0.07.$$

The FitzHugh-Nagumo equation (8.1) describes CO oxidation on a Pt(110) surface; see [25] and the references therein for more details. The travelling-wave ODE associated with (8.1) is

$$u_x = v \qquad v_x = cv + \frac{1}{\epsilon}u(u-1)(u - \frac{w+b}{a}) \qquad w_x = \frac{1}{c}(f(u) - w)$$

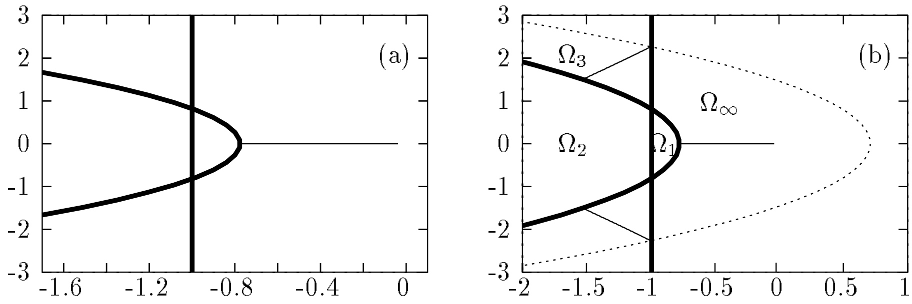


Figure 5. (a) We plotted the essential spectrum $\Sigma_{\rm ess}^0$ of p_0 (bold lines) and the extended spectrum $\Sigma_{\rm ext}^{\rm e}$ (thin line); theorem 3 implies that the union of these two sets is the spectrum of the bifurcating pulses with the exception of uniformly isolated eigenvalues. (b) We plotted the essential spectrum of p_0 (bold lines), the essential spectrum of p_1 (dotted line) and the absolute spectrum of p_1 (thin lines); note that the line $\mathrm{Re}\,\lambda = -1$ is contained in $\Sigma_{\rm ess}^0$ and in $\Sigma_{\rm ess}^1$. The sets Ω_j for $j = 1, 2, 3, \infty$ denote the four connected components of $\mathbb{C} \setminus \Sigma_{\rm ess}^0$.

where we assumed that the wave speed c is non-zero. Two hyperbolic equilibria are given by

$$p_0 = (0,0,0) \qquad p_1 = \Big(\frac{b}{a}, 0, 0\Big).$$

Using HOMCONT [4] within AUTO97 [8], we recovered the homoclinic pulses found in [25] that terminate onto a heteroclinic cycle formed by a front that connects p_0 to p_1 and a back that connects p_1 to p_0. The corresponding parameter values are $c = 1.73144$ and $\epsilon = 0.10744$. We do not attempt to prove the existence of a front or a back rigorously. Note, however, that hypothesis 1 is automatically met in three space dimensions once the front and the back exist.

We shall calculate the essential spectra of p_0 and p_1 as well as the absolute spectrum of p_1, and compare our findings with numerical simulations. Linearizing the PDE (8.1) about p_0 and p_1, and writing the associated eigenvalue problems as first-order ODEs, we obtain the constant-coefficient matrices A_0 and A_1, respectively, that are given by

$$A_0(\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ \lambda + \frac{b}{a\epsilon} & c & 0 \\ 0 & 0 & -\frac{1}{c}(\lambda+1) \end{pmatrix} \qquad A_1(\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ \lambda - \frac{b}{a\epsilon}(1-\frac{b}{a}) & c & \frac{b}{a^2\epsilon}(1-\frac{b}{a}) \\ 0 & 0 & -\frac{1}{c}(\lambda+1) \end{pmatrix}.$$

Thus, the essential spectra $\Sigma_{\rm ess}^0$ and $\Sigma_{\rm ess}^1$ of p_0 and p_1, respectively, are given by

$$\Sigma_{\rm ess}^0 = \{\mathrm{Re}\,\lambda = -1\} \cup \Big\{\lambda = -\tau^2 - \frac{b}{a\epsilon} + \mathrm{i}c\tau;\ \tau \in \mathbb{R}\Big\} \tag{8.2}$$
$$\Sigma_{\rm ess}^1 = \{\mathrm{Re}\,\lambda = -1\} \cup \Big\{\lambda = -\tau^2 + \frac{b}{a\epsilon}(1-\frac{b}{a}) + \mathrm{i}c\tau;\ \tau \in \mathbb{R}\Big\}.$$

In particular, p_0 is stable while p_1 is unstable, and hypothesis 3 is met. We plotted the essential spectrum of p_0 in figure 5(a).

Next, we compute the absolute spectrum $\Sigma_{\rm abs}^1$ of p_1. Note that $\mathbb{C} \setminus \Sigma_{\rm ess}^0$ has four disjoint connected components which we denote by Ω_j with $j = 1, 2, 3, \infty$ as indicated in figure 5(b). It is straightforward to calculate that

$$i_{\Omega_\infty} = 1 \qquad i_{\Omega_1} = i_{\Omega_3} = 2 \qquad i_{\Omega_2} = 3;$$

recall that these integers are equal to the number of unstable eigenvalues of $A_0(\lambda)$ for λ in the relevant connected component Ω. The eigenvalues of $A_1(\lambda)$ are given by

$$\frac{c}{2} \pm \Big(\frac{c^2}{4} + \lambda - \frac{b}{a\epsilon}(1 - \frac{b}{a})\Big)^{\frac{1}{2}} \qquad \text{and} \qquad -\frac{1}{c}(\lambda + 1).$$

Ordering these eigenvalues with decreasing real part so that

$$\operatorname{Re}\nu_1(\lambda) \geq \operatorname{Re}\nu_2(\lambda) \geq \operatorname{Re}\nu_3(\lambda)$$

and checking the definition

$$\Sigma^{1,\Omega}_{\text{abs}} := \{\lambda \in \Omega;\ \operatorname{Re}\nu_{i_\Omega}(\lambda) = \operatorname{Re}\nu_{i_\Omega+1}(\lambda)\},$$

we see that

$$\Sigma^{1,\Omega_\infty}_{\text{abs}} = \Big[-\frac{b}{a\epsilon}, -\frac{c^2}{4} + \frac{b}{a\epsilon}(1 - \frac{b}{a})\Big] \qquad \Sigma^{1,\Omega_1}_{\text{abs}} = \Sigma^{1,\Omega_2}_{\text{abs}} = \emptyset$$

$$\Sigma^{1,\Omega_3}_{\text{abs}} = \Omega_3 \cap \Big\{\lambda = c\tau - 1 - \frac{c^2}{2} \pm 2\mathrm{i}\tau\Big[(\tau - \frac{c}{2})^2 + 1 + \frac{b}{a\epsilon}(1 - \frac{b}{a})\Big]^{\frac{1}{2}};\ \tau \geq 0\Big\}.$$

In particular, at the bifurcation point, we have

$$\Sigma^{1,\Omega_\infty}_{\text{abs}} = [-0.77564, -0.03847]$$

so that the bifurcating pulses are stable as far as the absolute spectrum of p_1 is concerned.

In the last two sections, we have proved that the spectrum of the bifurcating pulses is the union of the essential spectrum of p_0, the extended spectral set $\Sigma^{\text{e}}_{\text{ext}}$ and all uniformly isolated eigenvalues. Theorem 3 states that, in each connected component Ω of $\mathbb{C} \setminus \Sigma^0_{\text{ess}}$, the extended spectral set $\Sigma^{\text{e},\Omega}_{\text{ext}}$ is equal to the absolute spectrum $\Sigma^{1,\Omega}_{\text{abs}}$ of p_1 provided hypotheses 4 and 5 are met. In the region Ω_∞, hypotheses 4 and 5 are

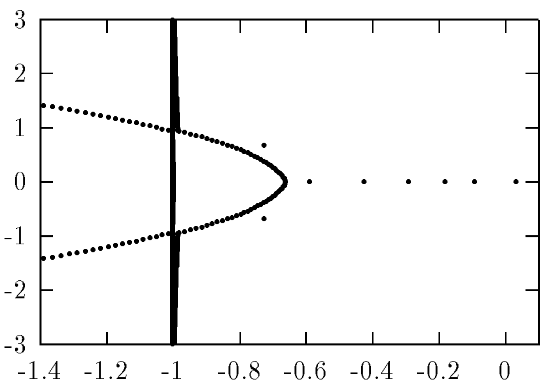


Figure 6. The picture shows the spectrum of the linearization about the pulse $h_L(x)$ at $L = 15$ computed for an overall interval length of 400 with periodic boundary conditions. We discretized the operator using centered finite differences with 1500 equi-distant mesh points and solved the resulting matrix eigenvalue problem using the routine DGEEV from the LAPACK package [2].

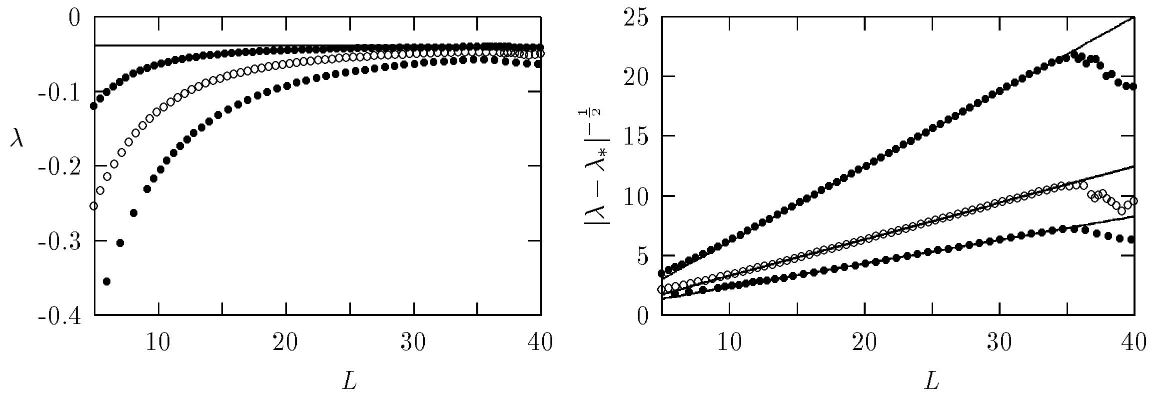


Figure 7. We continued three different eigenvalues of the linearization about the pulse $h_L(x)$ in the bifurcation parameter L. In the left picture, we plotted λ over L; the horizontal line is equal to the edge $\lambda_* = -0.038471$ of the absolute spectrum. The right picture is a plot of $\sqrt{|\lambda - \lambda_*|}^{-1}$ over L.

indeed both satisfied, and theorem 3 applies. Therefore, we have $\Sigma_{\rm abs}^{1,\Omega_\infty} = \Sigma_{\rm ext}^{\rm e,\Omega_\infty}$. In the region Ω_3, however, hypothesis 4 is violated as $D_{\rm b}(\lambda)$ vanishes identically in the triangular-shaped region that is bounded by $\operatorname{Re}\lambda = -1$, $\Sigma_{\rm ess}^0$ and $\Sigma_{\rm abs}^{1,\Omega_3}$; see figure 5(b). The reason is that the u-component along the back is less than 1/3 everywhere. In particular, we have $f(u) = 0$ along the back, and the equation for w decouples. It is then a consequence of the eigenvalue structure that, in the triangular-shaped region, the Evans function along the back vanishes as the w-component does not play any role there. Inspecting the proof of theorem 3, it follows that one of the coefficients $a_{i_\Omega}^{\rm b}(\lambda)$ and $a_{i_\Omega+1}^{\rm b}(\lambda)$ that appear in (7.4) vanishes identically; as a result, it can be shown that the pulses should not exhibit any spectrum inside the region Ω_3.

In summary, the spectrum about the bifurcating pulses is given by the union of the essential spectrum of p_0, the absolute spectrum of p_1 in the region Ω_∞ and a finite number of uniformly isolated eigenvalues; see figure 5(a).

Next, we compare these calculations with numerical simulations. We truncate the real line to an interval of length 400 and impose periodic boundary conditions. The results in [21] imply that the entire spectrum, including the essential spectrum, of the operator on the unbounded real line is then well approximated; note that this is no longer true if we use separated boundary conditions [21]. Figure 6 contains a plot of the PDE spectrum about the bifurcating pulse. The absolute spectrum $\Sigma_{\rm abs}^{1,\Omega_\infty}$ is resolved. To confirm that the edge of the absolute spectrum is indeed located to the left of the imaginary spectrum, we computed the first three eigenvalues on the real axis that are to the left of the trivial eigenvalue at $\lambda = 0$, and continued these three eigenvalues in the bifurcation parameter L that is the equal to half the length of the plateau where $h_L(x)$ is close to p_1. The results are shown in figure 7. The indications are that the eigenvalues indeed stop at the edge $\lambda_* = -0.038471$ of $\Sigma_{\rm abs}^{1,\Omega_\infty}$. In addition, these computations confirm that there is only one eigenvalue close to the origin uniformly in L as predicted by lemma 3.

We remark that the spectra of the front and the back cannot be computed numerically by truncating the domain to a bounded interval and imposing boundary conditions. Indeed, we cannot use periodic boundary conditions for fronts or backs. On the other hand, as shown in [21], separated boundary conditions will not recover the spectrum on the unbounded domain.

If we continue front and back to larger values of b, then the bifurcating pulses are eventually unstable. The reason is that the two unstable eigenvalues of the matrix $A_1(0)$ associated with the equilibrium p_1 merge and become non-real complex conjugated eigenvalues. Thus, $\lambda = 0$ is in the absolute spectrum $\Sigma_{\rm abs}^{1,\Omega_\infty}$, and the bifurcating pulses are unstable.

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