Interfaces between rolls in the Swift-Hohenberg equation

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Abstract

We study the existence of interfaces between stripe or roll solutions in the Swift-Hohenberg equation. We prove the existence of two different types of interfaces: corner-like interfaces, also referred to as knee solutions, and step-like interfaces. The analysis relies upon a spatial dynamics formulation of the existence problem and an equivariant center manifold reduction. In this setting, the interfaces are found as heteroclinic and homoclinic orbits of a reduced system of ODEs.

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1 Introduction

This paper is concerned with the existence of two-dimensional solutions in damped-driven pattern-forming systems. To be specific, we focus on the Swift-Hohenberg equation, which has been proposed as a prototypical example for pattern forming systems, in areas as diverse as nonlinear optics [15], Rayleigh-Bénard convection [2], granular media [1], chemical reactions [14], liquid crystals and solidification; see [4] and references there. Most of those systems exhibit stationary stripe (or roll) patterns, that is, planar patterns which are independent of x and periodic in y, where x and y are coordinates in the plane of observation. While the formation of these particular spatially periodic patterns is well understood close to onset, experimental patterns typically exhibit patches of stripe patterns with varying wavenumbers and orientations, separated by interfaces and defects. The present paper is a step towards a more systematic understanding of such two-dimensional patterns which are close to stripes in large parts of the physical domain.

We consider the Swift-Hohenberg equation

$$\partial_t u = -(1+\Delta)^2 u + \varepsilon u - u^3, \tag{1.1}$$

in which u depends upon two spatial variables $(x, y) \in \mathbb{R}^2$ and time $t \geq 0$, and ε is a small real parameter. Our results immediately carry over to more general pattern forming systems, such as reaction-diffusion systems, convection models, or Ginzburg-Landau-type models in nonlinear optics; we will comment on these extensions in the discussion at the end of this paper.

It is well-known that this equation possesses roll solutions

$$u_{\varepsilon,\kappa}(y) = U_{\varepsilon,\kappa}(ky) = \sqrt{4(\varepsilon-\kappa^2)/3}\cos(ky) + \mathrm{O}(|\varepsilon-\kappa^2|^{3/2}), \quad k = \sqrt{1+\kappa}, \tag{1.2}$$

for small $\varepsilon \in (0, \varepsilon_0]$ and $\kappa^2 < \varepsilon$. These are steady one-dimensional periodic solutions of (1.1). We will focus here on two-dimensional steady solutions of (1.1) that are periodic in y and close to roll solutions everywhere, that is,

$$u(x, y) = U_{\varepsilon,\kappa}(k(y + \xi(x))) + v(x, k(y + \xi(x))), \tag{1.3}$$

where $\xi : \mathbb{R} \to \mathbb{R}$ and v is 2π-periodic in its second argument. We say u is δ-close to $U_{\varepsilon,\kappa}$ if

$$\sup_{x\in\mathbb{R}} |\xi'(x)| < \delta, \quad \sup_{x\in\mathbb{R}} \|v(x,\cdot)\|_{H^1(0,2\pi)} < \delta.$$

A particularly interesting subclass of such solutions are *knee solutions*, for which $\xi'(x) \to \pm\eta_*$, as $x \to \pm\infty$; see Figure 1.1. We refer to [7, 8] and the references therein for a discussion of these

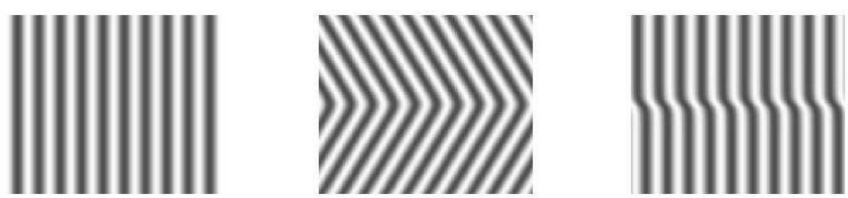

Figure 1.1: Plot of a roll solution (left), a knee solution (middle), and a step (right).

solutions within the approximation of Swift-Hohenberg dynamics through the Cross-Newell phase

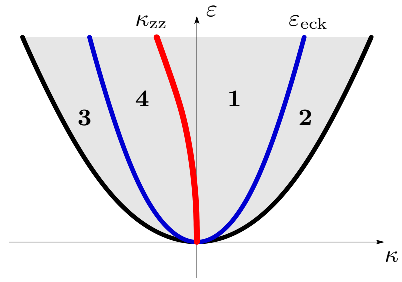

Figure 1.2: Rolls exist in the shaded region. They are Eckhaus unstable in the regions 2 and 3, and zigzag unstable in the regions 3 and 4. In region 1, rolls are stable in both one and two dimensions.

diffusion equation. Another subclass of solutions are *steps*, for which $\xi'(x) \to \eta_*$, as $x \to \pm\infty$; see Figure 1.1. Our main results concern the existence of these two types of solutions.

A crucial role in our analysis is played by the stability properties of rolls. Close to threshold, stability boundaries of rolls are given by two curves

$$\varepsilon = \varepsilon_{\rm eck}(\kappa) = 3\kappa^2 + {\rm O}(|\kappa|^3), \qquad \kappa = \kappa_{\rm zz}(\varepsilon) = -\frac{1}{512}\varepsilon^2 + {\rm O}(\varepsilon^3),$$

such that rolls with $\varepsilon < \varepsilon_{\rm eck}(\kappa)$ are Eckhaus unstable and rolls with $\kappa < \kappa_{\rm zz}(\varepsilon)$ are zigzag unstable [6, 3]. We also write $\kappa_{\rm eck}$ for the (positive) inverse of $\varepsilon_{\rm eck}$. The Eckhaus instability refers to longitudinal, one-dimensional perturbations, and the zigzag instability refers to transverse, two-dimensional perturbations. In the remaining region, where $\varepsilon \geq \varepsilon_{\rm eck}(\kappa)$ *and* $\kappa \geq \kappa_{\rm zz}(\varepsilon)$, the rolls are linearly stable [17]; see Figure 1.2.

The existence results in this paper concern the parameter region close to the zigzag instability curve. They are summarized in the following theorem.

Theorem 1 *Assume that ε is sufficiently small. Then for any $\delta > 0$ small, there exists $\kappa_*(\varepsilon) < \kappa_{\rm zz}(\varepsilon)$ such that for any $\kappa \in (\kappa_*(\varepsilon), \kappa_{\rm zz}(\varepsilon))$ the following properties hold.*

(i) The Swift-Hohenberg equation (1.1) possesses a pair $\pm u_\kappa$ of steady knee solutions *of the form (1.3), δ-close to $u_{\varepsilon,\kappa}$, which are periodic in y with period $2\pi/k$, $k = \sqrt{1+\kappa}$, and even in x. Furthermore,*

$$\xi'_\kappa(x) \to \pm\eta^\infty_\kappa = \pm\sqrt{-2\mu} + {\rm O}(\varepsilon|\mu|^{1/2} + |\mu|^{3/2}), \quad \textit{as } x \to \pm\infty, \quad \textit{where} \;\; \mu = \kappa - \kappa_{\rm zz}(\varepsilon).$$

(ii) In addition, there is an open subset I_κ of $\mathbb{R}$ and two one-parameter families $\{\pm u_{\kappa,\nu},\ \nu \in I_\kappa\}$ of steady steps *of the form (1.3), δ-close to $u_{\varepsilon,\kappa}$, which are $2\pi/k$–periodic in y, odd in x, and for which*

$$\xi'_{\kappa,\nu}(x) \to \eta^\infty_{\kappa,\nu} = {\rm O}(|\mu|^{1/2}), \quad \textit{as } x \to \pm\infty, \quad \textit{with} \;\; \eta^\infty_{\kappa,\nu} \neq \eta^\infty_\kappa.$$

We point out that near stable rolls, that is, for $\kappa_{\rm zz} < \kappa < \kappa_{\rm eck}$, there do not exist two-dimensional solutions which are δ-close to the roll solutions for some $\delta(\kappa)$ sufficiently small (see the discussion in Section 4).

Outline:

The remainder of this paper is occupied with the proof of Theorem 1. In Section 2, we simplify the existence problem using an equivariant spatial center-manifold reduction, inspired by the general approach to almost planar waves developed in [9] for reaction-diffusion systems; see also [10, 11]. In Section 3, we then analyze the reduced system of ordinary differential equations. Knee solutions and steps are found as reversible heteroclinic and homoclinic connections, respectively. We establish their existence via transversality arguments. We conclude with a discussion of our results and of possible extensions in Section 4.

2 Reduction to a spatial center-manifold

We are interested in steady solutions of (1.1) which are periodic in y with period $2\pi/k$, so that they satisfy the equation

$$-(1+\partial_{xx}+k^2\partial_{yy})^2u+\varepsilon u-u^3=0. \tag{2.1}$$

Here, we have normalized the period in y to 2π by replacing ky by y. Following the general approach in [9] we start by rewriting this equation as a first order system in which x is the time-like variable. This is easily achieved by taking $\mathbf{u}=(u,u_1,u_2,u_3)^t=(u,u_x,u_{xx},u_{xxx})^t$, so that (2.1) becomes

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}x}=\mathcal{A}(k,\varepsilon)\mathbf{u}+\mathcal{F}(\mathbf{u}), \tag{2.2}$$

in which

$$\mathcal{A}(k,\varepsilon)\mathbf{u}=(u_1,u_2,u_3,-(1+k^2\partial_{yy})^2u-2(1+k^2\partial_{yy})u_2+\varepsilon u)^t,\qquad \mathcal{F}(\mathbf{u})=(0,0,0,-u^3)^t.$$

We regard this system as an infinite dimensional dynamical system in the space of 2π-periodic functions $X:=H^3_{\mathrm{per}}(0,2\pi)\times H^2_{\mathrm{per}}(0,2\pi)\times H^1_{\mathrm{per}}(0,2\pi)\times L^2(0,2\pi)$.

The rolls $U_{\varepsilon,\kappa}$ in (1.2) are equilibria of this dynamical system, and due to the translation invariance in y we actually have a line of equilibria

$$\mathbf{U}^\xi_{\varepsilon,\kappa}(\cdot)=(U_{\varepsilon,\kappa}(\cdot+\xi),0,0,0)^t,\quad \xi\in\mathbb{R},$$

for $k=\sqrt{1+\kappa}$. We focus on solutions of (2.1) for parameter values close to the zigzag instability curve $\kappa=\kappa_{\mathrm{zz}}(\varepsilon)$. We therefore choose (ε_*,κ_*) with $\kappa_*=\kappa_{\mathrm{zz}}(\varepsilon_*)$, and consider (k,ε) in (2.2) with $\varepsilon=\varepsilon_*$ and $k=\sqrt{1+\kappa}$ close to $k_*=\sqrt{1+\kappa_*}$. For simplicity, we write from now on U_* instead of $U_{\varepsilon_*,\kappa_*}$ and $\mathbf{U}^\xi_*$ instead of $\mathbf{U}^\xi_{\varepsilon_*,\kappa_*}$.

We start with the linearization at the equilibrium $\mathbf{U}_*=\mathbf{U}^0_*$,

$$\mathcal{A}_*=\mathcal{A}(k_*,\varepsilon_*)+D\mathcal{F}(\mathbf{U}_*),\qquad (k_*=\sqrt{1+\kappa_*}),$$

which is a closed linear operator in X with domain $Y:=H^4_{\mathrm{per}}(0,2\pi)\times H^3_{\mathrm{per}}(0,2\pi)\times H^2_{\mathrm{per}}(0,2\pi)\times H^1_{\mathrm{per}}(0,2\pi)$. The spectral properties of $\mathcal{A}_*$ are determined by the linear stability of rolls, more precisely by the spectral properties of the linearization of (1.1) at U_*,

$$\mathcal{L}_*=-(1+\partial_{xx}+k_*^2\partial_{yy})^2+\varepsilon_*-3U_*^2.$$

This differential operator has coefficients which only depend upon y. Then with Fourier decomposition in x with wavenumbers ℓ we find the family of linear operators

$$\mathcal{L}(\ell) = -(1 - \ell^2 + k_*^2 \partial_{yy})^2 + \varepsilon_* - 3U_*^2,$$

acting in $L^2(0, 2\pi)$ with domain $H^4_{\mathrm{per}}(0, 2\pi)$. According to the spectral analysis in [17], the spectra of $\mathcal{L}(\ell)$ have the following properties:

(i) the spectrum of $\mathcal{L}(0)$ is contained in the open left complex half plane, except for a simple eigenvalue at the origin with associated eigenvector the derivative of the roll U_{*y};

(ii) for any $\ell \neq 0$, the spectrum of $\mathcal{L}(\ell)$ is contained in the open left complex half plane;

(iii) the smooth continuation of the simple zero eigenvalue of $\mathcal{L}(0)$ for small $\ell \neq 0$ is a simple eigenvalue $\lambda_*(\ell)$ of $\mathcal{L}(\ell)$ with expansion $\lambda_*(\ell) = -c_*\ell^4 + \mathrm{O}(\ell^6)$ in which $c_* > 0$ depends upon $(\varepsilon_*, \kappa_*)$. We denote by $v_*(\ell)$ the associated eigenvectors, $\mathcal{L}(\ell)v_*(\ell) = \lambda_*(\ell)v_*(\ell)$, where $v_*(0) = U_{*y}$. We refer to $\lambda = \lambda_*(\ell)$ as *linear dispersion relation.*

We summarize the properties of $\mathcal{A}_*$ in the next lemma.

Lemma 2.1 *Assume $(\varepsilon_*, \kappa_*)$ is such that $\kappa_* = \kappa_{\mathrm{zz}}(\varepsilon_*)$. Then the linear operator $\mathcal{A}_*$ acting in X with domain Y has the following properties.*

(i) The spectrum of $\mathcal{A}_*$ is purely point spectrum,

$$\mathrm{spec}(\mathcal{A}_*) = \{0\} \cup \sigma_1(\mathcal{A}_*), \quad \textit{with} \quad \sigma_1(\mathcal{A}_*) \subset \{\nu \in \mathbb{C}\,;\, |\operatorname{Re}\nu| \geq a_*,\ |\operatorname{Im}\nu| \leq b_*\},$$

for some positive constants a_* and b_*.

(ii) The zero eigenvalue of $\mathcal{A}_*$ is algebraically quadruple and the four-dimensional generalized kernel of $\mathcal{A}_*$ is spanned by the vectors

$$\mathbf{e}_0 = \begin{pmatrix} U_{*y} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_1 = \begin{pmatrix} 0 \\ U_{*y} \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} -\frac{1}{2}v_*''(0) \\ 0 \\ U_{*y} \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ -\frac{1}{2}v_*''(0) \\ 0 \\ U_{*y} \end{pmatrix}, \tag{2.3}$$

in which $v_*''(\ell)$ is the second derivative with respect to ℓ of the eigenvectors $v_*(\ell)$ of $\mathcal{L}(\ell)$, and $\mathcal{A}_*\mathbf{e}_0 = 0$, $\mathcal{A}_*\mathbf{e}_j = \mathbf{e}_{j-1}$, $j = 1, 2, 3$.

(iii) The spectral projection $P : X \to X$ onto the generalized kernel of $\mathcal{A}_*$ is given by

$$P\mathbf{u} = \langle \mathbf{u}, \mathbf{e}_0^{\mathrm{ad}} \rangle \mathbf{e}_0 + \langle \mathbf{u}, \mathbf{e}_1^{\mathrm{ad}} \rangle \mathbf{e}_1 + \langle \mathbf{u}, \mathbf{e}_2^{\mathrm{ad}} \rangle \mathbf{e}_2 + \langle \mathbf{u}, \mathbf{e}_3^{\mathrm{ad}} \rangle \mathbf{e}_3,$$

in which $\langle \cdot, \cdot \rangle$ denotes the scalar product in $(L^2(0, 2\pi))^4$, and $\{\mathbf{e}_0^{\mathrm{ad}}, \mathbf{e}_1^{\mathrm{ad}}, \mathbf{e}_2^{\mathrm{ad}}, \mathbf{e}_3^{\mathrm{ad}}\}$ is a dual basis with $\langle \mathbf{e}_j, \mathbf{e}_k^{\mathrm{ad}} \rangle = \delta_{jk}$, for $0 \leq j, k \leq 3$, and $\mathcal{A}_*^{\mathrm{ad}}\mathbf{e}_3^{\mathrm{ad}} = 0$, $\mathcal{A}_*^{\mathrm{ad}}\mathbf{e}_j = \mathbf{e}_{j+1}$, for $j = 0, 1, 2$, where $\mathcal{A}_*^{\mathrm{ad}}$ is the L^2-adjoint of $\mathcal{A}_*$. Furthermore, $\mathbf{e}_0^{\mathrm{ad}} = (V_{00}, 0, V_{02}, 0)^t$, $\mathbf{e}_1^{\mathrm{ad}} = (0, V_{11}, 0, V_{13})^t$, with V_{ij} smooth odd 2π-periodic functions, and

$$\mathbf{e}_2^{\mathrm{ad}} = \frac{1}{c_*(U_{*y}, U_{*y})} \begin{pmatrix} \frac{1}{2}\mathcal{L}''(0)U_{*y} \\ 0 \\ U_{*y} \\ 0 \end{pmatrix}, \quad \mathbf{e}_3^{\mathrm{ad}} = \frac{1}{c_*(U_{*y}, U_{*y})} \begin{pmatrix} 0 \\ \frac{1}{2}\mathcal{L}''(0)U_{*y} \\ 0 \\ U_{*y} \end{pmatrix}, \tag{2.4}$$

where $(\cdot,\cdot)$ *denotes the scalar product in* $L^2(0,2\pi)$, c_* *is the positive constant in the expansion of* $\lambda_*(\ell)$, *and* $\mathcal{L}''(\ell)$ *the second order derivative with respect to* ℓ *of* $\mathcal{L}(\ell)$.

Proof. (i) Since Y is compactly embedded in X, $\mathcal{A}_*$ has compact resolvent, so that its spectrum is purely point spectrum. For small (ε_*,κ_*), $\mathcal{A}_*$ is a small bounded perturbation of the operator with constant coefficients $\mathcal{A}(k_*,\varepsilon_*)$ with spectrum $\mathrm{spec}(A(k_*,\varepsilon_*)) = \{\pm\sqrt{k_*^2n^2-1\pm\sqrt{\varepsilon_*}},\ n\in\mathbb{Z}\}$. A standard perturbation argument then implies that

$$\mathrm{spec}(\mathcal{A}_*) \subset \mathrm{i}\mathbb{R}\cup\{\nu\in\mathbb{C}\,;\ |\operatorname{Re}\nu|\geq a_*,\ |\operatorname{Im}\nu|\leq b_*\},$$

for some positive constants a_* and b_*.

The eigenvalue problem $\mathcal{A}_*\mathbf{v}=\nu\mathbf{v}$ is equivalent to the equality

$$-(1+k_*^2\partial_{yy})^2v-2(1+k_*^2\partial_{yy})\nu^2v+\varepsilon_*v-3U_*^2v=\nu^4v.$$

Upon comparing this equality with the formula for $\mathcal{L}(\ell)$ we conclude that $\nu=\mathrm{i}\ell$ is an eigenvalue of $\mathcal{A}_*$ if and only if 0 is an eigenvalue of the operator $\mathcal{L}(\ell)$. Then the properties (i) and (ii) of $\mathcal{L}(\ell)$ imply that 0 is the only eigenvalue of $\mathcal{A}_*$ on the imaginary axis. In addition, we find that 0 is geometrically simple with associated eigenvector $\mathbf{e}_0$ as in (2.3).

(ii) The algebraic multiplicity of the zero eigenvalue of $\mathcal{A}_*$ turns out to be equal to the order of the root $\ell=0$ of $\lambda_*(\ell)=-c_*\ell^4+\mathrm{O}(\ell^6)=0$. First, it is easy to check that $\mathbf{e}_1$ given in (2.3) is a principal vector in the generalized kernel of $\mathcal{A}_*$, $\mathcal{A}_*\mathbf{e}_1=\mathbf{e}_0$. Next, notice that

$$\mathcal{A}_*\mathbf{u}=\left(u_1,u_2,u_3,\mathcal{L}(0)u-\frac{1}{2}\mathcal{L}''(0)u_2\right)^t,$$

and that differentiating the equality $\mathcal{L}(\ell)v_*(\ell)=\lambda_*(\ell)v_*(\ell)$ twice with respect to ℓ gives

$$\mathcal{L}''(0)v_*(0)+\mathcal{L}(0)v_*''(0)=0.$$

Together with the formula for $\mathbf{e}_1$ these equalities imply that $\mathcal{A}_*\mathbf{e}_2=\mathbf{e}_1$. In addition, $\mathcal{A}_*\mathbf{e}_3=\mathbf{e}_2$, so that the algebraic multiplicity of 0 is at least four. Now, the fourth order derivative of $\mathcal{L}(\ell)v_*(\ell)=\lambda_*(\ell)v_*(\ell)$ with respect to ℓ gives

$$\mathcal{L}^{(4)}(0)v_*(0)+6\mathcal{L}''(0)v_*''(0)+\mathcal{L}(0)v_*^{(4)}(0)=-4!\,c_*v_*(0).$$

Since $c_*\neq 0$ this equality implies that there is no solution $\mathbf{u}\in X$ of $\mathcal{A}_*\mathbf{u}=\mathbf{e}_3$. Consequently, the generalized kernel of $\mathbf{A}_*$ is four-dimensional, spanned by $\{\mathbf{e}_0,\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, so that 0 has algebraic multiplicity four, just as the order of the root $\ell=0$ of $\lambda_*(\ell)=0$.

(iii) We construct the spectral projection P with the help of the L^2-adjoint of $\mathcal{A}_*$ given by

$$\mathcal{A}_*^{\mathrm{ad}}\mathbf{u}=\left(\mathcal{L}(0)u_3,u,u_1-\frac{1}{2}\mathcal{L}''(0)u_3,u_2\right)^t.$$

Then the first part in (iii) follows from standard results on adjoint operators, and a direct calculation shows that $\mathbf{e}_2^{\mathrm{ad}}$ and $\mathbf{e}_3^{\mathrm{ad}}$ given in (2.4) have the required properties. ∎

Remark 2.2 *For the shifted equilibria* $\mathbf{U}_*^\xi$, *the linearization gives the shifted linear operators* $\mathcal{A}_*^\xi=\mathcal{A}(k_*,\varepsilon_*)+D\mathcal{F}(\mathbf{U}_*^\xi)$. *Clearly, these operators have the same properties as* $\mathcal{A}_*$, *and we then introduce the shifted vectors* $\mathbf{e}_j^\xi$, $\mathbf{e}_j^{\mathrm{ad},\xi}$, *the shifted adjoints* $\mathcal{A}_*^{\mathrm{ad},\xi}$, *and the shifted spectral projections* P^ξ.

We now go back to the system (2.2) in which we take $\varepsilon = \varepsilon_*$ and $k^2 = k_*^2 + \mu$, with μ small. Following the general approach to almost planar waves in [9, 11], we set

$$\mathbf{u} = \mathbf{U}_*^\xi + \eta_1 \mathbf{e}_1^\xi + \eta_2 \mathbf{e}_2^\xi + \eta_3 \mathbf{e}_3^\xi + \mathbf{w}^\xi, \quad \text{with } P^\xi \mathbf{w}^\xi = P\mathbf{w} = 0, \tag{2.5}$$

where ξ, η_1, η_2, and η_3 are real-valued functions depending upon x. By substituting (2.5) into (2.2), we find

$$\begin{aligned} &\xi_x \mathbf{e}_0 + \eta_{1x}\mathbf{e}_1 + \eta_{2x}\mathbf{e}_2 + \eta_{3x}\mathbf{e}_3 + \xi_x\eta_1\mathbf{e}_{1y} + \xi_x\eta_2\mathbf{e}_{2y} + \xi_x\eta_3\mathbf{e}_{3y} + \mathbf{w}_x + \xi_x\mathbf{w}_y \\ &\quad = \mathcal{A}_*(\eta_1\mathbf{e}_1 + \eta_2\mathbf{e}_2 + \eta_3\mathbf{e}_3 + \mathbf{w}) + (\mathcal{A}(k,\varepsilon_*) - \mathcal{A}(k_*,\varepsilon_*))(\mathbf{U}_* + \eta_2\mathbf{e}_2 + \mathbf{w}) + \mathcal{G}(\eta_2,\mathbf{w}), \end{aligned} \tag{2.6}$$

where

$$\mathcal{G}(\eta_2,\mathbf{w}) = \mathcal{F}(\mathbf{U}_* + \eta_2\mathbf{e}_2 + \mathbf{w}) - \mathcal{F}(\mathbf{U}_*) - D\mathcal{F}(\mathbf{U}_*)(\eta_2\mathbf{e}_2 + \mathbf{w}).$$

We obtain a system for the variables ξ, η_1, η_2, η_3, and $\mathbf{w}$, by taking the scalar product of (2.6) with $\mathbf{e}_0^{\rm ad}$, $\mathbf{e}_1^{\rm ad}$, $\mathbf{e}_2^{\rm ad}$, $\mathbf{e}_3^{\rm ad}$, successively, and then projecting with $\mathrm{id} - P$. We use the explicit formulas for $\mathbf{e}_{jy}$, $\mathbf{e}_k^{\rm ad}$ in Lemma 2.1, and the fact that the components of all these vectors are odd functions, so that, in particular, the scalar products $\langle \mathbf{e}_{jy}, \mathbf{e}_k^{\rm ad}\rangle$ vanish. Then after straightforward calculations we find a scalar equation for ξ,

$$\xi_x = (1 - \langle \mathbf{w}, \mathbf{e}_{0y}^{\rm ad}\rangle)^{-1}\eta_1, \tag{2.7}$$

and the system

$$\begin{aligned} \eta_{1x} &= \eta_2 + \mathrm{O}(|\eta_1|^2 + |\eta_2|^2 + \|\mathbf{w}\|_Y^2 + |\mu|) \\ \eta_{2x} &= \eta_3 + \mathrm{O}(|\eta_1|^2 + \|\mathbf{w}\|_X^2) \\ \eta_{3x} &= \mathrm{O}(|\eta_1|^2 + |\eta_2|^2 + \|\mathbf{w}\|_Y^2 + |\mu|) \\ \mathbf{w}_x &= \mathcal{A}_*\mathbf{w} + \mathrm{O}(|\eta_1|^2 + |\eta_2|^2 + |\eta_3|^2 + \|\mathbf{w}\|_Y^2 + |\mu|) \end{aligned} \tag{2.8}$$

in which we have replaced ξ_x in the right hand sides of (2.8) from (2.7). In particular, the equation (2.7) decouples, so that we first solve (2.8), and afterwards determine ξ from (2.7). The precise form of the nonlinear terms in this system is not essential at this point, we shall discuss it more detail in the next section.

The system (2.8) is a quasilinear system in the Hilbert space $\mathbb{R}^3 \times X_h$, where $X_h = (\mathrm{id} - P)X$, with smooth nonlinear vector field defined on $\mathbb{R}^3 \times Y_h$, where $Y_h = (\mathrm{id} - P)Y$. At $\mu = 0$ its linearization about zero decomposes into the direct sum of a matrix on $\mathbb{R}^3$ having a triple zero eigenvalue and the restriction of $\mathcal{A}_*$ to X_h. According to Lemma 2.1, the latter operator is hyperbolic, so that we can apply the center manifold theorem for quasilinear systems in [16, Theorem 1], and conclude that the small bounded solutions of (2.8) for μ small are of the form

$$(\eta_1,\eta_2,\eta_3,\mathbf{w}) = (\eta_1,\eta_2,\eta_3,\mathbf{h}(\eta_1,\eta_2,\eta_3;\mu)), \quad \mathbf{h}(\eta_1,\eta_2,\eta_3;\mu) = \mathrm{O}(|\eta_1|^2 + |\eta_2|^2 + |\eta_3|^2 + |\mu|),$$

in which the reduction function $\mathbf{h} : U \to Y_h$ is of class C^k, for an arbitrary positive integer k, on a neighborhood U of the origin in $\mathbb{R}^4$. As a consequence, we can determine the small bounded solutions of (2.8) by solving the system of ODEs

$$\begin{aligned} \eta_{1x} &= \eta_2 + f_1(\eta_1,\eta_2,\eta_3;\mu) \\ \eta_{2x} &= \eta_3 + f_2(\eta_1,\eta_2,\eta_3;\mu) \\ \eta_{3x} &= f_3(\eta_1,\eta_2,\eta_3;\mu) \end{aligned} \tag{2.9}$$

obtained by substituting $\mathbf{w} = \mathbf{h}(\eta_1, \eta_2, \eta_3; \mu)$ into (2.8), with $f_j(\eta_1, \eta_2, \eta_3; \mu) = \mathrm{O}(|\eta_1|^2 + |\eta_2|^2 + |\eta_3|^4 + |\mu|)$, $j = 1, 2, 3$. Notice also that the equation (2.7) now becomes

$$\xi_x = \eta_1(1 + f_0(\eta_1, \eta_2, \eta_3; \mu)), \quad f_0(\eta_1, \eta_2, \eta_3; \mu) = \mathrm{O}(|\eta_1|^2 + |\eta_2|^2 + |\eta_3|^2 + |\mu|). \tag{2.10}$$

3 The reduced equations

In this section, we discuss the reduced system (2.9). More precisely, we focus on the existence of heteroclinic and homoclinic orbits of (2.9), which correspond to knees and steps, respectively, for the Swift-Hohenberg equation. The result in Theorem 1 is a consequence of the following proposition.

Proposition 3.1 *Assume ε_* is sufficiently small. There exists $\mu_* > 0$ such that for any $\mu \in (-\mu_*, 0)$ the following properties hold.*

(i) The reduced system (2.9) possesses a pair of heteroclinic orbits $\pm(\eta_1^\mu, \eta_2^\mu, \eta_3^\mu)$ with

$$\eta_1^\mu = \eta_*^\mu + \mathrm{O}(|\mu|^{3/2}) \ \rightarrow \ \pm\,\eta_\infty^\mu, \ \text{as } x \to \pm\infty, \tag{3.1}$$

$$\eta_2^\mu = \eta_{*x}^\mu + \mathrm{O}(|\mu|^2) \ \rightarrow \ 0, \ \text{as } x \to \pm\infty, \qquad \eta_3^\mu = \eta_{*xx}^\mu + \mathrm{O}(|\mu|^{5/2}) \ \rightarrow \ 0, \ \text{as } x \to \pm\infty,$$

in which

$$\eta_*^\mu(x) = \sqrt{-\frac{\alpha_*\mu}{\beta_*}}\,\tanh\left(\sqrt{-\frac{\alpha_*\mu}{2}}\,x\right), \quad \alpha_* = \frac{2}{c_*}\left(1 + \mathrm{O}(\varepsilon_*)\right), \quad \beta_* = \frac{1}{c_*}\left(1 + \mathrm{O}(\varepsilon_*)\right).$$

Moreover, η_1^μ, η_3^μ are odd and η_2^μ is even in x.

(ii) There exist open subsets I_μ and O_μ of $\mathbb{R}$, such that the reduced system (2.9) possesses a pair of one-parameter families of homoclinic orbits $\{\pm(\eta_1^{\mu,\nu}, \eta_2^{\mu,\nu}, \eta_3^{\mu,\nu}),\ \nu \in I_\mu\}$, with

$$(\eta_1^{\mu,\nu},\ \eta_2^{\mu,\nu},\ \eta_3^{\mu,\nu}) \ \rightarrow \ (\eta_\infty^{\mu,\nu}, 0, 0), \ \text{as } x \to \pm\infty, \quad \eta_\infty^{\mu,\nu} = \mathrm{O}(|\mu|^{1/2}).$$

Here $\eta_\infty^{\mu,\nu} \in O_\mu$ and $\eta_\infty^{\mu,\nu} \neq \eta_\infty^\mu$, η_∞^μ being the constant in (3.1). Furthermore, η_1^μ, η_3^μ are even and η_2^μ is odd in x, and the map $\nu \in I_\mu \ \mapsto \eta_\infty^{\mu,\nu} \in O_\mu$ is one-to-one and onto.

Proof. A standard normal form transformation allows to change the variables η_1, η_2, and η_3, such that the equations (2.9), (2.10) become

$$\eta_{1x} = \eta_2, \quad \eta_{2x} = \eta_3, \quad \eta_{3x} = f(\eta_1, \eta_2, \eta_3; \mu), \tag{3.2}$$

with $f(\eta_1, \eta_2, \eta_3; \mu) = \mathrm{O}(|\eta_1|^2 + |\eta_2|^2 + |\eta_3|^2 + |\mu|)$, and

$$\xi_x = \eta_1. \tag{3.3}$$

Here, we have used the same notation for the variables, for simplicity. Our purpose is to show that (3.2) possesses heteroclinic and homoclinic orbits for sufficiently small $\mu < 0$.

We claim that (3.2) has a line of equilibria $(\eta_1, 0, 0)$ near the origin, for any μ. In order to see this, we go back to the Swift-Hohenberg equation (1.1) and notice that for any $\varepsilon \in (0, \varepsilon_0]$ in addition to the x-independent roll solutions $u_{\varepsilon,\kappa}(y)$ in (1.2) the equation (1.1) has rotated rolls $u_{\varepsilon,\kappa}(\sin(\varphi)x + \cos(\varphi)y)$

for angles $\varphi \in [-\pi/2, \pi/2)$, due to rotation invariance. These rotated rolls are periodic in y with periods $2\pi/(k\cos(\varphi))$, and provide us with a family of solutions of the dynamical system (2.2),

$$\mathbf{u}_\varphi = (u_\varphi, \partial_x(u_\varphi), \partial_{xx}(u_\varphi), \partial_{xxx}(u_\varphi)), \quad u_\varphi = U_{\varepsilon,\kappa}(k\tanh(\varphi)x + y), \quad k = \sqrt{1+\kappa}\cos(\varphi),$$

for any pair (k, ε). With the Ansatz (2.5) we obtain a family of solutions to (2.7)–(2.8) with $\xi_x = k\tanh(\varphi)$. After the center manifold reduction and the normal form transformation we find the line of equilibria $(\eta_1, 0, 0) = (k\tanh(\varphi), 0, 0)$, for small φ, of (3.2), which proves the claim.

Next, the reflection invariances $x \mapsto -x$ and $y \mapsto -y$ of the equation (1.1), together with the parity properties of $U_{\varepsilon,\kappa}$ and $\mathbf{e}_j$, imply that the vector field in (3.2) commutes with the symmetry $\mathcal{S}$ defined by

$$\mathcal{S}(\eta_1, \eta_2, \eta_3) = (-\eta_1, -\eta_2, -\eta_3),$$

and that it has a reversibility symmetry $\mathcal{R}$, i.e., it anti-commutes with

$$\mathcal{R}(\eta_1, \eta_2, \eta_3) = (-\eta_1, \eta_2, -\eta_3).$$

In particular, these properties imply that the truncation at order 3 of the system (3.2) is of the form

$$\eta_{1x} = \eta_2, \quad \eta_{2x} = \eta_3, \quad \eta_{3x} = \alpha_*\mu\eta_2 + \beta_1\eta_1^2\eta_2 + \beta_2\eta_2^3 + \beta_3\eta_1\eta_2\eta_3 + \beta_4\eta_2\eta_3^2. \tag{3.4}$$

The important coefficients in this system are α_* and β_1 which are given by the formulas

$$\alpha_* = \frac{2}{c_*}(1 + \mathrm{O}(\varepsilon_*)) > 0, \quad \beta_1 = \frac{3}{c_*}(1 + \mathrm{O}(\varepsilon_*)) > 0,$$

for sufficiently small ε_*. Here c_* is the positive coefficient in the expansion of the eigenvalue $\lambda_*(\ell)$. Indeed, going back the reduction procedure, we find that α_* is the coefficient of the linear term in μ in the scalar product $\langle(\mathcal{A}(k, \varepsilon_*) - \mathcal{A}(k_*, \varepsilon_*))\mathbf{e}_2, \mathbf{e}_3^{\mathrm{ad}}\rangle$ (with $k^2 = k_*^2 + \mu$). Using the explicit formulas in Lemma 2.1 and the expansion (1.2) we obtain the expression for α_* above. Next, for the second coefficient we find $\beta_1 = \langle \mathbf{h}_{12}, \partial_y(\mathbf{e}_3^{\mathrm{ad}})\rangle$, where $\mathbf{h}_{12} \in Y_h$ is the coefficient of $\eta_1\eta_2$ in the expansion of the reduction function $\mathbf{h}$. In order to compute $\mathbf{h}_{12}$ we replace $\mathbf{w}$ by the expansion of $\mathbf{h}(\eta_1, \eta_2, \eta_3; 0)$ in the last equation in (2.8) for $\mu = 0$, and then substitute the derivatives η_{jx} from (3.4). Collecting the quadratic terms in the resulting expression we find that $\mathbf{h}_{12}$ is the unique solution in Y_h of the equation $\mathcal{A}_*\mathbf{h}_{12} = \mathbf{e}_{2y} + 2\mathbf{h}_{11}$, where $\mathbf{h}_{11}$ is the unique solution in Y_h of $\mathcal{A}_*\mathbf{h}_{11} = \mathbf{e}_{1y}$. After lengthy calculations we obtain the formula for β_1 above.

We now introduce the scaling

$$\widetilde{x} = (\alpha_*|\mu|)^{1/2}x, \quad \eta_1 = \frac{(\alpha_*|\mu|)^{1/2}}{\beta_*^{1/2}}\widetilde{\eta}_1, \quad \eta_2 = \frac{\alpha_*|\mu|}{\beta_*^{1/2}}\widetilde{\eta}_2, \quad \eta_3 = \frac{(\alpha_*|\mu|)^{3/2}}{\beta_*^{1/2}}\widetilde{\eta}_3,$$

where $\beta_* = \beta_1/3$, and find the scaled system

$$\eta_{1x} = \eta_2, \quad \eta_{2x} = \eta_3, \quad \eta_{3x} = \mathrm{sign}(\mu)\eta_2 + 3\eta_1^2\eta_2 + \mathrm{O}(|\mu|), \tag{3.5}$$

in which we have dropped the tilde, for notational simplicity. At zeroth order in μ, in the case $\mathrm{sign}(\mu) = -1$, this system has a pair of heteroclinic orbits with

$$\eta_1(x) = \pm\tanh\left(\frac{x}{\sqrt{2}}\right). \tag{3.6}$$

These heteroclinic orbits are embedded in two one-parameter families of homoclinic orbits

$$\pm(\eta_1^q(x), \eta_2^q(x), \eta_3^q(x)) \to \pm(q, 0, 0), \text{ as } x \to \pm\infty, \text{ for } \frac{1}{\sqrt{3}} < q < \frac{2}{\sqrt{3}}, \ q \neq 1.$$

To end the proof we show that these heteroclinic and homoclinic orbits persist for small $\mu < 0$. Due to the symmetry $\mathcal{S}$, we can restrict to the orbits with the sign $+$ in the above formulas.

Recall that the reduced system (3.2) has a line of equilibria $(\eta_1, 0, 0)$ near the origin. After the scaling, we then find a line of equilibria $(q, 0, 0)$ to (3.5) for $|q| \leq q_*$, $q_* \gg 1$. When $\mu = 0$ the linearization about the equilibrium $(q, 0, 0)$ has the eigenvalues 0, $\pm\sqrt{3q^2 - 1}$, so that equilibria with $q > 1/\sqrt{3}$ have a one-dimensional unstable manifold. For small μ, this property still holds for equilibria with $q_\mu < q \leq q_*$, for some $q_\mu < 1$. In order to show the persistence of the heteroclinic orbit, we consider the surface $\mathcal{U}_\mu$ which is the union of the unstable manifolds of the equilibria $(q, 0, 0)$, $q_\mu < q \leq q_*$. In view of the reversibility $\mathcal{R}$ of the system, it suffices to show that the surface $\mathcal{U}_\mu$ and the line $L_2 = \{(0, \eta_2, 0),\ \eta_2 \in \mathbb{R}\}$, which is invariant under $\mathcal{R}$, intersect for small μ. Indeed, by taking the intersection point as initial data in (3.5) we obtain an orbit which converges backwards in time x to an equilibrium $(q, 0, 0)$, by construction, and which converges forward in time x to $\mathcal{R}(q, 0, 0) = (-q, 0, 0)$, due to reversibility.

The surface $\mathcal{U}_\mu$ and the line L_2 intersect for small μ when they intersect transversely at $\mu = 0$. When $\mu = 0$, the intersection point is $P_0 = (0, 1/\sqrt{2}, 0)$, as it belongs to the heteroclinic orbit (3.6). We compute the normal n_0 to $\mathcal{U}_0$ at this point and show that the scalar product $\langle n_0, (0, 1, 0)\rangle$ does not vanish, which implies that $\mathcal{U}_0$ and L_2 intersect transversely. Using the first integrals

$$A = \eta_3 + \eta_1 - \eta_1^3, \quad B = \eta_2^2 + \eta_1^2 - \frac{1}{2}\eta_1^4 - 2A\eta_1,$$

we parameterize $\mathcal{U}_0$ by q, which gives the line of equilibria $(q, 0, 0)$, and $p = \eta_1$. We obtain the parametric equations

$$\eta_1 = p, \quad \eta_2 = \sqrt{-p^2 + \frac{1}{2}p^4 + 2p(q - q^3) - q^2 + \frac{3}{2}q^4}, \quad \eta_3 = -p + p^3 + q - q^3. \tag{3.7}$$

The intersection point P_0 is found for $(p, q) = (0, 1)$, and a straightforward calculation gives $n_0 = (-2\sqrt{2}, 2, 2\sqrt{2})$, so that $\langle n_0, (0, 1, 0)\rangle \neq 0$. This shows that $\mathcal{U}_0$ and L_2 intersect transversely, and proves the existence of the heteroclinic orbit in part (i).

We use a similar transversality argument to prove the persistence of the family of homoclinic orbits in which we replace the surface $\mathcal{U}_\mu$ by the unstable manifold to a given equilibrium and the line L_2 by the plane $P_2 = \{(\eta_1, 0, \eta_3),\ \eta_1, \eta_3 \in \mathbb{R}\}$, which is invariant under the second reversibility $\mathcal{SR}$. At $\mu = 0$, the unstable manifold to an equilibrium $(q, 0, 0)$ is given by the parametric equations (3.7) and it intersects transversely the plane P_2 when

$$-p + p^3 + q - q^3 \neq 0,$$

at the intersection point where

$$-p^2 + \frac{1}{2}p^4 + 2p(q - q^3) - q^2 + \frac{3}{2}q^4 = 0.$$

A straightforward calculation shows that these conditions hold for $1/\sqrt{3} < q < 2\sqrt{3}$ provided $q \neq 1$. This proves the second part of the proposition. ∎

4 Discussion

We showed existence of solutions to the two-dimensional Swift-Hohenberg equation which are spatially periodic in one direction and asymptotic to roll solutions in the other direction. These solutions are parameterized either by the wavenumber in the horizontal direction, k_y, or by the effective wavenumber of the rolls in the far field, k_{eff}, both close to the wavenumber of the zigzag instability, so that the wavenumber of the rolls in the far field lies in the zigzag-stable regime. In the following, we comment on some generalizations and possible extensions of our result.

Large amplitude and amplitude equations. The proofs in the present paper immediately generalize to more general dissipative systems, for instance to reaction-diffusion systems or to problems in convection. Technically, we only require the existence of spatial center-manifolds. The results also generalize to not necessarily small-amplitude roll solutions, which possess a zigzag-type stability boundary. For instance, one can consider a reaction-diffusion system $u_t = D\Delta u + f(u)$ on $\mathbb{R}^2$, which possesses y-periodic stripe solutions, such as the Gray-Scott or Gierer-Meinhardt models; see [12], or [5, 13] for a recent account. A formal expansion of solutions close to a zigzag instability, with periodic boundary conditions in y and long-wavelength modulations in x typically yields a fourth-order equation of Cahn-Hillard type,

$$\xi_t = (-\xi_{xxx} + \sigma_1 \xi_x + \sigma_2 \xi_x^3)_x,$$

where $\xi(t, x)$ is the location of the rolls as in (1.3). The sign of σ_1 is determined by the period in y: $\sigma_1 < 0$ indicates that the vertical stripe pattern is zigzag unstable. The sign of σ_2 can be understood in terms of a linear stability analysis of rotated rolls. Consider therefore a solution with $\xi_x = \alpha$, which corresponds to a roll solution rotated by an angle $\theta = \arctan \alpha$. Since the y-period now is fixed, this solution possesses a wavenumber $k(\theta) = k_*/\cos\theta \sim k_*(1 + (\alpha^2/2))$. The linearization of the Cahn-Hillard approximation at this rotated roll therefore becomes

$$\xi_t = -\xi_{xxxx} + (\sigma_1 + 3\sigma_2\alpha^2)\xi_{xx} = -\xi_{xxxx} + (\sigma_1 + 6\frac{\sigma_2}{k_*}(k - k_*))\xi_{xx}.$$

In particular, the sign of σ_2 distinguishes two types of zigzag instabilities: for $\sigma_2 > 0$, the instability is supercritical and wavenumbers larger than κ_{zz} are stable (as in the Swift-Hohenberg equation), and for $\sigma_2 < 0$, the instability is subcritical and wavenumbers smaller than κ_{zz} are stable. We use the terms sub- and supercritical in the sense of local pitchfork bifurcations, where supercritical refers to existence of nontrival branches in the regime where the trivial pattern is unstable, and linear instability is typically saturated by nonlinearity; subcritical branches exist where the primary branch is stable and nonlinearity amplifies the linear instability. We are not aware of an example for this latter, subcritical case. In the supercritical case, the knee solution corresponds to the kink in the Cahn-Hillard appoximation. In the subcritical case, we would find localized pulse solutions for ξ_x, which correspond to dislocated vertical rolls.

More parameters. Adding an additional parameter, one can study the emergence of this sideband instability in a family of wavetrains. As a specific example, one can think of a family of rolls with $k \sim 0$, which limit on a localized stripe at $k = 0$. The stripe itself, "homoclinic" in the y-direction, may undergo a transverse sideband instability upon increasing a system parameter μ above zero.

An example for such an onset has been observed experimentally and shown to exist analytically; see [13]. Analytically, one finds the periodic rolls near the localized stripe as reversible periodic solutions near a homoclinic orbit in the y-dynamics [19]. The linearization at the pulse possesses an algebraically simple zero eigenvalue which continues to an algebraically simple eigenvalue for the periodic roll patterns. For non-zero x-Fourier modes, $\mathrm{e}^{\mathrm{i}\ell x}$, this eigenvalue moves along a curve $\lambda(\ell^2) \sim \alpha(\mu, k)\ell^2$, where we can assume that $\alpha(\mu, 0) = \mu$. An analysis as in [18, Theorem 5.5] shows that typically $\alpha(\mu, k) = \alpha(\mu, 0) + (M\mathrm{e}^{-4\pi\nu/k} + \mathrm{o}_k(1))$, where ν is the spatial decay rate of the pulse. For $M < 0$, $\mu > 0$, the band of zigzag-unstable wavenumbers is approximately given by

$$|k| < k_{\mathrm{zz}}(\mu) = \frac{-4\pi\nu}{\log|\mu/M|}.$$

One can now investigate the existence of knees close to $k \sim 0$ following our approach. While our analysis applies to the case $k = 0$, there do not exist knees in localized stripes: the coefficient of the cubic nonlinearity vanishes for $k = 0$. To see this, note that the rotated localized stripe, unlike the periodic stripes, possesses the same zero wavenumber and therefore the linearization in the constant $\xi_x \equiv \eta_*$ does not change the diffusion coefficient. For $k \sim 0$, one expects an accompanying sideband instability with a small cubic coefficient, so that one would find knees at a finite wavenumber.

Non-existence. We focussed here on parameter values close to the zigzag instability curve $\kappa_{\mathrm{zz}}(\varepsilon)$, but the same approach can be used in the other parameter regions, as well; see Figure 1.2. In particular, in region 1 where rolls are stable in both one and two-dimensions the linear dispersion relation is quadratic $\lambda_*(\ell) = -c_*\ell^2 + \mathrm{O}(\ell^4)$, $c_* > 0$, so that the zero eigenvalue of the linearization $\mathcal{A}_*$ is now double. Then, instead of a four-dimensional center manifold, we find a two-dimensional manifold in the directions ξ and η_1. This manifold is filled with equilibria corresponding to translated and rotated rolls, which shows that there are no solutions of the form (1.3) in this case.

Stability. We expect that the knee solutions from Theorem 1 are asymptotically stable, steps would be unstable. A first indication of this is the Cahn-Hillard approximation, where knees are minimizers of the bistable energy. A perturbation analysis taking into account higher-order terms and modulations in the direction of y will be the subject of future work.

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