Spectral stability of modulated travelling waves bifurcating near essential instabilities

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Abstract

Localized travelling waves to reaction-diffusion systems on the real line are investigated. The issue addressed in this work is the transition to instability which arises when the essential spectrum crosses the imaginary axis. In the first part of this work, it has been shown that large modulated pulses bifurcate near the onset of instability; they are a superposition of the primary pulse with spatially-periodic Turing patterns of small amplitude. The bifurcating modulated pulses can be parametrized by the wavelength of the Turing patterns. Furthermore, they are time-periodic in a moving frame. In this second part, spectral stability of the bifurcating modulated pulses is addressed. It is shown that the modulated pulses are spectrally stable if, and only if, the small Turing patterns are spectrally stable, that is, if their continuous spectrum only touches the imaginary axis at zero. This requires an investigation of the period map associated with the time-periodic modulated pulses.

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1. Introduction

Pattern formation in reaction-diffusion equations on unbounded domains has attracted much interest. Patterns are often generated at bifurcation points where a primary pattern destabilizes. The issue is then to determine which patterns arise through the particular destabilization mechanism at hand and what their stability might be. If the instability is caused by point spectrum, it can be investigated utilizing reductions to finite-dimensional equations. If, on the other hand, parts of the essential spectrum cross the imaginary axis, such reductions are in general no longer available.

Arguably, the simplest scenario in which the essential spectrum generates new patterns is the Turing bifurcation. Imagine a reaction-diffusion system on the real line such that $u = 0$, say, is a homogeneous stationary solution. If the homogeneous steady state destabilizes, its linearization accommodates waves of the form $\mathrm{e}^{\mathrm{i}(k_0 x - \omega_0 t)}$ for certain values of k_0 and ω_0. Typically, near this transition to instability, small spatially-periodic travelling waves arise for any wavenumber close to k_0. Their wave speed is approximately equal to ω_0/k_0. In this article, we focus exclusively on the situation where $\omega_0 = 0$ and $k_0 \neq 0$. The bifurcation with $\omega_0 = 0$ and $k_0 \neq 0$ is known as the Turing bifurcation, and the bifurcating spatially-periodic steady patterns are often referred to as Turing patterns. Note, however, that Turing bifurcations can be analyzed by investigating ordinary differential equations since the bifurcating Turing patterns are stationary in time. As far as the existence of stationary bifurcating patterns is concerned, there exists therefore a reduction to finite dimensions.

Another class of patterns that arise on the real line are localized travelling waves, which we call pulses. Instabilities caused by their point spectrum lead to new localized solutions that are periodic in time in an appropriate moving frame. They resemble the original pulse but have a non-uniform wave speed; in addition, their shape changes periodically in time. As mentioned before, this transition can be analyzed by means of finite-dimensional center-manifold reductions. A more complicated situation arises if the localized travelling-wave solution destabilizes due to a Turing bifurcation of the asymptotic homogeneous state. We call this transition to instability an essential instability since, for the linearized equation about the travelling wave, the essential spectrum crosses the imaginary axis. In the first part of this work [13], we have proved that an essential instability leads to the bifurcation of modulated travelling waves. These solutions resemble a superposition of the small stationary Turing patterns and the localized pulse; they are time-periodic in an appropriate moving frame. We refer to Theorem 1. below for more details; see also Figure 1. It should be emphasized that this transition is genuinely infinite-dimensional.

The issue addressed in this work is the spectral stability of the bifurcating modulated pulses. We show that a modulated time-periodic pulse is linearly stable provided the asymptotic small-amplitude periodic pattern is linearly stable, i.e. if its continuous spectrum only touches the imaginary axis at zero. In fact, if the Turing patterns bifurcate supercritically,

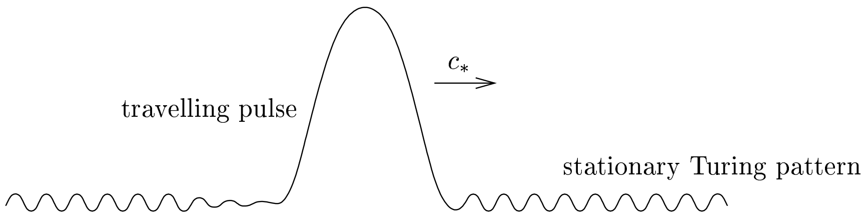

Figure 1: A schematic picture of a modulated pulse. The Turing patterns are stationary while the pulse moves through them with constant velocity, leaving behind a recovery zone. Note that the Turing patterns to the right and left may differ by a phase.

there is an open interval of wavenumbers for which they are stable. At the boundary of the interval, the Turing patterns destabilize in the so-called Eckhaus instability [2]. Accordingly, linearly stable modulated pulses exist for a continuum of asymptotic wavenumbers even though there existed only one stable pulse before bifurcation.

For the stability analysis, we have to understand the linearization of the time-period map about a modulated pulse in an appropriate moving frame; recall that modulated pulses are time-periodic in a moving frame and not stationary. We have to locate the essential spectrum of the relevant linear operator and exclude the existence of unstable isolated eigenvalues. Such isolated eigenvalues could pop out of the essential spectrum near the bifurcation point since the essential spectrum touches the unit circle. In the context of travelling waves that satisfy an ordinary differential equation, the Evans function provides an efficient technique to deal with such eigenvalues; see [3, 7] for recent advances. The advantage of such an approach is that information from the particular bifurcation scenario can be used efficiently in the stability analysis; also, isolated eigenvalues can be found as solutions to regular perturbation problems. The analogous approach for modulated pulses leads to an elliptic equation in the spatial variable on an appropriate space of time-periodic functions. In contrast to the situation for ordinary differential equations, however, the elliptic equation is ill-posed as a dynamical system in the spatial variable; it cannot be solved by standard semigroup theory. We utilize recent results [12] on the existence of exponential dichotomies for elliptic equations on unbounded cylinders to study the elliptic eigenvalue problem. In particular, exponential dichotomies allow us to find two infinite-dimensional subspaces which contain all solutions to the elliptic equation that decay in either forward or backward direction of the spatial variable. Eigenfunctions are then contained in the intersection of these subspaces. Besides ill-posedness of the elliptic equation, there are other difficulties which we have to resolve; the eigenvalue problem, for instance, is not always a regular perturbation of the $\mu = 0$ limit.

Before we can state our main result, we shall collect the hypotheses and results from [13]. We consider the semilinear parabolic equation

$$u_t = Du_{xx} + f(u, \mu), \qquad x \in \mathbb{R}, \tag{1.1}$$

where $u \in \mathbb{R}^n$, D is a diagonal matrix with positive entries, and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a smooth nonlinearity such that $f(0, \mu) = 0$ for all μ.

Equation (1.1) is well-posed on the space $X := C^0_{\mathrm{unif}}(\mathbb{R}, \mathbb{R}^n)$ of bounded and uniformly continuous functions on $\mathbb{R}$. We consider strong solutions $u(t)$ of (1.1) which are differentiable as functions into X, continuous with values in C^2_{unif} and satisfy (1.1) in X.

We assume the existence of a pulse to (1.1).

Hypothesis (TW) *Assume that $h(x - c_0 t)$ is a travelling-wave solution of (1.1) for $\mu = 0$ and some $c_0 \neq 0$ such that $h(\xi)$ tends to zero exponentially as $\xi \to \pm\infty$.*

The next assumption is on the linearization about the equilibrium $u = 0$. We assume that the equilibrium is neutrally stable with a critical eigenvalue at zero and an associated non-trivial wavenumber $k_0 \neq 0$. To be precise, consider the linearized equation

$$w_t = L^0_\infty w,$$

where

$$L^0_\infty w := D w_{xx} + \partial_u f(0, 0) w. \tag{1.2}$$

The spectrum $\mathrm{spec}(L^0_\infty)$ of the constant-coefficient operator L^0_∞ can be computed using the Fourier transform. Indeed, $\lambda \in \mathrm{spec}(L^0_\infty)$ if, and only if,

$$d^0(\lambda, \nu) := \det(\nu^2 D + \partial_u f(0, 0) - \lambda) = 0 \tag{1.3}$$

for some purely imaginary $\nu = \mathrm{i}k$ with $k \in \mathbb{R}$. The dispersion relation $\lambda^0_*(k)$ is obtained by solving (1.3).

Hypothesis (P1) *Assume that* $\mathrm{spec}(L^0_\infty) \cap \mathrm{i}\mathbb{R} = \{0\}$, *and that there are constants $k_0 \neq 0$ and $C_r > 0$ such that the following is true: $d^0(\lambda, \mathrm{i}k) = 0$ for λ close to zero if, and only if, either*

$$\lambda = \lambda^0_*(k) = -C_r (k - k_0)^2 + \mathrm{O}(|k - k_0|^3), \tag{1.4}$$

for k close to k_0, or else $\lambda = \lambda^0_*(-k)$ for k close to $-k_0$. Finally, we assume that $\partial_\lambda d^0(\lambda, \nu)|_{(0, \mathrm{i}k_0)} \neq 0$.

Quadratic tangency (1.4) of the dispersion relation is a generic assumption. Generically, under the above assumption, small stationary spatially-periodic patterns bifurcate for any wavenumber k close to k_0 when the critical situation is unfolded by the parameter μ. This is precisely the aforementioned Turing bifurcation.

Since we are interested in stable patterns arising through this bifurcation, we assume supercriticality. Consider L^0_∞ on the space of $2\pi/k_0$-periodic functions. Note that the zero of the function $d^0(0, \mathrm{i}k_0)$ corresponds to an isolated double eigenvalue at zero of L^0_∞. The two eigenvectors are related by the underlying O(2)-symmetry, generated by translations and reflection in the spatial variable. We can therefore continue this double eigenvalue to a curve $\lambda_{\rm bif}(\mu)$ of isolated double eigenvalues of $D\partial_{xx} + \partial_u f(0,\mu)$ for any μ close to zero. If the double eigenvalue crosses the imaginary axis transversely upon varying μ, it can be shown that spatially periodic solutions bifurcate which are invariant under reflection. Indeed, we can restrict the steady-state equation

$$Du_{xx} + f(u,\mu) = 0$$

associated with (1.1) to the space of even $2\pi/k_0$-periodic functions. Lyapunov-Schmidt reduction then leads to a one-dimensional bifurcation problem with remaining $\mathbb{Z}_2$-symmetry induced by the translation of half the period. We expect a pitchfork bifurcation $\mu z + az^3 + \mathrm{O}(z^5) = 0$, where the sign of the cubic coefficient a determines the bifurcation direction.

Hypothesis (P2) *We assume that the double eigenvalue $\lambda_{\rm bif}(\mu)$ crosses the imaginary axis transversely with $\partial_\mu \lambda_{\rm bif}(0) > 0$. Moreover, assume that the bifurcating steady-state solutions exist for $\mu > 0$, that is, we assume $a < 0$.*

Transforming (1.1) into the moving frame $(\xi, t) = (x - ct, t)$, we obtain

$$u_t = Du_{\xi\xi} + cu_\xi + f(u,\mu), \qquad \xi \in \mathbb{R}, \tag{1.5}$$

which then admits the equilibrium $h(\xi)$ for $(c,\mu) = (c_0, 0)$. In this moving coordinate frame, the stationary spatially-periodic patterns described above become spatially and temporally periodic wave trains. In other words, the Turing bifurcation of the origin translates into a Hopf bifurcation. Algebraically, this effect is seen in a modified dispersion relation.

Setting $(c,\mu) = (c_0, 0)$, we linearize (1.5) about $u = 0$ and obtain the linear constant-coefficient operator

$$L_\infty w := Dw_{\xi\xi} + c_0 w_\xi + \partial_u f(0,0)w.$$

Define

$$d(\lambda,\nu) := \det(\nu^2 D + \nu c_0 + \partial_u f(0,0) - \lambda) = d^0(\lambda - \nu c_0, \nu). \tag{1.6}$$

Hypothesis (P1) is then equivalent to the following: assume that $\mathrm{spec}(L_\infty) \cap \mathrm{i}\mathbb{R} = \{\pm \mathrm{i}\omega_0\}$ where $\omega_0 = c_0 k_0 > 0$; moreover, assume that $d(\lambda, \mathrm{i}k) = 0$ for λ close to $\mathrm{i}\omega_0$ if, and only if, either

$$\lambda = \lambda_*(k) = \mathrm{i}\omega_0 + \mathrm{i}c_0(k - k_0) - C_r(k-k_0)^2 + \mathrm{O}(|k-k_0|^3) \tag{1.7}$$

for k close to k_0.

We remark that $\partial_\nu d(\lambda,\nu)|_{(\mathrm{i}\omega_0, \mathrm{i}k_0)} \neq 0$. To see this observe that $\partial_\nu d = \partial_\nu d^0 - c_0 \partial_\lambda d^0$ and $d^0(\lambda^0_*(k), \mathrm{i}k) = 0$. Differentiation yields

$$\partial_\lambda d^0 \frac{\partial \lambda^0_*}{\partial k} + \mathrm{i}\partial_\nu d^0 = 0.$$

Since $\frac{\partial\lambda_*^0}{\partial k} = 0$ at $k = k_0$, we have $\partial_\nu d^0 = 0$, and therefore

$$\partial_\nu d = -c_0 \partial_\lambda d^0 \neq 0. \tag{1.8}$$

Next, we linearize (1.5) about the travelling wave $h(\xi)$

$$Lw = Dw_{\xi\xi} + c_0 w_\xi + \partial_u f(h(\xi), 0)w, \tag{1.9}$$

for $w \in X$. The following hypothesis, formulated in the moving coordinate frame, is a generic assumption on a marginally stable pulse that undergoes an essential instability induced by a Turing bifurcation at the equilibrium.

Hypothesis (S1)

(i) $\lambda = 0 \in \operatorname{spec}(L)$ *is a simple eigenvalue.*

(ii) $(L - \mathrm{i}\omega_0)w = 0$ *has a unique (up to constant complex multiples) non-zero bounded solution* $w^{\mathrm{c}}(\xi)$, *and we have* $|w^{\mathrm{c}}(\xi) - \mathrm{e}^{\mathrm{i}k_0(\xi+\varphi_\pm)}w_H^\pm| \to 0$ *as* $\xi \to \pm\infty$ *for appropriate constants* $\varphi_\pm$ *and non-zero vectors* $w_H^\pm \in \mathbb{C}^n$.

(iii) $\lambda \in \operatorname{spec}(L)$ *with* $\operatorname{Re}\lambda \geq 0$ *if, and only if, either* $\lambda = \pm\mathrm{i}\omega_0$ *or* $\lambda = 0$.

In [13], we proved the following theorem.

Theorem 1. ([13]) *Assume that Hypotheses (P1), (P2), (S1) and (TW) are satisfied. There is then a smooth function* $\mu_{\mathrm{bif}}(\omega) \geq 0$ *with* $\mu_{\mathrm{bif}}(\omega_0) = \mu'_{\mathrm{bif}}(\omega_0) = 0$ *and* $\mu''_{\mathrm{bif}}(\omega_0) > 0$ *such that, for any* ω *close to* ω_0 *and any small* $\mu > \mu_{\mathrm{bif}}(\omega)$, *the following is true. For a unique wave speed* $c = c_*(\mu, \omega)$ *close to* c_0, *equation (1.5) has a unique solution* $h_{\mu,\omega}(\xi, t)$ *with the following properties:*

(i) $h_{\mu,\omega}(\xi, t)$ *is periodic in* t *with period* $2\pi/\omega$. *In other words, the bifurcating pulse is time-periodic in the frame moving with speed* c_*. *The family* $h_{\mu,\omega}(\cdot,\cdot)$ *is continuous in* (μ, ω) *with values in* $C^0(\mathbb{R}^2, \mathbb{R}^n)$ *provided with the local topology.*

(ii) *We have* $c_*(0, \omega_0) = c_0$ *and* $h_{0,\omega_0}(\xi, t) = h(\xi)$.

(iii) *There exist a constant* $\delta > 0$ *and functions* $\gamma_{\mu,k}(x)$, *which have amplitude of the order* $\sqrt{\mu - \mu_{\mathrm{bif}}(\omega)}$ *and period* $2\pi/k$ *in* x, *such that, for* $c_0 < 0$,

$$\begin{aligned} |h_{\mu,\omega}(\xi, t) - \gamma_{\mu,k_*}(\xi + c_* t + \varphi_-)| &\leq K\mathrm{e}^{-\kappa|\xi|} && \xi \to -\infty \\ |h_{\mu,\omega}(\xi, t) - \gamma_{\mu,k_*}(\xi + c_* t + \varphi_+)| &\leq K\mathrm{e}^{-\delta\mu|\xi|} && \xi \to \infty. \end{aligned}$$

Here, $\varphi_\pm = \varphi_\pm(\mu, \omega)$ *is independent of* ξ *and* t, *and the spatial wavenumber is given by* $k_*(\mu, \omega) = \omega/c_*(\mu, \omega)$. *For* $c_0 > 0$, *replace* ξ *by* $-\xi$.

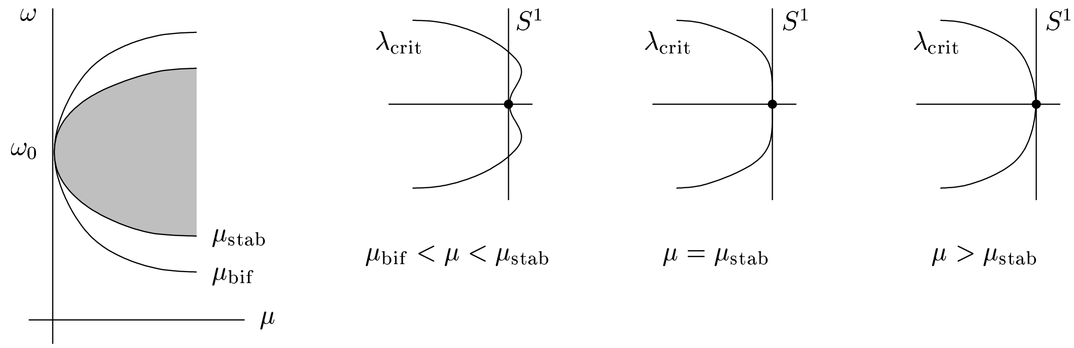

Figure 2: The left picture shows the (μ, ω)-plane. For fixed ω close to ω_0, Turing patterns and modulated pulses exist for $\mu > \mu_{\mathrm{bif}}(\omega)$; they are unstable as long as $\mu_{\mathrm{bif}}(\omega) < \mu < \mu_{\mathrm{stab}}(\omega)$ and stabilize at $\mu = \mu_{\mathrm{stab}}(\omega)$. Stable modulated pulses exist inside the shaded area. The three pictures to the right show the spectrum of the modulated pulses in the complex plane close to $\lambda = 1$ for different values of μ; the vertical line symbolizes the unit circle.

Note that the solutions $\gamma_{\mu,k}$ are stationary in time in the steady coordinate frame (x, t). They are precisely the small Turing patterns mentioned above. The modulated pulse $h_{\mu,\omega}(x - c_* t, t)$ can be thought of as a superposition of the steady Turing pattern $\gamma_{\mu,k}(x)$ and the primary pulse $h(x - c_0 t)$; see Figure 1. It is a relative periodic orbit with respect to translation in x. We refer to [13] for more details about the spatial structure of the modulated pulses.

We turn to the stability of the bifurcating modulated pulses. Spectral stability of a modulated pulse with time period $2\pi/\omega$ means the following: the spectrum of the linearization of the time-$2\pi/\omega$ map associated with (1.1), considered on the space $X = C^0_{\mathrm{unif}}$, about the modulated pulse is strictly contained inside the unit circle with the exception of $\lambda = 1$, which is always in the spectrum due to translation invariance. The main result is then contained in the following theorem; see also Figure 2.

Theorem 2. *Suppose that Hypotheses (TW), (P1), (P2) and (S1) are met. There exists a smooth curve $\mu_{\mathrm{stab}}(\omega)$ with $\mu_{\mathrm{stab}}(\omega_0) = \mu'_{\mathrm{stab}}(\omega_0) = 0$ and $\mu''_{\mathrm{stab}}(\omega_0) > \mu''_{\mathrm{bif}}(\omega_0)$ such that the following holds: the bifurcating modulated pulses $h_{\mu,\omega}(\xi, t)$ described in Theorem 1. are spectrally stable if, and only if, $\mu > \mu_{\mathrm{stab}}(\omega)$.*

The modulated pulses destabilize at $\mu = \mu_{\mathrm{stab}}(\omega)$ due to an Eckhaus instability of the small asymptotic Turing patterns. Thus, the modulated time-periodic pulses are linearly stable provided the asymptotic Turing pattern are linearly stable. More details on the spectrum of the Turing patterns and the modulated pulses can be found in the next section.

This paper is organized as follows. Existence and stability of Turing patterns in a steady frame is investigated in Section 3.. In Section 4., we formulate the spectral problem for the modulated pulse as a bifurcation problem for an appropriate elliptic equation. We then relate the spectra of Turing patterns in a moving and a steady frame in Section 5.. Finally, in Section 6., we show that the spectrum of the modulated pulses coincides with the spectrum of the Turing patterns with the possible exception of a finite number of stable isolated eigenvalues.

2. Spectral stability

Let $u(\xi, t)$ be any bounded and uniformly continuous function with period $2\pi/\omega$ in t. The linearized equation about $u(\xi, t)$ with $(c, \mu) = (c_*(\mu), \mu)$ is given by

$$v_t = Dv_{\xi\xi} + cv_\xi + \partial_u f(u(\xi, t), \mu)v.$$

The evolution operator associated with this equation on the space $X = C^0_{\text{unif}}(\mathbb{R}, \mathbb{R}^n)$ is then denoted by $\Psi_{u,\mu}(t, s)$ for $t \geq s \geq 0$. Since the function $u(\xi, t)$ is periodic in t with period $T = 2\pi/\omega$, we shall investigate the operator

$$\mathcal{T}_{u,\mu} = \Psi_{u,\mu}(T, 0),$$

that is, the time-T map induced by the linearized equation. We say that a T-periodic solution $u(\xi, t)$ of (1.5) is *spectrally stable* if the spectrum $\text{spec}(\mathcal{T}_{u,\mu})$ of $\mathcal{T}_{u,\mu}$ as an operator in X is strictly contained in the unit ball in $\mathbb{C}$ with the exception of $\lambda = 1$. In other words, $\lambda \in \text{spec}(\mathcal{T}_{u,\mu})$ implies $|\lambda| < 1$ or $\lambda = 1$.

2.1. Stability of Turing patterns

Consider the linearization

$$v_t = Dv_{xx} + \partial_u f(\gamma_{\mu,k}(x), \mu)v.$$

about the Turing patterns in the steady frame. The associated time-T map is denoted by $\mathcal{T}^0_{\gamma_{\mu,k},\mu}$. The same linearization in a moving frame is given by

$$v_t = Dv_{\xi\xi} + cv_\xi + \partial_u f(\gamma_{\mu,k}(\xi + ct), \mu)v.$$

where $c = \omega/k$. The associated period map is then denoted by $\mathcal{T}_{\gamma_{\mu,k},\mu}$. We have the following result on the relation between the spectra of $\mathcal{T}^0$ and $\mathcal{T}$.

Proposition 1. *The spatially periodic patterns $\gamma_{\mu,k}$ are spectrally stable in the steady frame if, and only if, they are spectrally stable in a moving frame. More precisely, there is a unique (up to shifts by 2π) real-valued function $\theta(\lambda)$ such that, for λ close to one,*

$$\lambda \in \text{spec}(\mathcal{T}_{\gamma_{\mu,k},\mu}) \Longleftrightarrow \lambda e^{i\theta(\lambda)} \in \text{spec}(\mathcal{T}^0_{\gamma_{\mu,k},\mu}).$$

It suffices therefore to investigate the stability of the spatially-periodic time-independent solutions which bifurcate from the spatially homogeneous equilibrium. Though the following theorem seems to be well known, at least as far as formal computations are concerned, we were unable to locate a mathematically rigorous derivation; see however [1, 10].

Theorem 3. *Assume that Hypotheses (P1) and (P2) are met. For a generic nonlinearity $f = f(u, \mu)$, there are curves $\mu_{\rm tur}(k)$ and $\mu_{\rm eck}(k)$ defined for k close to k_0 such that*

(i) $\mu_{\rm tur}(k_0) = \mu_{\rm eck}(k_0) = \mu'_{\rm tur}(k_0) = \mu'_{\rm eck}(k_0) = 0$,

(ii) $\mu''_{\rm eck}(k_0) = 3\mu''_{\rm tur}(k_0) > 0$,

(iii) equation (1.1) has non-trivial spatially-periodic time-independent patterns $\gamma_{\mu,k}(x)$ with spatial period $2\pi/k$ if, and only if, $\mu > \mu_{\rm tur}(k)$ for μ sufficiently small, and

(iv) the patterns $\gamma_{\mu,k}$ are spectrally stable if, and only if, $\mu > \mu_{\rm eck}(k)$.

Corollary 1. *Under the assumptions of Theorem 3., the Turing pattern with wavenumber k is nonlinearly stable in the sense of [15, Theorem 1.1] provided $\mu > \mu_{\rm eck}(k)$.*

We give the proof of Proposition 1. in Section 5.. The proof of Theorem 3. is outlined in Section 3..

2.2. Stability of modulated pulses

The following sharper version of Theorem 2. shows that spectral stability of the modulated pulses follows from stability of the asymptotic Turing patterns. It also gives more details about the spectrum of $\mathcal{T}_{h_{\mu,\omega},\mu}$ near $\lambda = 1$.

Theorem 4. *Suppose that Hypotheses (TW), (P1), (P2) and (S1) are met. The bifurcating modulated pulses $h_{\mu,\omega}(\xi,t)$ described in Theorem 1. are spectrally stable if, and only if, the associated asymptotic states $\gamma_{\mu,k_*(\mu,\omega)}$ are spectrally stable. The latter is true if*

$$\mu > \mu_{\rm stab}(\omega) = \frac{1}{2c_0^2}\mu''_{\rm eck}(k_0)(\omega-\omega_0)^2 + \mathrm{O}(\omega-\omega_0)^3.$$

Furthermore, $\lambda \in \mathrm{spec}(\mathcal{T}_{h_{\mu,\omega},\mu})$ with λ in a small, possibly μ-depending, neighborhood of one if, and only if, $\lambda = \lambda_{\rm crit}(\rho)$ for some real ρ close to zero, where $\lambda_{\rm crit}$ is defined in (5.4) below. In this case, there exists a function $v_\rho \in X$ such that $\mathcal{T}_{h_{\mu,\omega},\mu} v_\rho = \lambda v_\rho$, and v_ρ behaves like $\mathrm{e}^{\mathrm{i}(k_*+\rho)\xi}$ as $\xi \to \pm\infty$ where $k_* = k_*(\mu,\omega)$. If $\lambda \neq 1$, the function v_ρ is unique up to scalar multiples; if $\lambda = 1$, there exists in addition a unique localized eigenfunction.

If we parametrize the modulated pulses by the wavenumber k of the asymptotic patterns, the stability boundary is determined by the Eckhaus curve $\mu_{\rm eck}(k)$. Hence, the existence and stability curves $\mu_{\rm bif}(\omega)$ and $\mu_{\rm stab}(\omega)$, see Theorem 1. and 4., are implicitly defined by

$$\mu_{\rm bif}(\omega) = \mu_{\rm tur}(\omega/c_*(\mu_{\rm bif}(\omega),\omega)), \qquad \mu_{\rm stab}(\omega) = \mu_{\rm eck}(\omega/c_*(\mu_{\rm stab}(\omega),\omega)).$$

Since $\partial_\omega \mu_{\rm bif}(\omega_0) = 0$, we can also parametrize by $k = \omega/c_*(\mu,\omega)$. Hence, using the implicit relations, Taylor expansions of $\mu_{\rm bif}$ and $\mu_{\rm stab}$ can be derived.

3. The spectrum of Turing patterns in a steady frame

We study small spatially-periodic, time-independent solutions of (1.1)

$$u_t = Du_{xx} + f(u,\mu), \qquad x \in \mathbb{R},$$

under the spectral hypothesis (P1) for generic nonlinearities $f(u,\mu)$. We first recall the existence proof [6] which uses center-manifold reduction and normal-form theory. Afterwards, we investigate the linearized equation about the small patterns.

3.1. Existence of Turing patterns

We formulate the spectral assumption (P1) in terms of the dynamics of the linear ODE

$$\begin{aligned} u_x &= v \\ v_x &= -D^{-1}(\partial_u f(0,\mu)u - \lambda u). \end{aligned} \tag{3.1}$$

For $\mu = 0$, bounded solutions of this equation are eigenfunctions of the operator L^0_∞ corresponding to the eigenvalue λ. Hypothesis (P1) implies that $\mathrm{i}k_0$ is a double eigenvalue of the matrix on the right-hand side of (3.1) with $\lambda = 0$; otherwise, we could solve the dispersion relation (1.7) for k as a function of λ. Furthermore, there are no other purely imaginary eigenvalues. Equivalently, we have that k_0^2 is an eigenvalue of $D^{-1}\partial_u f(0,0)$ with geometric multiplicity one and algebraic multiplicity equal to two. Let u_0 be the eigenvector of $D^{-1}\partial_u f(0,0)$ corresponding to the eigenvalue k_0^2 and u_1 the associated generalized eigenvector.

We seek periodic solutions of the ordinary differential equation

$$\begin{aligned} u_x &= v \\ v_x &= -D^{-1}f(u,\mu). \end{aligned} \tag{3.2}$$

Small bounded solutions lie on the four-dimensional center manifold which is tangent to the critical eigenspace corresponding to the eigenvalues $\pm\mathrm{i}k_0$.

Vectors (u, v) in the tangent space of the center manifold at the origin can be written in the form

$$(u, v) = (A_0 u_0 + A_1 u_1, B_0 u_0 + B_1 u_1).$$

In these coordinates, the linearized equation at $\mu = 0$ is

$$A_{0,x} = B_0, \quad A_{1,x} = B_1, \quad B_{0,x} = -k_0^2 A_0 - A_1, \quad B_{1,x} = -k_0^2 A_1.$$

The linear change of coordinates

$$A = -2\mathrm{i}k_0^2 A_0 + \mathrm{i}(k_0 - 1)A_1 - 2k_0 B_0 + B_1, \qquad B = k_0 A_1 - \mathrm{i}B_1$$

transforms the linear part into complex Jordan normal form

$$A_x = \mathrm{i}k_0 A + B, \qquad B_x = \mathrm{i}k_0 B.$$

The reflection symmetry $x \to -x$ of (1.1) translates into reversibility of the equation on the center manifold: replacing x by $-x$ and applying $(A, B) \to (\bar{A}, -\bar{B})$ maps orbits into orbits.

Following [6], we introduce the invariants $R = |A|^2$ and $Q = \mathrm{i}(A\bar{B} - \bar{A}B)$. After a suitable smooth nonlinear change of coordinates, the equation on the center manifold can be written in the following simpler form

$$\begin{aligned} A_x &= \mathrm{i}k_0 A + B + \mathrm{i}A\mathrm{O}(|\mu| + R + |Q|) + \mathrm{O}(|A| + |B|)^m \\ B_x &= \mathrm{i}k_0 B + A(-q_1\mu + q_2 R + q_3 Q)(1 + \mathrm{O}(|\mu| + R + |Q|)) \\ &\quad + \mathrm{i}B\mathrm{O}(|\mu| + R + |Q|) + \mathrm{O}(|A| + |B|)^m, \end{aligned} \tag{3.3}$$

where $m \in \mathbb{N}$ is arbitrarily large but fixed. We seek periodic solutions with prescribed period $2\pi/k$ for k close to k_0. For the truncated equation, neglecting the higher-order terms, these solutions are explicitly given by

$$A(x) = r_0 \mathrm{e}^{\mathrm{i}kx}, \qquad B(x) = \mathrm{i}(k - k_0) r_0 \mathrm{e}^{\mathrm{i}kx}$$

with

$$r_0^2 = \frac{1}{q_2}(q_1\mu - (k - k_0)^2),$$

where we should choose μ such that the right-hand side is positive.

Using the reversibility of the equation and the fact that k is close to k_0, it is not difficult to see that, even for the full equation, there exists a branch of periodic solutions with prescribed period $2\pi/k$. In general, these periodic solution are no longer given as relative equilibria with respect to the normal-form symmetry defined by diagonal complex rotations acting upon (A, B). Indeed, the Floquet exponents of the periodic solutions are a double eigenvalue at zero, associated with the trivial time shift and reversibility, respectively, and simple eigenvalues at $\pm\sqrt{2q_1\mu} \neq 0$. Therefore, the periodic orbits are non-degenerate as reversible periodic solutions and hence persist. We refer to [6] for more details. An alternative existence proof would use Lyapunov-Schmidt reduction.

Summarizing, we have shown the existence of a family of periodic solutions on the center manifold with expansion

$$\begin{aligned} A(x;\mu,k) &= r_0 \mathrm{e}^{\mathrm{i}kx}(1+\mathrm{O}(|k-k_0|+|q_1\mu-(k-k_c)^2|)), \\ B(x;\mu,k) &= \mathrm{i}(k-k_0) r_0 \mathrm{e}^{\mathrm{i}kx}(1+\mathrm{O}(|k-k_0|+|q_1\mu-(k-k_c)^2|)), \end{aligned}$$

where

$$r_0^2 = \frac{1}{q_2}(q_1\mu-(k-k_0)^2).$$

3.2. The normal-form coefficients

Note that the sign of the normal-form coefficient q_2 determines the bifurcation direction of the Turing patterns. If we assume that the quadratic terms of the Taylor expansion of f at the origin vanish at $\mu=0$, q_2 can be easily calculated by evaluating and projecting f onto the center eigenspace. Indeed, the normal-form transformation acts like a projection onto the space of cubic polynomials, thereby leaving the image alias the monomials of the normal form invariant. Assume that P_0^{c} is the spectral projection of $D^{-1}\partial_u f(0,0)$ onto the center eigenspace $u = A_0 u_0 + A_1 u_1$. Furthermore, assume that f admits the expansion

$$\begin{aligned} &-P_0^{\mathrm{c}} D^{-1} f(A_0u_0 + A_1u_1, 0) = \\ &\quad -k_0^2 A_0 u_0 - A_1 u_0 - k_0^2 A_1 u_1 + \sum_{l=0}^{3} f_l^0 A_0^{3-l} A_1^l u_0 + \sum_{l=0}^{3} f_l^1 A_0^l A_1^{3-l} u_1 + \mathrm{O}(A_0^4 + A_1^4), \end{aligned}$$

then

$$B_x = \mathrm{i}k_0 B - \mathrm{i}\sum_l f_l^1 A_0^l A_1^{3-l},$$

where $A_0 = \mathrm{i}(A-\bar{A})/(4k_0^2) + (k_0-1)(B+\bar{B})/(4k_0^3)$ and $A_1 = (B+\bar{B})/(2k_0)$. Hence, $q_2 = \frac{3}{64}k_0^{-6} f_3^1$.

In order to obtain the coefficient q_1 of the linear unfolding in μ, we compare the determinants of the linear part of the original equation and the equation in normal form. Note that we have to add the complex conjugate equation, however, in order to obtain the correct result. Since the determinant is invariant under the linear coordinate changes, we obtain

$$q_1 = -\frac{1}{2k_0^2}\frac{\partial_\mu \det(D^{-1}\partial_u f(0,\mu) - k_0^2)}{\partial_\lambda \det(D^{-1}(\partial_u f(0,0) - \lambda) - k_0^2)}.$$

3.3. Linear stability of the Turing patterns

In the following, we assume that $q_1 > 0$ and $q_2 > 0$. This implies that, for $\mu > 0$, the origin is linearly (neutrally) stable and nonlinearly unstable for the x-dynamics; this corresponds to the usual picture of a supercritical bifurcation.

Consider the linearized eigenvalue equation

$$\begin{aligned} u_x &= v \\ v_x &= -D^{-1}(\partial_u f(\gamma_\mu, \mu) - \lambda)u \end{aligned} \tag{3.4}$$

about the Turing patterns γ_μ. In order to put (3.4) into normal form, we apply the aforementioned center-manifold reduction together with the subsequent transformation into normal-form to the equation

$$\begin{aligned} u_x &= v \\ v_x &= -D^{-1}(f(u, \mu) - \lambda u). \end{aligned}$$

Linearizing the resulting normal-form equation at $\lambda = 0$ about the Turing pattern with period k, we obtain

$$\begin{aligned} A_x &= \mathrm{i}k_0 A + B + \mathrm{O}(|\mu|(|A| + |B|)) \\ B_x &= \mathrm{i}k_0 B + A(-q_1\mu + 2q_2 r_0^2) + \bar{A} q_2 r_0^2 \mathrm{e}^{2\mathrm{i}kx} + q_1' \lambda A \\ &\quad + \mathrm{O}(|\mu|(|A|(|\mu| + |k - k_0|) + |B|)), \end{aligned} \tag{3.5}$$

where k is close to k_0. Here, we had to account for the additional parameter $\lambda \in \mathbb{C}$. As a result, an additional term of the form $q_1'\lambda$ appears in the second equation of (3.3); the coefficient q_1' is given by

$$q_1' = \frac{1}{2k_0^2}.$$

Note that (3.5) is the normal form of (3.4). We now explore (3.5) for various scalings of μ, $k - k_0$ and λ in order to capture all solutions to (3.4).

In a co-rotating frame, we rescale in ϵ according to

$$q_1\mu = \epsilon^2, \quad q_1'\lambda = \epsilon^2\tilde{\lambda}, \quad k - k_0 = \tilde{k}\epsilon, \quad x = \epsilon^{-1}\xi, \quad A = \epsilon \mathrm{e}^{\mathrm{i}kx}\tilde{A}, \quad B = \epsilon^2 \mathrm{e}^{\mathrm{i}kx}\tilde{B}. \tag{3.6}$$

Note that existence of periodic solutions is equivalent to $\tilde{k}^2 < 1$. We obtain the perturbed linear Ginzburg-Landau equation

$$\begin{aligned} \tilde{A}_\xi &= -\mathrm{i}\tilde{k}\tilde{A} + \tilde{B} + \mathrm{O}(\epsilon) \\ \tilde{B}_\xi &= -\mathrm{i}\tilde{k}\tilde{B} + (\tilde{A} + \overline{\tilde{A}})(1 - \tilde{k}^2) - \tilde{k}^2\tilde{A} + \tilde{\lambda}\tilde{A} + \mathrm{O}(\epsilon), \end{aligned} \tag{3.7}$$

plus the complex conjugated equation. The error terms are small rapid oscillations with period $2\pi\epsilon/k$; they are linear in $(\tilde{A}, \tilde{B})$.

A complex number λ is in the spectrum if, and only if, the time-periodic differential equation (3.7) has purely imaginary Floquet exponents. First, we neglect the error terms and calculate the Floquet exponents for the resulting truncated equation using the normal-form symmetry. Afterwards, we comment on the effect of the error terms.

Purely imaginary Floquet exponents correspond to purely imaginary eigenvalues of the matrix

$$M(\tilde{\lambda}, \tilde{k}) = \begin{pmatrix} -\mathrm{i}\tilde{k} & 0 & 1 & 0 \\ 0 & \mathrm{i}\tilde{k} & 0 & 1 \\ \tilde{\lambda} + 1 - 2\tilde{k}^2 & 1 - \tilde{k}^2 & -\mathrm{i}\tilde{k} & 0 \\ 1 - \tilde{k}^2 & \tilde{\lambda} + 1 - 2\tilde{k}^2 & 0 & \mathrm{i}\tilde{k} \end{pmatrix}.$$

Calculating the characteristic polynomial, we obtain

$$P(\rho, \tilde{\lambda}, \tilde{k}) = \det(M(\tilde{\lambda}, \tilde{k}) - \mathrm{i}\rho\,\mathrm{id}) = \rho^4 - 2(3\tilde{k}^2 - \tilde{\lambda} - 1)\rho^2 + \tilde{\lambda}^2 + 2\tilde{\lambda}(1 - \tilde{k}^2).$$

Solutions of $P(\rho, \tilde{\lambda}, \tilde{k}) = 0$ with $\rho \in \mathbb{R}$ do not exist for non-real λ. If $\tilde{k}^2 < 1$, which is necessary for existence of the Turing patterns, and $\tilde{\lambda} > 0$, the polynomial P is positive everywhere if, and only if, $\tilde{k}^2 < \frac{1}{3}$.

For $\tilde{k}^2 > \frac{1}{3}$, zeros appear for real ρ. This bifurcation is referred to as the Eckhaus instability [2]. Furthermore, for any $\tilde{k}^2 < \frac{1}{3}$, $\lambda = 0$ is possible only if $\rho = 0$ and we obtain the asymptotic dispersion relation

$$\tilde{\lambda}_{\text{crit}}(\rho) = -\frac{1 - 3\tilde{k}^2}{1 - \tilde{k}^2}\rho^2 + \mathrm{O}(\rho^4). \tag{3.8}$$

Similarly, we obtain a second curve of eigenvalues given by

$$\tilde{\lambda}_{\text{stab}}(\rho) = -2(1 - \tilde{k}^2) - \frac{5 + 2\tilde{k}^2}{2(1 - \tilde{k}^2)}\rho^2 + \mathrm{O}(\rho^4). \tag{3.9}$$

Standard averaging implies stability of the Floquet exponents under time-periodic small perturbations, i.e., for (3.7) with ϵ small but non-zero; see, for instance, [4]. Floquet exponents are therefore given as zeros of an ϵ-dependent equation $\hat{P}(\rho, \tilde{\lambda}, \tilde{k}, \epsilon) = 0$ such that $\hat{P} = P$ at $\epsilon = 0$.

We claim that eigenvalues lie on two curves $\tilde{\lambda}_{\text{crit}}(\rho)$ and $\tilde{\lambda}_{\text{stab}}(\rho)$ for ρ close to zero, even for $\epsilon > 0$. Moreover, $\partial_\rho \tilde{\lambda}_{\text{crit}} = 0$ at $\rho = 0$. This claim then proves stability for $\tilde{k}^2 < \frac{1}{3}$.

The representation as curves follows from the implicit function theorem since we have

$$\left.\frac{\partial P}{\partial \tilde{\lambda}}\right|_{(\rho, \tilde{\lambda}) = (\rho, \tilde{\lambda}(\rho))} \neq 0$$

for ρ small and $\tilde{k}^2 < 1$ along both curves $\tilde{\lambda}_{\text{crit}}$ and $\tilde{\lambda}_{\text{stab}}$. Furthermore,

$$\frac{\partial \tilde{\lambda}}{\partial \rho} = \left[\frac{\partial P}{\partial \tilde{\lambda}}\right]^{-1} \frac{\partial P}{\partial \rho}.$$

The second factor is evaluated at $\tilde{\lambda} = 0$ and corresponds to the original equation without λ; zeros correspond to purely imaginary Floquet exponents. But, due to reversibility, which is preserved under the perturbation, $\rho = 0$ is always double as an exponent and therefore $\partial_\rho P|_{(\tilde{\lambda},\rho)=0} = 0$.

Finally, we remark that it indeed suffices to consider $\tilde{\lambda} = \mathrm{O}(1)$ in ϵ. This can be seen by rescaling with respect to λ: let $k - k_0 = \tilde{k}\sqrt{|\lambda|}$ and replace ϵ by $\sqrt{|\lambda|}$ in the scaling (3.6). Substituting this scaling into (3.5), dividing by $|\lambda|$ and setting $|\lambda| = 0$, it can be easily seen that the resulting equation has only stable eigenvalues corresponding to $\arg\lambda = -1$ for any value of $\tilde{k}$. We omit the details.

Nonlinear stability as asserted in Corollary 1. is a consequence of [15, Theorem 1.1]; see also [16]. The assumptions in [15] are met due to the shape of the critical eigenvalue curve (3.8).

4. Elliptic characterization of the spectrum

In this section, we consider the eigenvalue problem for the operators $\mathcal{T}_{h_0,0}$ and $\mathcal{T}_{h_\mu,\mu}$ on the space $X = C^0_{\mathrm{unif}}(\mathbb{R},\mathbb{R}^n)$. We write h_μ for $h_{\mu,\omega}$ whenever the dependence on ω is not important. Similarly, $c_*(\mu)$ denotes the associated wave speed. Finally, $T = 2\pi/\omega$ is the temporal period of the modulated pulse $h_{\mu,\omega}$.

For $\mu = 0$, the pulse $h_0(\xi,t) = h_0(\xi)$ is independent of t. The linearization about $h_0(\xi)$ is given by

$$v_t = Dv_{\xi\xi} + c_0 v_\xi + \partial_u f(h_0(\xi),0)v.$$

Using the definition

$$L = D\partial_{\xi\xi} + c_0\partial_\xi + \partial_u f(h_0(\xi),0),$$

we have

$$\mathcal{T}_{h_0,0} = \mathrm{e}^{LT}.$$

It follows from the Spectral Theorem and Hypothesis (S1) that

$$\mathrm{spec}(\mathcal{T}_{h_0,0}) \cap \{\lambda \in \mathbb{C};\ |\lambda| \geq 1\} = \{\lambda = 1\}.$$

In other words, the spectrum of $\mathcal{T}_{h_0,0}$ touches the unit circle at $\lambda = 1$ with the rest of the spectrum being strictly contained inside the unit circle. Since the spectrum of $\mathcal{T}_{h_\mu,\mu}$ is upper semi-continuous with respect to μ, it suffices to consider a small neighborhood of $\lambda = 1$ to detect possible instabilities in the spectrum of $\mathcal{T}_{h_\mu,\mu}$.

A complex number λ is in the resolvent set of $\mathcal{T}_{h_\mu,\mu}$ if, and only if, the operator $(\mathcal{T}_{h_\mu,\mu} - \lambda)$ has a bounded inverse on X. The latter is true if, for any $g \in X$, the linearization

$$v_t = Dv_{\xi\xi} + c_*(\mu)v_\xi + \partial_u f(h_\mu(\xi,t),\mu)v$$

about h_μ has a unique solution $v(\xi,t)$ such that

$$v(\xi,T)-\lambda v(\xi,0)=g(x)$$

and $|v(\cdot,0)|_X \le C_\lambda\,|g|_X$. In order to study the spectrum of $\mathcal{T}_{h_\mu,\mu}$ near $\lambda=1$, we use the transformation

$$w(\xi,t)=\mathrm{e}^{-\alpha t}v(\xi,t).$$

In the variable $w(\xi,t)$, the linearization about h_μ is given by

$$w_t = Dw_{\xi\xi}+c_*(\mu)w_\xi+\partial_u f(h_\mu(\xi,t),\mu)w-\alpha w. \tag{4.1}$$

Therefore, a complex number $\lambda=\mathrm{e}^{\alpha T}$ is in the resolvent set of $\mathcal{T}_{h_\mu,\mu}$ if, and only if, for any $g\in X$, equation (4.1) has a unique solution $w(\xi,t)$ such that

$$w(\xi,T)-w(\xi,0)=\mathrm{e}^{-\alpha T}g(x)$$

and $|w(\cdot,0)|_X \le C_\alpha\,|g|_X$ for some constant C_α.

In particular, we see that $\lambda=\mathrm{e}^{\alpha T}$ is an eigenvalue of $\mathcal{T}_{h_\mu,\mu}$ if, and only if, the eigenvalue problem

$$\begin{aligned} w_t &= Dw_{\xi\xi}+c_*(\mu)w_\xi+\partial_u f(h_\mu(\xi,t),\mu)w-\alpha w \\ w(\xi,T) &= w(\xi,0) \end{aligned} \tag{4.2}$$

has a bounded solution $w(\xi,t)$. We cast this equation as an elliptic problem in the spatial variable ξ. Using the notation $W=(w,w_\xi)$, we obtain

$$W_\xi=\begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\partial_t-\partial_u f(h_\mu(\xi,t),\mu)+\alpha) & D^{-1}c_*(\mu)\end{pmatrix}W = A_{h_\mu,\mu,\alpha}(\xi)W. \tag{4.3}$$

Here, $W(\xi)\in Y$ with $Y=H^{\frac12}(S^1)\times L^2(S^1)$ where $S^1=[0,T]/\sim$; see [13, Section 3.1]. We say that $W(\xi)$ is a solution of (4.3) if $W(\xi)$ is differentiable in ξ as a function into Y, continuous with values in $H^1(S^1)\times H^{\frac12}(S^1)$ and satisfies (4.3) in Y. Note that the equation

$$w(\xi,T)=w(\xi,0)$$

has been taken into account by the choice of the Hilbert space Y.

For future reference, we define

$$A_{u,\mu,\alpha}(\xi):=\begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}(\partial_t-\partial_u f(u(\xi,t),\mu)+\alpha) & D^{-1}c_*(\mu)\end{pmatrix} \tag{4.4}$$

for any function $u(\xi,t)$ which has period $T=2\pi/\omega$ in t.

Note that the initial-value problem for (4.3) is not well-posed on Y. Under certain circumstances, however, (4.3) can be solved in forward or backward ξ-direction for initial values in certain ξ-depending subspaces of Y. We say that (4.3) has an exponential dichotomy on $\mathbb{R}^+$ if there are projections $P^{\rm s}(\xi)$ defined for $\xi \geq 0$ with the following property: for any $W_0 \in \mathrm{R}(P^{\rm s}(0))$, there exists a unique solution $W(\xi)$ of (4.3) which is defined for $\xi > 0$ such that $W(0) = W_0$. Moreover, $W(\xi)$ tends to zero exponentially as $\xi \to \infty$, and $W(\xi) \in \mathrm{R}(P^{\rm s}(\xi))$ for all $\xi > 0$. Similarly, for any $W_0 \in \mathrm{N}(P^{\rm s}(\xi_0))$, there is a unique solution $W(\xi)$ of (4.3) which is defined for $0 < \xi < \xi_0$ such that $W(\xi_0) = W_0$; furthermore, $W(\xi)$ decays exponentially for decreasing $0 \geq \xi \leq \xi_0$. In other words, for $\xi \geq 0$, there are two complementary subspaces, $\mathrm{R}(P^{\rm s}(\xi))$ and $\mathrm{N}(P^{\rm s}(\xi))$, such that we can solve the elliptic equation forward and backward in ξ for initial values in $\mathrm{R}(P^{\rm s}(\xi))$ and $\mathrm{N}(P^{\rm s}(\xi))$, respectively. Exponential dichotomies on $\mathbb{R}^-$ are defined analogously; solutions in $\mathrm{R}(P^{\rm u}(0))$ decay exponentially as $\xi \to -\infty$. For elliptic equations, the existence and roughness of dichotomies has recently been established [12]. The relation between (4.2) and (4.3) is as follows: the time-T map $\mathcal{T}_{h_\mu,\mu}$ has an isolated eigenvalue $\lambda = \mathrm{e}^{\alpha T}$ close to one whenever the elliptic equation (4.3) has a bounded solution. In fact, we have the following lemma.

Lemma 1. *The complex number $\lambda = \exp(\alpha T) = \exp(2\pi\alpha/\omega)$ is in the spectrum of the operator $T_{h_\mu,\mu}$ for λ close to one if, and only if, one of the following two conditions is met:*

(i) The equation $W_\xi = A_{\gamma_\mu,\mu,\alpha}(\xi)W$ about the asymptotic Turing pattern does not have an exponential dichotomy on $\mathbb{R}$.

(ii) The equation $W_\xi = A_{h_\mu,\mu,\alpha}(\xi)W$ about the modulated pulse has exponential dichotomies $P^{\rm s}(\xi)$ and $P^{\rm u}(\xi)$ on $\mathbb{R}^+$ and $\mathbb{R}^-$, respectively, but not on $\mathbb{R}$; i.e. we have $\mathrm{R}(P^{\rm s}(0)) \cap \mathrm{R}(P^{\rm u}(0)) \neq \{0\}$ or $\mathrm{R}(P^{\rm s}(0)) + \mathrm{R}(P^{\rm u}(0)) \neq Y$.

Below, we shall see that if (i) is not met so that $W_\xi = A_{\gamma_\mu,\mu,\alpha}(\xi)W$ has an exponential dichotomy on $\mathbb{R}$, then the equation $W_\xi = A_{h_\mu,\mu,\alpha}(\xi)W$ has exponential dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$.

Proof. First, suppose that (i) and (ii) are not satisfied for some λ close to one. We shall prove that λ is in the resolvent set. Since (i) is not met, the equation

$$W_\xi = A_{\gamma_\mu,\mu,\alpha}(\xi)W \tag{4.5}$$

about the Turing pattern has exponential dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$. Using [12, Theorem 1], these dichotomies can be extended to exponential dichotomies $P^{\rm s}(\xi)$ and $P^{\rm u}(\xi)$, defined for $\xi \geq 0$ and $\xi \leq 0$, respectively, of the equation

$$W_\xi = A_{h_\mu,\mu,\alpha}(\xi)W. \tag{4.6}$$

In particular, if (i) is not met, then (4.6) has exponential dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$. Since (ii) is not satisfied by assumption, (4.6) has an exponential dichotomy on the real line $\mathbb{R}$

in the Banach space Y. Therefore, for any $\hat{g}(\xi) \in C^0_{\mathrm{unif}}(\mathbb{R}, L^2([0,T]))$, we obtain a mild solution to the elliptic equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi} W = A_{h_\mu,\mu,\alpha}(\xi) W + \begin{pmatrix} 0 \\ \hat{g}(\xi,t) \end{pmatrix} \tag{4.7}$$

given by the standard variation-of-constant formula

$$W(\xi) = \int_{-\infty}^{\xi} \Phi^{\mathrm{u}}(\xi,\eta)\hat{G}(\eta)\,\mathrm{d}\eta + \int_{\infty}^{\xi} \Phi^{\mathrm{s}}(\xi,\eta)\hat{G}(\eta)\,\mathrm{d}\eta$$

with $\hat{G}(\xi) = (0, \hat{g}(\xi,t))$. Here, $\Phi^{\mathrm{s}}(\xi,\eta)$ and $\Phi^{\mathrm{u}}(\xi,\eta)$ denote the evolutions of (4.6) on the spaces $\mathrm{R}(P^{\mathrm{s}}(\eta))$ and $\mathrm{R}(P^{\mathrm{u}}(\eta))$, respectively; see [12]. It follows that $W \in C^0(\mathbb{R}, H^{\frac{\epsilon+1}{2}} \times H^{\frac{\epsilon}{2}})$ for any $0 < \epsilon < 1$; the proof uses the regularity properties in [5, Theorem 7.1.3] which also hold for $\Phi^{\mathrm{s}}(\xi,\eta)$ and $\Phi^{\mathrm{u}}(\xi,\eta)$ due to [12, Theorem 3]. Hence, the mild solution $W = (w, w_\xi)$ satisfies $w \in C^0(\mathbb{R}, C^0(S^1))$ and

$$|w(\cdot,0)|_{C^0} \le |w|_{C^0(\mathbb{R},C^0(S^1))} \le \text{Const.}\, |\hat{g}|_{C^0(\mathbb{R},L^2([0,T]))}. \tag{4.8}$$

After this preparations, we return to the parabolic equation. We have to show that $(\mathcal{T}_{h_\mu,\mu} - \lambda)$ is invertible, that is, we have to solve the equation

$$v_t = Dv_{\xi\xi} + cv_\xi + \partial_u f(h_\mu(\xi,t),\mu)v - \alpha v$$

with boundary conditions $v(\xi,T) - v(\xi,0) = \mathrm{e}^{-\alpha T} g(\xi)$. First, we solve

$$\tilde{g}_t = D\tilde{g}_{\xi\xi} + c\tilde{g}_\xi + \partial_u f(h_\mu(\xi,t),\mu)\tilde{g} - \alpha\tilde{g} - m\tilde{g} \tag{4.9}$$

$$\tilde{g}(\xi,T) - \tilde{g}(\xi,0) = \mathrm{e}^{-\alpha T} g(\xi) \tag{4.10}$$

for some large constant $m > 0$. Note that the time-T map of (4.9) is a contraction provided m is sufficiently large. We therefore obtain a solution $\tilde{g}$ to (4.9-4.10) which satisfies $\tilde{g}(\cdot,0) \in C^0_{\mathrm{unif}}(\mathbb{R})$ and $\tilde{g}(\cdot,\cdot) \in C^0_{\mathrm{unif}}(\mathbb{R}, L^2(S^1))$. In fact, due to regularity properties of parabolic equations, we have $\tilde{g}(\cdot,\cdot) \in C^k(\mathbb{R}, C^k(0,T))$ for any finite k. Next, we set $v = \tilde{g} + w$, and seek w as a solution to

$$w_t = Dw_{\xi\xi} + cw_\xi + \partial_u f(h_\mu(\xi,t),\mu)w - \alpha w + m\tilde{g}(\xi,t), \qquad w(\xi,T) = w(\xi,0). \tag{4.11}$$

Substituting $\hat{g} = m\tilde{g}$ into (4.7), we obtain a mild solution $\tilde{w} \in C^0(\mathbb{R}, C^0(S^1))$ of (4.7). We claim that $\tilde{w}$ is a strong solution of (4.11) which, using the estimate (4.8), would prove that $(\mathcal{T}_{h_\mu,\mu} - \lambda)$ is invertible. In order to prove the claim, it suffices to show that $\tilde{w}$ is a mild solution of (4.11) since it is then automatically a strong solution owing to $\tilde{g}(\cdot,\cdot) \in C^k(\mathbb{R}, C^k(0,T))$; see [5]. By definition, mild solutions to (4.11) satisfy the integral equation

$$w(t) = \Psi_{h_\mu,\mu,\alpha}(t,0)w(0) + m\int_0^t \Psi_{h_\mu,\mu,\alpha}(t,s)\tilde{g}(\xi,s)\,\mathrm{d}s, \tag{4.12}$$

where $\Psi_{h_\mu,\mu,\alpha}(t,s)$ denotes the evolution of (4.9) with $m = 0$. We approximate $\tilde{g}$ in $C^0_{\text{unif}}(\mathbb{R}, L^2(S^1))$ by a sequence of functions $\tilde{g}_n \in C^k(\mathbb{R}, H^k(S^1))$, substitute $\tilde{g}_n$ into (4.7) and denote the resulting mild solution of (4.7) by $\tilde{w}_n$. Since $\tilde{g}_n$ is smooth, it follows that $\tilde{w}_n \in C^{k-1}(\mathbb{R}, H^{k-1}(S^1))$; hence, it is a strong solution of (4.11) with $\tilde{g}$ replaced by $\tilde{g}_n$. In particular, $\tilde{w}_n$ satisfies (4.12), with $\tilde{g}$ replaced by $\tilde{g}_n$, for any n. Moreover, $\tilde{w}_n$ converges to $\tilde{w}$ in $C^0_{\text{unif}}(\mathbb{R}, C^0(S^1))$ as $n \to \infty$ since $\tilde{g}_n$ converges to $\tilde{g}$ in $C^0_{\text{unif}}(\mathbb{R}, L^2(S^1))$. Thus, $\tilde{w}$ satisfies (4.12), which proves the claim.

It remains to show that λ is in the spectrum if either (i) or (ii) is met.

First, we assume that (i) is satisfied. Thus, suppose that (4.5) does not have an exponential dichotomy. For $\mu = 0$, we have $\gamma_\mu = 0$, and (4.5) is given by

$$W_\xi = A_{0,0,\alpha} W = \begin{pmatrix} 0 & \text{id} \\ D^{-1}(\partial_t - \partial_u f(0,0) + \alpha) & D^{-1} c_0 \end{pmatrix} W.$$

The operator $A_{0,0,0}$ has two eigenvalues $\pm i k_0$ on the imaginary axis, while the rest of its spectrum is bounded away from the imaginary axis. Therefore, the operator $A_{0,0,\alpha}$ has two eigenvalues close to the imaginary axis for α close to zero with the rest of its spectrum uniformly bounded away from the imaginary axis. On account of [13], there exists a trichotomy, that is, projections $Q^{\text{s}}_{0,\alpha}$, $Q^{\text{c}}_{0,\alpha}$ and $Q^{\text{u}}_{0,\alpha}$ such that the equation

$$W_\xi = A_{0,0,\alpha} W$$

can be solved for initial values $W(0) \in \mathrm{R}(Q^{\text{s}}_{0,\alpha})$, $W(0) \in \mathrm{R}(Q^{\text{c}}_{0,\alpha})$ and $W(0) \in \mathrm{R}(Q^{\text{u}}_{0,\alpha})$ on the intervals $\mathbb{R}^+$, $\mathbb{R}$ and $\mathbb{R}^-$, respectively. Moreover, $\mathrm{R}(Q^{\text{s}}_{0,\alpha}) \oplus \mathrm{R}(Q^{\text{c}}_{0,\alpha}) \oplus \mathrm{R}(Q^{\text{u}}_{0,\alpha}) = Y$. Initial values in $\mathrm{R}(Q^{\text{s}}_{0,\alpha})$ or $\mathrm{R}(Q^{\text{u}}_{0,\alpha})$ lead to solutions which decay exponentially with a uniform exponential rate for $\xi \to \infty$ or $\xi \to -\infty$, respectively. On the other hand, solutions with initial values in $\mathrm{R}(Q^{\text{c}}_{0,\alpha})$ may not decay at all. Using [12, Theorem 1], we conclude that the equation

$$W_\xi = A_{\gamma_\mu,\mu,\alpha} W, \tag{4.13}$$

admits projections $Q^{\text{s}}_{\mu,\alpha}(\xi)$, $Q^{\text{c}}_{\mu,\alpha}(\xi)$ and $Q^{\text{u}}_{\mu,\alpha}(\xi)$ defined for μ sufficiently close to zero and $\xi \in \mathbb{R}^+$, $\xi \in \mathbb{R}$ and $\xi \in \mathbb{R}^-$, respectively, which have the same properties as the projections for $\mu = 0$ described above. Furthermore, the projections are $2\pi/k$-periodic in ξ. Therefore, we may consider the 2×2 matrix

$$\begin{array}{rccl} B_{\mu,\alpha} & : & \mathrm{R}(Q^{\text{c}}_{\mu,\alpha}(0)) & \longrightarrow & \mathrm{R}(Q^{\text{c}}_{\mu,\alpha}(\frac{2\pi}{k})) = \mathrm{R}(Q^{\text{c}}_{\mu,\alpha}(0)) \\ & & W(0) & \longmapsto & W(\frac{2\pi}{k}). \end{array} \tag{4.14}$$

Note that $B_{\mu,\alpha}$ is continuous in μ and analytic in α. Due to our assumption of nonexistence of dichotomies, we conclude that $B_{\mu,\alpha}$ has spectrum on the unit circle. Thus, there is a bounded solution $W = (w, w_\xi)$ of (4.13); since f is smooth, we have $W \in C^1(H^3 \times H^{\frac{5}{2}})$ due to the regularizing properties proved in [12]. We conclude that $w(\xi, 0) \in X$ is a strong solution of (4.2), with h_μ replaced by γ_μ, and λ is in the spectrum of the small Turing pattern. Finally, it follows from [14, Lemma 6.3] and its proof that λ is then also in the spectrum of the modulated pulse. Note that the latter is a relative periodic orbit and the arguments given in [14] readily apply.

Finally, assume that (ii) is met; it follows that we have $\mathrm{R}(P^{\mathrm{s}}(0)) \cap \mathrm{R}(P^{\mathrm{u}}(0)) \neq \{0\}$ or $\mathrm{R}(P^{\mathrm{s}}(0)) + \mathrm{R}(P^{\mathrm{u}}(0)) \neq Y$. In the first case, any non-zero element in the intersection $\mathrm{R}(P^{\mathrm{s}}(0)) \cap \mathrm{R}(P^{\mathrm{u}}(0))$ generates an eigenfunction of $\mathcal{T}_{h_\mu,\mu}$; see the previous paragraph. It remains to consider the case where $\mathrm{R}(P^{\mathrm{s}}(0)) + \mathrm{R}(P^{\mathrm{u}}(0)) \neq Y$. In this case, we apply the arguments given above to the adjoint operator $\mathcal{T}^*_{h_\mu,\mu}$ of $\mathcal{T}_{h_\mu,\mu}$ defined on the dual space X^*. The associated elliptic problem is then still defined on Y since Y is a Hilbert space. The exponential dichotomies of the adjoint elliptic problem are the adjoints $P^{\mathrm{s}}(\xi)^*$ and $P^{\mathrm{u}}(\xi)^*$ of the projections $P^{\mathrm{s}}(\xi)$ and $P^{\mathrm{u}}(\xi)$; see [12, 13]. Hence, their ranges intersect, $\mathrm{R}(P^{\mathrm{s}}(0)^*) \cap \mathrm{R}(P^{\mathrm{u}}(0)^*) \neq \{0\}$, and we obtain an eigenfunction of the adjoint operator $\mathcal{T}^*_{h_\mu,\mu}$. Since $\mathcal{T}_{h_\mu,\mu}$ is bounded in X, its spectrum is equal to the spectrum of its adjoint. This proves the lemma. ■

Remark 2. *It follows from the proof of Lemma 1. that* $\lambda = \exp(\alpha T)$ *is in the spectrum of the Turing pattern* γ_μ *for* α *close to zero if, and only if, the equation* $W_\xi = A_{\gamma_\mu,\mu,\alpha}(\xi)W$ *does not have an exponential dichotomy.*

5. The spectrum of the Turing patterns in the moving frame

5.1. Spectra in moving versus steady-state coordinates

In this section, we prove Proposition 1. – and a bit more. Key to the proof is Floquet theory. As we shall see, the function $\theta(\lambda)$ is characterized by Floquet exponents.

We consider small spatially-periodic steady-state solutions $\gamma_{\mu,k}(x)$ of (1.1)

$$u_t = Du_{xx} + f(u,\mu).$$

The functions $\gamma_{\mu,k}(\xi + ct)$ then satisfy

$$u_t = Du_{\xi\xi} + cu_\xi + f(u,\mu);$$

they have temporal period $T = 2\pi/ck$. In this section, we fix the wavenumber k and omit the corresponding index k.

Linearizing the nonlinear PDE (1.1) about a Turing pattern, we obtain the linear equation

$$v_t = Dv_{xx} + \partial_u f(\gamma_\mu(x),\mu)v \tag{5.1}$$

in a steady coordinate frame and

$$w_t = Dw_{\xi\xi} + cw_\xi + \partial_u f(\gamma_\mu(\xi + ct),\mu)w \tag{5.2}$$

in the moving frame. We compare the spectra of the time-T maps $\mathcal{T}^0_{\gamma_\mu,\mu}$ and $\mathcal{T}_{\gamma_\mu,\mu}$ of these two equations. Since the equation in the steady frame is autonomous and the generator L^0 of the semigroup is sectorial, the Spectral Mapping theorem holds, and the spectrum of the time-T map $\mathcal{T}^0_{\gamma_\mu,\mu} = \mathrm{e}^{L^0 T}$ is determined by L^0:

$$\mathrm{spec}(\mathrm{e}^{L^0 T}) = \mathrm{e}^{\mathrm{spec}(L^0)T}.$$

First suppose that $\alpha \in \mathrm{spec}(L^0)$; we shall prove that $\mathrm{e}^{\alpha T + \mathrm{i}\theta}$ is in the spectrum of the time-T map $\mathcal{T}_{\gamma_\mu,\mu}$ in the moving frame for a suitable number $\theta \in \mathbb{R}$. It follows from Floquet theory that, for any such eigenvalue α of L^0, there exist a $\rho \in \mathbb{R}$ and a bounded solution v of

$$\alpha v = D v_{xx} + \partial_u f(\gamma_\mu(x), \mu) v$$

such that $v(x + 2\pi/k) = \mathrm{e}^{\mathrm{i}\rho} v(x)$. Let $w(\xi, t) = v(\xi + ct)\exp(\alpha t)$. A straightforward computation shows that w satisfies the linearized equation (5.1) in the moving coordinate frame and

$$w(\xi, T) = v(\xi + cT)\mathrm{e}^{\alpha T} = v(\xi)\mathrm{e}^{\alpha T + \mathrm{i}\rho} = w(\xi, 0)\mathrm{e}^{\alpha T + \mathrm{i}\rho}.$$

Therefore, w is an eigenfunction to the eigenvalue $\exp(\alpha T + \mathrm{i}\rho)$, and $\theta = \rho$.

Conversely, suppose that $\mathrm{e}^{\alpha T}$ belong to the spectrum of $\mathcal{T}_{\gamma_\mu,\mu}$. Due to Lemma 1. and Remark 2., we know that there exists an eigenfunction which satisfies

$$w_t = D w_{\xi\xi} + c w_\xi + \partial_u f(\gamma_\mu(\xi + ct), \mu) w - \alpha w$$

with boundary conditions $w(x, T) = w(x, 0)$. Exploiting Floquet theory on $\mathrm{R}(Q^{\mathrm{c}}_{\mu,\alpha}(0))$, c.f. (4.14), we see that $w(\xi + 2\pi/k, t) = w(\xi, t)\exp(\mathrm{i}\rho)$ for some $\rho \in \mathbb{R}$. Let $\tilde{w}(x, t) := w(x - ct, t)$, which then satisfies

$$\tilde{w}_t = D\tilde{w}_{xx} + \partial_u f(\gamma_\mu(x), \mu)\tilde{w} - \alpha\tilde{w}$$

and

$$\tilde{w}(x, T) = w(x - cT, T) = w(x - cT, 0) = \mathrm{e}^{-\mathrm{i}\rho} w(x, 0) = \mathrm{e}^{-\mathrm{i}\rho}\tilde{w}(x, 0),$$

where we used $T = 2\pi/ck$. Hence, $\exp(\alpha t)\,\tilde{w}(x, t)$ is the desired eigenfunction of $\exp(L^0 T)$.

This proves Proposition 1..

5.2. The dispersion relation in the moving frame

In Section 3., we derived the dispersion relations

$$\begin{aligned}
\alpha_{\mathrm{crit}}(\rho) &= -\frac{q_1\mu(q_1\mu - 3(k-k_0)^2)}{q_1'(q_1\mu - (k-k_0)^2)}\rho^2 + \mathrm{O}(\rho^2(|\mu|^{\frac{3}{2}} + \rho^2)) \\
\alpha_{\mathrm{stab}}(\rho) &= -\frac{2}{q_1'}(q_1\mu - (k-k_0)^2) - \frac{q_1\mu(5q_1\mu + 2(k-k_0)^2)}{2q_1'(q_1\mu - (k-k_0)^2)}\rho^2 \\
&\quad + \mathrm{O}(|\mu|^{\frac{3}{2}} + \rho^2(|\mu|^{\frac{3}{2}} + \rho^2))
\end{aligned} \tag{5.3}$$

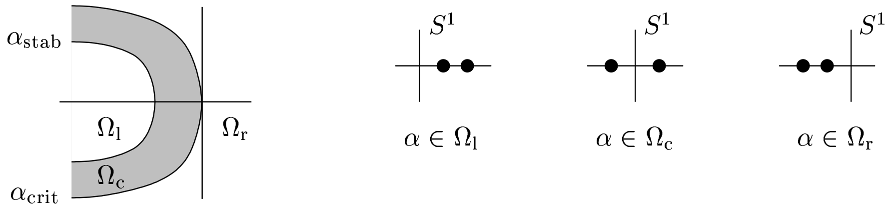

Figure 3: The left picture shows the regions $\Omega_{\rm l}$, $\Omega_{\rm c}$, and $\Omega_{\rm r}$ in the complex plane; the three pictures to the right show the critical Floquet multipliers for α in the aforementioned regions in the case where $c_0 < 0$. The vertical axis symbolizes the unit circle.

for the essential spectrum of the Turing patterns $\gamma_{\mu,k}$ in the steady coordinate frame; see (3.8) and (3.9). As before, k is the wavenumber of the Turing pattern $\gamma_{\mu,k}$. Using the results of the last section, we obtain the dispersion relations

$$\begin{aligned} \lambda_{\rm crit}(\rho) &= \exp\Big(\frac{2\pi\alpha_{\rm crit}(\rho)}{ck} + {\rm i}\rho\Big) \\ \lambda_{\rm stab}(\rho) &= \exp\Big(\frac{2\pi\alpha_{\rm stab}(\rho)}{ck} + {\rm i}\rho\Big) \end{aligned} \tag{5.4}$$

of the Turing patterns $\gamma_{\mu,k}$ in a frame moving with speed c.

We can therefore distinguish between three different regions associated with the spectrum of the Turing patterns in the complex plane near $\lambda = 1$. The region $\Lambda_{\rm r}$ is the connected components of the spectrum to the right of the curve $\lambda_{\rm crit}(\rho)$; The set $\Lambda_{\rm c}$ denotes the connected component of the area between the curves $\lambda_{\rm crit}(\rho)$ and $\lambda_{\rm stab}(\rho)$; and finally, $\Lambda_{\rm l}$ is the connected component to the left of the curve $\lambda_{\rm stab}(\rho)$. Using the relation $\lambda = \exp(\alpha T)$, we see that the sets Λ_j for $j = {\rm r, c, l}$ correspond to sets Ω_j for the Floquet exponents α with α near zero where $j = {\rm r, c, l}$; see Figure 3. In the following, we use these latter sets. According to Remark 2., the elliptic equation

$$W_\xi = A_{\gamma_\mu,\mu,\alpha}(\xi)W \tag{5.5}$$

has no Floquet multipliers on the unit circle for α in any of the regions $\Omega_{\rm r}$, $\Omega_{\rm c}$ or $\Omega_{\rm l}$. It follows from the proof of Lemma 1. that the reduced operator $B_{\mu,\alpha}$, see (4.14), has two Floquet multipliers close to one for any α close to zero. The next lemma gives the location of these critical multipliers of (5.5) for any α in one of the three regions defined above.

Lemma 1. *Suppose that $c_0 < 0$. The two critical multipliers of (5.5) are then both inside the unit circle for $\alpha \in \Omega_{\rm r}$; if $\alpha \in \Omega_{\rm c}$, then one of the critical multipliers is inside and the other one is outside the unit circle; finally, if $\alpha \in \Omega_{\rm l}$, then both multipliers are outside the unit circle. See Figure 3.*

If $c_0 > 0$, we change $\xi \to -\xi$ and can then apply the lemma.

Proof. Consider the linearized operator $A_{0,\mu,\alpha}$ about the homogeneous trivial equilibrium; see (4.4). For $\mu = 0$ and $\alpha = 0$, it has precisely two eigenvalues given by $\nu = \pm \mathrm{i}k_0$ on the imaginary axis. These eigenvalues are simple zeros of the dispersion relation $d(\mathrm{i}\omega_0, \nu) = 0$. Hence, they persist as zeros $\nu = \nu(\mu)$ for non-zero μ, and we have

$$\partial_\mu \nu = -\frac{\partial_\mu d}{\partial_\nu d} = -\frac{\partial_\mu d}{\partial_\lambda d^0}\frac{\partial_\lambda d^0}{\partial_\nu d} = -\frac{\partial_\mu d}{\partial_\lambda d}\frac{\partial_\lambda d^0}{\partial_\nu d} = -\frac{C_1}{c_0} =: \tilde{C}_1,$$

where we used (1.8) and the definition

$$C_1 := -\frac{\partial_\mu d}{\partial_\lambda d} > 0;$$

the constant C_1 is positive due to Hypothesis (P2); see [13, Sect. 3.4]. Therefore, $\tilde{C}_1 > 0$ since $c_0 < 0$ by assumption. In [13], we applied the elliptic center-manifold theory developed in [9] and reduced the elliptic system

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}_\xi = \begin{pmatrix} w_2 \\ D^{-1}(\partial_t w_1 + c w_2 - \partial_u f(w_1, \mu)) \end{pmatrix}$$

near $(w_1, w_2) = 0$ to a two-dimensional center manifold. The vector field on the center manifold is given by

$$U_\xi = (\tilde{C}_1 \mu + \mathrm{i}k_0)U - C_2 |U|^2 U$$

upon omitting higher-order terms. The small Turing patterns are of the form

$$U(\xi) = \sqrt{\frac{\tilde{C}_1 \mu}{C_2}} \mathrm{e}^{\mathrm{i}(k_0 \xi + \varphi)}$$

for arbitrary $\varphi \in \mathbb{R}$, and the linearization about them is given by

$$V_\xi = (-\tilde{C}_1 \mu + \mathrm{i}k_0)V - \tilde{C}_1 \mu \mathrm{e}^{2\mathrm{i}(k_0 \xi + \varphi)} \bar{V}.$$

Next, we account for the parameter α in the linearization. For the linearization at $U = 0$, we obtain

$$\partial_\alpha \nu = -\frac{\partial_\lambda d}{\partial_\nu d} = \frac{1}{c_0}$$

since the dependence on α and λ is the same. Hence, we get

$$V_\xi = (\frac{1}{c_0}\alpha - \tilde{C}_1 \mu + \mathrm{i}k_0)V - \tilde{C}_1 \mu \mathrm{e}^{2\mathrm{i}(k_0 \xi + \varphi)} \bar{V}.$$

for the linearization about the wave train. Upon introducing $V = \mathrm{e}^{\mathrm{i}(k_0 \xi + \varphi)} \tilde{V}$ and dropping the tilde, we obtain the system

$$V_\xi = (\frac{1}{c_0}\alpha - \tilde{C}_1 \mu)V - \tilde{C}_1 \mu \bar{V}.$$

Separating into real and imaginary part, and using $\tilde{C}_1 = -C_1/c_0$, we get

$$\frac{\mathrm{d}}{\mathrm{d}\xi} V_\mathrm{r} = \frac{1}{c_0}(\alpha + 2C_1 \mu)V_\mathrm{r}, \qquad \frac{\mathrm{d}}{\mathrm{d}\xi} V_\mathrm{i} = \frac{1}{c_0}\alpha V_\mathrm{i}.$$

Recall that $C_1 > 0$ and $c_0 < 0$. The continuous spectrum corresponds to the lines $\operatorname{Re}\alpha = 0$ and $\operatorname{Re}\alpha = -2C_1\mu$. The Floquet exponents are negative for $\alpha > 0$ and positive for $\alpha < -2C_1\mu$. For $-2C_1\mu < \alpha < 0$, one exponent is positive, the other one is negative. Using the results obtained in Section 3.3., it is straightforward to show that the situation does not change if the higher-order terms are taken into account. ■

6. Absence of point spectrum

In this section, we consider the spectrum of the modulated pulses. The spectrum of the Turing patterns near $\lambda = 1$ is contained in the two curves λ_{crit} and λ_{stab}. It follows from [14, Lemma 6.3] and its proof that any point in the spectrum of the Turing patterns is also in the spectrum of the modulated pulse. Here, we prove that the regions in the complex plane bounded by the essential spectrum of the Turing patterns are not filled with spectrum. Furthermore, it is shown that there are no isolated unstable eigenvalues near $\lambda = 1$ in the spectrum of the modulated pulses. Throughout this section, we assume that $c_0 < 0$. The case where $c_0 > 0$ can be reduced to $c_0 < 0$ by setting $\xi \to -\xi$.

6.1. Floquet exponents in the set Ω_{r}.

Let $\alpha \in \Omega_{\mathrm{r}}$. Hence, due to Lemma 1., the operator $A_{\gamma_\mu,\mu,\alpha}$ has its critical multipliers inside the unit circle. We have to identify any points $\lambda \in \Lambda_{\mathrm{r}}$ in the spectrum of $\mathcal{T}_{h_\mu,\mu}$. First, we prove that any such λ is an isolated eigenvalue. We then use the elliptic formulation to detect isolated eigenvalues. Note that such eigenvalues could pop out of the essential spectrum at $\mu = 0$; we refer to [7] for travelling-wave solutions to the Ginzburg-Landau equation whose spectrum exhibit this behavior.

Lemma 1. *If $\alpha \in \Omega_{\mathrm{r}}$ corresponds to an element λ in the spectrum of the modulated pulse, then λ is an isolated eigenvalue (with finite multiplicity).*

Proof. Consider the linearized reaction-diffusion equation

$$v_t = Dv_{\xi\xi} + c_*(\mu)v_\xi + \partial_u f(\tilde{\gamma}_\mu(\xi,t),\mu)v,$$

where

$$\tilde{\gamma}_\mu(\xi,t) := \begin{cases} \gamma_\mu(\xi + c_* t + \varphi_-) & \text{for } \xi \le 0 \\ \gamma_\mu(\xi + c_* t + \varphi_+) & \text{for } \xi > 0. \end{cases}$$

Since $h_\mu(\xi,t)$ converges to $\tilde{\gamma}_\mu(\xi,t)$ as $\xi \to \pm\infty$ uniformly in t, see Theorem 1., it follows as in [5, Exc. A.2, p. 137] that

$$\partial_u f(h_\mu(\xi,t),\mu) - \partial_u f(\tilde{\gamma}_\mu(\xi,t),\mu) : C^2_{\text{unif}} \longrightarrow C^0_{\text{unif}}$$

is compact. Following the arguments in [14, Lemma 6.2], we see that the difference of the operators $\mathcal{T}_{\tilde{\gamma}_\mu,\mu}$ and $\mathcal{T}_{h_\mu,\mu}$ is compact. We can then apply [5, Theorem A.1]; as a consequence, any connected component of the resolvent set of $\mathcal{T}_{\tilde{\gamma}_\mu,\mu}$ consists either entirely of spectrum of $\mathcal{T}_{h_\mu,\mu}$ or else consists of points in the resolvent set of $\mathcal{T}_{h_\mu,\mu}$ with the possible exception of at most finitely many isolated eigenvalues with finite multiplicity.

Next, we calculate the spectrum of $\mathcal{T}_{\tilde{\gamma}_\mu,\mu}$. Consider the eigenvalue problem

$$W_\xi = A_{\gamma_\mu,\mu,\alpha}(\xi)W$$

for the small patterns. This equation admits a trichotomy given by projections $Q^{\mathrm{s}}_{\mu,\alpha}(\xi)$, $Q^{\mathrm{c}}_{\mu,\alpha}(\xi)$ and $Q^{\mathrm{u}}_{\mu,\alpha}(\xi)$ defined for μ sufficiently close to zero and $\xi \in \mathbb{R}^+$, $\xi \in \mathbb{R}$ and $\xi \in \mathbb{R}^-$, respectively. Note that the eigenvalues of the matrix

$$\begin{array}{rcccl} B_{\mu,\alpha} & : & \mathrm{R}(Q^{\mathrm{c}}_{\mu,\alpha}(0)) & \longrightarrow & \mathrm{R}(Q^{\mathrm{c}}_{\mu,\alpha}(\frac{2\pi}{k})) = \mathrm{R}(Q^{\mathrm{c}}_{\mu,\alpha}(0)) \\ & & W(0) & \longmapsto & W(\frac{2\pi}{k}) \end{array}$$

are strictly contained inside the unit circle by Lemma 1.. We conclude that solutions with initial values in $\mathrm{R}(Q^{\mathrm{c}}_{\mu,\alpha}(\xi))$ decay for $\xi \to \infty$. Therefore, for $\alpha \in \Omega_{\mathrm{r}}$, the equation

$$W_\xi = A_{\gamma_\mu,\mu,\alpha}(\xi)W$$

has an exponential dichotomy given by the projections $Q^{\mathrm{cs}}_{\mu,\alpha}(\xi) := Q^{\mathrm{c}}_{\mu,\alpha}(\xi) + Q^{\mathrm{s}}_{\mu,\alpha}(\xi)$ and $Q^{\mathrm{u}}_{\mu,\alpha}(\xi)$ defined for $\xi \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^-$, respectively. These projections are close in norm, uniformly in ξ, to the projections $Q^{\mathrm{cs}}_{0,0}$ and $Q^{\mathrm{u}}_{0,0}$ for the equation

$$W_\xi = A_{0,0,0}W.$$

Furthermore, on account of [13, Lemma 3.4], the projections $Q^{\mathrm{cs}}_{0,0}$ and $Q^{\mathrm{u}}_{0,0}$ satisfy

$$\mathrm{R}(Q^{\mathrm{cs}}_{0,0}) \oplus \mathrm{R}(Q^{\mathrm{u}}_{0,0}) = Y.$$

Hence, the projections $Q^{\mathrm{cs}}_{\mu,\alpha}(\xi)$ and $Q^{\mathrm{u}}_{\mu,\alpha}(\xi)$ satisfy

$$\mathrm{R}(Q^{\mathrm{cs}}_{\mu,\alpha}(\xi)) \oplus \mathrm{R}(Q^{\mathrm{u}}_{\mu,\alpha}(\xi + \varphi)) = Y$$

for arbitrary ξ and φ. Using the arguments in the proof of Lemma 1., we conclude that the connected set Ω_{r} is contained in the resolvent set of $\mathcal{T}_{\tilde{\gamma}_\mu,\mu}$. Recalling the discussion above, we conclude that Ω_{r} consists either entirely of spectrum of $\mathcal{T}_{h_\mu,\mu}$ or else consists of points in the resolvent set of $\mathcal{T}_{h_\mu,\mu}$ together with at most finitely many isolated eigenvalues. The first possibility, however, has been excluded in Section 4.. This proves the lemma. ∎

It suffices therefore to calculate isolated eigenvalues. First, we show that the eigenvalue problem for the modulated pulse in the elliptic formulation can be considered as a regular perturbation of the eigenvalue problem for the original pulse at $\mu = 0$. We then investigate the eigenvalue problem for the original pulse and carry out the perturbation analysis.

The relevant eigenvalue problem for the modulated wave cast as an elliptic equation is given by

$$W_\xi = A_{h_\mu,\mu,\alpha}(\xi)W$$

where $\alpha \in \Omega_{\mathrm{r}}$. We claim that this equation has exponential dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$ given by projections $P^{\mathrm{cs}}_{\mu,\alpha}(\xi)$ and $P^{\mathrm{u}}_{\mu,\alpha}(\xi)$ defined for $\xi \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^-$, respectively. Indeed, the limiting problem

$$W_\xi = A_{\gamma_\mu,\mu,\alpha}(\xi)W$$

has an exponential dichotomy, see Lemma 1. and the proof of Lemma 1., and the claim follows from [12]. Isolated eigenvalues correspond to values of α such that

$$Y^{\mathrm{r}}_{\mu,\alpha} := \mathrm{R}(P^{\mathrm{cs}}_{\mu,\alpha}(0)) \cap \mathrm{R}(P^{\mathrm{u}}_{\mu,\alpha}(0)) \neq \{0\}. \tag{6.1}$$

We are interested in obtaining information from the limit $\mu \to 0$.

Notice that the projections $P^{\mathrm{cs}}_{\mu,\alpha}(\xi)$ and $P^{\mathrm{u}}_{\mu,\alpha}(\xi)$ are actually defined for all α near zero, and not just for $\alpha \in \Omega_{\mathrm{r}}$. Indeed, this follows from [12] by seeking those initial values leading to solutions of

$$W_\xi = A_{h_\mu,\mu,\alpha}(\xi)W$$

which grow not faster than $\mathrm{e}^{\eta\xi}$ as $\xi \to \infty$ for some small but fixed $\eta > 0$ and which decay exponentially with rate larger than η for $\xi \to -\infty$. Moreover, as $\mu \to 0$, it the projections $P^{\mathrm{cs}}_{\mu,\alpha}(\xi)$ and $P^{\mathrm{u}}_{\mu,\alpha}(\xi)$ converge to projections $P^{\mathrm{cs}}_{0,\alpha}(\xi)$ and $P^{\mathrm{u}}_{\mu,\alpha}(\xi)$ which constitute dichotomies for the eigenvalue problem

$$W_\xi = A_{h_0,0,\alpha}(\xi)W \tag{6.2}$$

of the original travelling pulse h_0 at $\mu = 0$. In fact, solutions to (6.2) associated with initial values in $\mathrm{R}(P^{\mathrm{cs}}_{0,\alpha}(0))$ exist for $\xi \geq 0$ and grow not faster that $\mathrm{e}^{\eta\xi}$ as $\xi \to \infty$ for some small but fixed $\eta > 0$; similarly, solutions with initial values in $\mathrm{R}(P^{\mathrm{u}}_{0,\alpha}(0))$ exist for $\xi \leq 0$ and decay exponentially with rate larger than η for $\xi \to -\infty$.

Upon setting $\mu = 0$ in (6.1), we obtain the limiting subspace

$$Y^{\mathrm{r}}_{0,\alpha} = \mathrm{R}(P^{\mathrm{cs}}_{0,\alpha}(0)) \cap \mathrm{R}(P^{\mathrm{u}}_{0,\alpha}(0))$$

which we have to calculate. It follows from [13, Lemma 3.4] that

$$Y^{\mathrm{r}}_{0,\alpha} = \mathrm{span}\{W_0(0)\},$$

where

$$W_0(\xi) = \Big(h_0(\xi), \frac{\mathrm{d}}{\mathrm{d}\xi}h_0(\xi)\Big).$$

Moreover, on account of [13, Lemma 3.8], the spaces $\mathrm{R}(P^{\mathrm{cs}}_{0,\alpha}(0))$ and $\mathrm{R}(P^{\mathrm{u}}_{0,\alpha}(0))$ cross transversely upon perturbing to $\alpha \neq 0$. Since the projections depend continuously upon μ and are smooth with respect to α, we can conclude that, for any $\mu > 0$ sufficiently small, there is a unique value $\alpha = \alpha(\mu)$ close to zero such that

$$\mathrm{R}(P^{\mathrm{cs}}_{\mu,\alpha}(0)) \cap \mathrm{R}(P^{\mathrm{u}}_{\mu,\alpha}(0)) \neq \{0\}.$$

Hence, there is the possibility of an instability occurring if $\alpha(\mu) \in \Omega_{\mathrm{r}}$. We show that in fact $\alpha(\mu) = 0$ for all μ.

For $\mu > 0$, two bounded solutions of

$$v_t = Dv_{\xi\xi} + c_*(\mu)v_\xi + \partial_u f(h_\mu(\xi, t), \mu)v$$

are given by

$$v_0(\xi, t) = \frac{\mathrm{d}}{\mathrm{d}\xi} h_\mu(\xi, t), \qquad v_1(\xi, t) = \frac{\mathrm{d}}{\mathrm{d}t} h_\mu(\xi, t).$$

This follows readily by differentiating the equation

$$u_t = Du_{\xi\xi} + c_*(\mu)u_\xi + f(u, \mu),$$

which is satisfied by $u(\xi, t) = h_\mu(\xi, t)$, with respect to ξ and t. These solutions correspond to the translation symmetries in ξ and t. Note that we have $v_1(\xi, t) = \partial_t h_0(\xi, t) = 0$ at $\mu = 0$ since $h_0(\xi)$ does not depend on t.

On account of Theorem 1., we have

$$\begin{aligned} |h_\mu(\xi, t) - \gamma_\mu(\xi + ct + \varphi_-)| &\leq K\mathrm{e}^{-\kappa|\xi|} && \text{for } \xi \to -\infty \\ |h_\mu(\xi, t) - \gamma_\mu(\xi + ct + \varphi_+)| &\leq K\mathrm{e}^{-\delta\mu|\xi|} && \text{for } \xi \to \infty. \end{aligned}$$

Hence, upon using the differential equation and the regularity properties of the solutions, we see that

$$v_0(\xi, t) \to \gamma'_\mu(\xi + ct + \varphi_\pm), \qquad v_1(\xi, t) \to c\gamma'_\mu(\xi + ct + \varphi_\pm)$$

as $\xi \to \pm\infty$ exponentially in ξ and uniformly in t with rates as above. Therefore, we conclude that

$$|v_1(\xi, \cdot) - cv_0(\xi, \cdot)| \leq \begin{cases} K\mathrm{e}^{-\kappa|\xi|} & \text{for } \xi \to -\infty \\ K\mathrm{e}^{-\delta\mu|\xi|} & \text{for } \xi \to \infty. \end{cases}$$

Note that the function $v_{\mathrm{loc}} = v_1 - cv_0$ is not equal to zero; at $\xi = 0$, we have $v_0(0, t) \approx h'_0(0)$ while $v_1(0, t)$ is of the order $\sqrt{\mu}$.

Thus, the function

$$W_{\mathrm{loc}}(\xi) = \left(v_{\mathrm{loc}}(\xi, \cdot), \frac{\mathrm{d}}{\mathrm{d}\xi} v_{\mathrm{loc}}(\xi, \cdot)\right) \in Y$$

associated with $v(\xi, t)$ converges to zero exponentially as $\xi \to \pm\infty$ and

$$\mathrm{span}\{W_{\mathrm{loc}}(0)\} = \mathrm{R}(P^{\mathrm{cs}}_{\mu,0}(0)) \cap \mathrm{R}(P^{\mathrm{u}}_{\mu,0}(0)), \tag{6.3}$$

leading to a non-trivial intersection. Hence, we obtain $\alpha(\mu)=0$ for all μ close to zero.

The localized eigenfunction $W_{\mathrm{loc}}(\xi)$ has the following interpretation: consider the reaction-diffusion equation in a frame which moves with the velocity of the asymptotic Turing patterns. In this frame, the Turing patterns are stationary. The modulated pulse evolves in time as follows: it consists of a localized pulse that resembles the original travelling wave and moves through the Turing pattern with a certain velocity. The derivative of the modulated pulse with respect to time is then localized and corresponds to the eigenfunction $v_{\mathrm{loc}}(\xi,t)$. In formulas, we have that $h_\mu(x-ct,t)$ converges to the time-independent patterns $\gamma_\mu(x+\varphi_\pm)$ as $x\to\pm\infty$; thus, the time derivative of $h_\mu(x-ct,t)$ converges to zero as $x\to\pm\infty$.

Summarizing, we have proved that there are no isolated eigenvalues in the region Λ_{r}, that is, in the connected component of the resolvent set to the right of the essential spectrum.

6.2. Floquet exponents in the set Ω_{c}.

In this section, we take $\alpha\in\Omega_{\mathrm{c}}$. We shall show that there exists an $\alpha\in\Omega_{\mathrm{c}}$ such that $\lambda=\exp(\alpha T)\in\Lambda_{\mathrm{c}}$ is not in the spectrum of $\mathcal{T}_{h_\mu,\mu}$. The strategy is similar to the one pursued in the previous section. One additional difficulty is that the eigenvalue problem is not a regular perturbation of the $\mu=0$ limit since the modulated pulses do not converge to zero with uniform exponential rate.

Consider the eigenvalue equation

$$W_\xi = A_{\gamma_\mu,\mu,\alpha}(\xi)W$$

about the Turing patterns. Due to Lemma 1., for $\alpha\in\Omega_{\mathrm{c}}$ and $\mu>0$, the operator $A_{\gamma_\mu,\mu,\alpha}$ has one of its critical multipliers inside and the other one outside the unit circle. We denote the associated stable and unstable critical Floquet exponents by $\nu^{\mathrm{s}}_{\mu,\alpha}$ and $\nu^{\mathrm{u}}_{\mu,\alpha}$, respectively. It also follows from Lemma 1. that $\mathrm{Re}\,\nu^{\mathrm{s}}_{\mu,0}<0$ and $\nu^{\mathrm{u}}_{\mu,0}=0$ for any $\mu>0$. In particular, there exist real numbers $\tilde\nu^{\mathrm{s}}_{\mu,\alpha}<0$ and $\tilde\nu^{\mathrm{u}}_{\mu,\alpha}$ such that

$$\mathrm{Re}\,\nu^{\mathrm{s}}_{\mu,\alpha}<\tilde\nu^{\mathrm{s}}_{\mu,\alpha}<\tilde\nu^{\mathrm{u}}_{\mu,\alpha}<\mathrm{Re}\,\nu^{\mathrm{u}}_{\mu,\alpha}$$

for all $\mu>0$ and any α in a small possibly μ-dependent neighborhood of zero. For such values of (μ,α), we can then construct exponential dichotomies $\tilde Q^{\mathrm{s}}_{\mu,\alpha}(\xi)$ and $\tilde Q^{\mathrm{u}}_{\mu,\alpha}(\xi)$ defined for $\xi\in\mathbb{R}^+$ and $\xi\in\mathbb{R}^-$, respectively, such that the following is true: solutions with initial values in $\mathrm{R}(\tilde Q^{\mathrm{s}}_{\mu,\alpha}(0))$ exist for positive ξ and decay exponentially to zero with rate $\tilde\nu^{\mathrm{s}}_{\mu,\alpha}$ as $\xi\to\infty$; analogously, solutions with initial values in $\mathrm{R}(\tilde Q^{\mathrm{u}}_{\mu,\alpha}(0))$ exist for negative ξ and grow at most with the exponential rate $\tilde\nu^{\mathrm{u}}_{\mu,\alpha}$ as $\xi\to-\infty$. In fact, for $\alpha\in\Omega_{\mathrm{c}}$, solutions in $\mathrm{R}(\tilde Q^{\mathrm{u}}_{\mu,\alpha}(0))$ decay exponentially as $\xi\to-\infty$ since then $\mathrm{Re}\,\nu^{\mathrm{u}}_{\mu,\alpha}>0$. Also, the projections depend continuously upon μ and smoothly upon α. We refer to Figure 4 for a summary of the decay rates associated with the various projections.

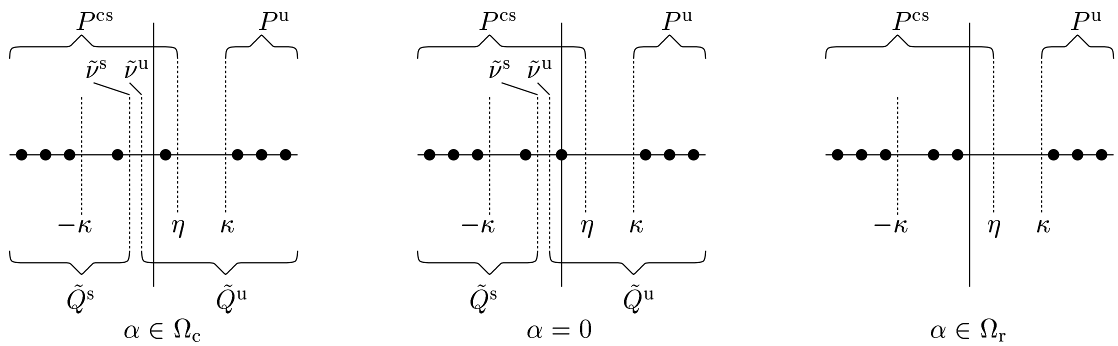

Figure 4: The pictures show the real part of the Floquet exponents of the eigenvalue problem associated with the asymptotic Turing pattern and the modulated pulse as well as the decay rates associated with the exponential dichotomies. The projections $\tilde{P}^{s}$ and $\tilde{P}^{u}$ associated with the eigenvalue problem of the modulated pulse admit the same decay rates as $\tilde{Q}^{s}$ and $\tilde{Q}^{u}$, respectively.

As in the last section, the exponential dichotomies defined for the linearization about the Turing patterns can be extended to dichotomies $\tilde{P}^{s}_{\mu,\alpha}(\xi)$ and $\tilde{P}^{u}_{\mu,\alpha}(\xi)$ for the eigenvalue equation

$$W_\xi = A_{h_\mu,\mu,\alpha}(\xi)W \tag{6.4}$$

about the modulated travelling wave. The exponential dichotomies $\tilde{P}^{s}_{\mu,\alpha}$ and $\tilde{P}^{u}_{\mu,\alpha}$ enjoy the same properties as $\tilde{Q}^{s}_{\mu,\alpha}$ and $\tilde{Q}^{u}_{\mu,\alpha}$. Therefore, on account of Lemma 1., it follows that $\lambda = \exp(\alpha T)$ with $\alpha \in \Omega_c$ is in the spectrum of $T_{h_\mu,\mu}$ if, and only if,

$$Y^{c}_{\mu,\alpha} := \mathrm{R}(\tilde{P}^{s}_{\mu,\alpha}(0)) \cap \mathrm{R}(\tilde{P}^{u}_{\mu,\alpha}(0)) \neq \{0\} \tag{6.5}$$

or

$$\mathrm{R}(\tilde{P}^{s}_{\mu,\alpha}(0)) + \mathrm{R}(\tilde{P}^{u}_{\mu,\alpha}(0)) \neq Y.$$

Our strategy is as follows. First, we show that the space $Y^{c}_{\mu,0}$ appearing in (6.5) is one-dimensional for $\alpha = 0$ and any small $\mu > 0$. Afterwards, we prove that the dimension of $Y^{c}_{\mu,\alpha}$ decreases to zero upon varying α. It follows from the proof given below that $\mathrm{R}(\tilde{P}^{s}_{\mu,\alpha}(0)) + \mathrm{R}(\tilde{P}^{u}_{\mu,\alpha}(0)) = Y$ whenever $Y^{c}_{\mu,\alpha} = \{0\}$. We can then conclude that any $\alpha \in \Omega_c$ close to zero corresponds to an element in the resolvent set.

Lemma 2. *For any small $\mu > 0$, we have* $\dim Y^{c}_{\mu,0} = 1$.

Proof. Let $\alpha = 0$ and consider

$$Y^{c}_{\mu,0} = \mathrm{R}(\tilde{P}^{s}_{\mu,0}(0)) \cap \mathrm{R}(\tilde{P}^{u}_{\mu,0}(0)).$$

This intersection consists of all solutions of (6.4) which decay exponentially as $\xi \to \infty$ and are bounded as $\xi \to -\infty$. We conclude that the function W_{loc} defined in the last section is contained in $Y^{c}_{\mu,0}$, since it actually decays to zero exponentially for $\xi \to \pm\infty$.

Let

$$W_0(\xi) = (h_\xi, h_{\xi\xi})(\xi), \tag{6.6}$$

and note that $W_0(0)$ is not contained in $Y^{\mathrm{c}}_{\mu,0}$ since it does not decay to zero as $\xi \to \infty$ for $\mu > 0$. However, $W_0(0) \in \mathrm{R}(\tilde{P}^{\mathrm{u}}_{\mu,0}(0))$ since it is bounded as $\xi \to -\infty$. Note that $W_0(\xi)$ does not converge to zero as $\xi \to -\infty$. Therefore, we have

$$\mathrm{R}(\tilde{P}^{\mathrm{u}}_{\mu,0}(0)) = \mathrm{span}\{W_0(0)\} \oplus \mathrm{R}(P^{\mathrm{u}}_{\mu,0}(0));$$

recall from the last section that $\mathrm{R}(P^{\mathrm{u}}_{\mu,0}(0))$ contains those solutions which decay to zero exponentially as $\xi \to -\infty$ with some rate $\kappa > 0$ independent of μ; see Figure 4. It follows that any element $W(0)$ of $\mathrm{R}(\tilde{P}^{\mathrm{u}}_{\mu,0}(0))$ can be written as

$$W(0) = aW_0(0) + W_{\mathrm{u}}(0)$$

for some number a where $W_{\mathrm{u}}(0) \in \mathrm{R}(P^{\mathrm{u}}_{\mu,0}(0))$. Next, suppose that

$$W(0) = aW_0(0) + W_{\mathrm{u}}(0) \subset \mathrm{R}(P^{\mathrm{cs}}_{\mu,0}(0)) \cap \mathrm{R}(\tilde{P}^{\mathrm{u}}_{\mu,0}(0)).$$

Since

$$W_0(0) \in \mathrm{R}(P^{\mathrm{cs}}_{\mu,0}(0)) \cap \mathrm{R}(\tilde{P}^{\mathrm{u}}_{\mu,0}(0)),$$

we conclude that

$$W_{\mathrm{u}}(0) \in \mathrm{R}(P^{\mathrm{cs}}_{\mu,0}(0)) \cap \mathrm{R}(\tilde{P}^{\mathrm{u}}_{\mu,0}(0)).$$

However, we had seen above that $W_{\mathrm{u}}(0) \in \mathrm{R}(P^{\mathrm{u}}_{\mu,0}(0))$. Thus,

$$W_{\mathrm{u}}(0) \in \mathrm{R}(P^{\mathrm{cs}}_{\mu,0}(0)) \cap \mathrm{R}(P^{\mathrm{u}}_{\mu,0}(0)),$$

and it follows that $W_u(0) \in \mathrm{span}\{W_{\mathrm{loc}}(0)\}$ on account of the results of the last section; see (6.3). Since

$$Y^{\mathrm{c}}_{\mu,0} \subset \mathrm{R}(P^{\mathrm{cs}}_{\mu,0}(0)) \cap \mathrm{R}(\tilde{P}^{\mathrm{u}}_{\mu,0}(0)) = \mathrm{span}\{W_0(0), W_{\mathrm{loc}}(0)\},$$

this shows that $Y^{\mathrm{c}}_{\mu,0} = \mathrm{span}\{W_{\mathrm{loc}}(0)\}$, and therefore $\dim Y^{\mathrm{c}}_{\mu,0} = 1$. ■

The previous lemma shows that the space $Y^{\mathrm{c}}_{\mu,0}$ is spanned by the localized eigenfunction W_{loc}. In the next step, we prove that the space $Y^{\mathrm{c}}_{\mu,\alpha}$ has dimension zero for $\alpha \neq 0$. In fact, we shall see that the space $Y^{\mathrm{c}}_{\mu,0}$ disappears in the same fashion as the space $Y^{\mathrm{r}}_{\mu,0}$ upon varying α; the latter space has already been investigated in the previous section.

Lemma 3. *For any small $\mu > 0$, and any $\alpha \in \Omega_{\mathrm{c}}$ in a small neighborhood of zero which may depend upon μ, we have* $\dim Y^{\mathrm{c}}_{\mu,\alpha} = 0$ *and* $\mathrm{R}(\tilde{P}^{\mathrm{s}}_{\mu,\alpha}(0)) + \mathrm{R}(\tilde{P}^{\mathrm{u}}_{\mu,\alpha}(0)) = Y$.

Proof. As mentioned before, for $\mu > 0$, the projections $\tilde{P}^{\mathrm{s}}_{\mu,\alpha}(0))$ and $\tilde{P}^{\mathrm{u}}_{\mu,\alpha}(0)$ depend continuously on μ and smoothly upon α, and we have

$$\dim\,[\mathrm{R}(\tilde{P}^{\mathrm{s}}_{\mu,0}(0)) \cap \mathrm{R}(\tilde{P}^{\mathrm{u}}_{\mu,0}(0))] = 1.$$

First, we consider the adjoint eigenvalue equation

$$W_\xi = -A_{h_\mu,\mu,\alpha}(\xi)^* W. \tag{6.7}$$

Note that we can regard the adjoint operator $A_{h_\mu,\mu,\alpha}(\xi)^*$ as a closed operator defined in the Hilbert space Y; see [8]. In particular, the adjoint equation has well-defined exponential dichotomies given by the adjoint projections $\tilde{P}^{\mathrm{s}}_{\mu,\alpha}(\xi)^*$ and $\tilde{P}^{\mathrm{u}}_{\mu,\alpha}(\xi)^*$ for $\xi \geq 0$ and $\xi \leq 0$, respectively; see [12]. It follows that

$$Y^{\mathrm{c},*}_{\mu,0} := [\mathrm{R}(\tilde{P}^{\mathrm{s}}_{\mu,0}(0)) + \mathrm{R}(\tilde{P}^{\mathrm{u}}_{\mu,0}(0))]^\perp$$

is finite-dimensional since

$$[\mathrm{R}(\tilde{P}^{\mathrm{s}}_{\mu,0}(0)) + \mathrm{R}(\tilde{P}^{\mathrm{u}}_{\mu,0}(0))]^\perp \subset [\mathrm{R}(P^{\mathrm{s}}_{\mu,0}(0)) + \mathrm{R}(P^{\mathrm{u}}_{\mu,0}(0))]^\perp \cong \mathbb{R}^3;$$

see [13, Lemma 3.4] and Section 6.1.. In fact, any element $\psi_0 \in Y^{\mathrm{c},*}_{\mu,0}$ leads to a bounded solution $\psi(\xi)$ of the adjoint eigenvalue equation (6.7) which satisfies $\psi(0) = \psi_0$ and decays exponentially to zero as $\xi \to -\infty$. Any such solution satisfies

$$\psi(\xi) \perp [\mathrm{R}(\tilde{P}^{\mathrm{s}}_{\mu,0}(\xi)) + \mathrm{R}(\tilde{P}^{\mathrm{u}}_{\mu,0}(\xi))]$$

for all ξ.

Before we continue with the proof, we investigate the asymptotic behavior of the solutions $\psi(\xi)$ in more detail. We claim that, for any such $\psi(\xi)$,

$$\psi(\xi) \perp [\mathrm{R}(P^{\mathrm{cs}}_{\mu,0}(\xi)) + \mathrm{R}(P^{\mathrm{u}}_{\mu,0}(\xi))] \tag{6.8}$$

for all ξ. In order to prove this claim, recall that

$$\mathrm{R}(\tilde{P}^{\mathrm{u}}_{\mu,0}(0)) = \mathrm{span}\{W_0(0)\} \oplus \mathrm{R}(P^{\mathrm{u}}_{\mu,0}(0)).$$

Similarly, we have

$$\mathrm{R}(P^{\mathrm{cs}}_{\mu,0}(0)) = \mathrm{span}\{W_0(0)\} \oplus \mathrm{R}(\tilde{P}^{\mathrm{s}}_{\mu,0}(0))$$

since $W_0(\xi)$ does not decay to zero as $\xi \to \infty$. By definition, any $\psi(0) \in Y^{\mathrm{c},*}_{\mu,0}$ satisfies

$$\psi(0) \perp W_0(0)$$

since $\psi(0) \perp \mathrm{R}(\tilde{P}^{\mathrm{u}}_{\mu,0}(0))$. Hence, we conclude that

$$\psi(0) \perp \mathrm{R}(P^{\mathrm{cs}}_{\mu,0}(0))$$

which proves (6.8). In the last section, see (6.3), we proved that

$$\dim\,[\mathrm{R}(P^{\mathrm{cs}}_{\mu,0}(\xi)) + \mathrm{R}(P^{\mathrm{u}}_{\mu,0}(\xi))]^{\perp} = 1.$$

Thus, there exists a unique (up to scalar multiples) solution $\psi(\xi)$ of (6.7) with $\psi(0) \in Y^{\mathrm{c},*}_{\mu,0}$. We return to the proof of the lemma. Using Lyapunov-Schmidt reduction, see for instance [11] or [13, Lemma 3.8], we see that

$$\dim Y^{\mathrm{c}}_{\mu,\alpha} \geq 1 \quad \text{or} \quad \dim Y^{\mathrm{c},*}_{\mu,\alpha} \geq 1$$

for $\mu > 0$ if, and only if,

$$E^{\mathrm{c}}(\mu,\alpha) = \alpha \int_{-\infty}^{\infty} \langle \psi(\xi), BW_{\mathrm{loc}}(\xi)\rangle_Y \,\mathrm{d}\xi + \mathrm{O}(\alpha^2) = 0, \tag{6.9}$$

where

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Note that the integral exists due to the convergence properties of $\psi(\xi)$ and $W_{\mathrm{loc}}(\xi)$. It suffices therefore to prove that the integral appearing in (6.9) is non-zero. In the last section, we have considered the intersection

$$Y^{\mathrm{r}}_{\mu,\alpha} = \mathrm{R}(P^{\mathrm{cs}}_{\mu,\alpha}(0)) \cap \mathrm{R}(P^{\mathrm{u}}_{\mu,\alpha}(0)).$$

For $\alpha = 0$, this intersection was spanned by $W_{\mathrm{loc}}(0)$. Using Lyapunov-Schmidt reduction, it can be shown that the subspace $Y^{\mathrm{r}}_{\mu,\alpha}$ is non-trivial if, and only if, the function

$$E^{\mathrm{r}}(\mu,\alpha) = \alpha \int_{-\infty}^{\infty} \langle \psi(\xi), BW_{\mathrm{loc}}(\xi)\rangle_Y \,\mathrm{d}\xi + \mathrm{O}(\alpha^2) = 0 \tag{6.10}$$

vanishes. Indeed, the fact that the integrands in (6.9) and (6.10) coincide follows from (6.8). Note that the function $E^{\mathrm{r}}(\mu,\alpha)$ is continuous in μ and smooth in α. Moreover, it is well-defined for $\mu = 0$ where it measures the distance of the spaces $\mathrm{R}(P^{\mathrm{cs}}_{0,\alpha}(0))$ and $\mathrm{R}(P^{\mathrm{u}}_{0,\alpha}(0))$. It has been shown in [13, Proof of Lemma 3.8] that

$$E^{\mathrm{r}}(0,\alpha) = \alpha M + \mathrm{O}(\alpha^2),$$

for some non-zero constant M. Exploiting continuity in μ and differentiability in α, we can therefore conclude that the integral appearing in (6.9) is non-zero. ■

Summarizing the results we have obtained in this section, we are able to conclude that there is a small, possibly μ-dependent, neighborhood of zero such that none of the $\alpha \in \Omega_{\mathrm{c}}$ in this neighborhood is a Floquet exponent for the operator $T_{h_\mu,\mu}$.

6.3. Proof of Theorem 4.

We collect the arguments presented in the preceding sections. In Section 5., we calculated the spectrum of the Turing patterns in the moving frame. The spectrum of the Turing patterns near $\lambda = 1$ is contained in the two curves λ_{crit} and λ_{stab}. Moreover, any point in the spectrum of the Turing patterns is also in the spectrum of the modulated pulse. In Section 6.1., we proved that any point to the right of the spectrum of the Turing patterns, in the moving frame, is not in the spectrum of the modulated pulse. We then showed that any point close to $\lambda = 1$ which is in the region Λ_c between the two curves constituting the spectrum of the Turing patterns is contained in the resolvent set of the modulated pulse. The remaining statements in Theorem 4. follow easily using the results established in the previous two sections. This completes the proof of Theorem 4..

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