Basin boundaries and bifurcations near convective instabilities: A case study

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June 17, 2003

Abstract

Using a scalar reaction-diffusion equation with drift and a cubic nonlinearity as a simple model problem, we investigate the effect of domain size on stability and bifurcations of steady states. We focus on two parameter regimes, namely the regions where the steady state is convectively or absolutely unstable. In the convective-instability regime, the trivial stationary solution is asymptotically stable on any bounded domain but unstable on the real line. To measure the degree to which the trivial solution is stable, we estimate the distance of the trivial solution to the boundary of its basin of attraction: We show that this distance is exponentially small in the diameter of the domain for subcritical nonlinearities, while it is bounded away from zero uniformly in the domain size for supercritical nonlinearities. Lastly, at the onset of the absolute instability where the trivial steady state destabilizes on large bounded domains, we discuss bifurcations and amplitude scalings.

1 Introduction

The purpose of this paper is to illustrate, via a case study, the dependence of stability properties of stationary solutions of PDEs on the diameter of the underlying spatial domain. We focus on the scalar reaction-diffusion equation

$$u_t = u_{xx} + u_x + \mu u + \kappa u^3, \qquad u(x,t) \in \mathbb{R} \tag{1.1}$$

where $\kappa = \pm 1$ is fixed, and $\mu \in \mathbb{R}$ is a parameter. We consider (1.1) either on the unbounded domain $\mathbb{R}$ or else on the bounded interval $(0, \ell)$ in which case we supplement (1.1) with Dirichlet boundary conditions $u(0,t) = u(\ell,t) = 0$ for all t. We are particularly interested in large intervals so that $\ell \gg 1$.

We begin by briefly recalling some basic facts about (1.1). First, note that $u(x,t) = 0$ is an equilibrium of (1.1) for every μ which we refer to as the trivial steady state. On the real line $\mathbb{R}$, the trivial steady state is asymptotically stable for $\mu < 0$ and unstable for $\mu > 0$. In contrast, it is asymptotically stable for $\mu < 1/4$ and unstable for $\mu > 1/4$ when considered on the bounded interval $(0, \ell)$ with $\ell \gg 1$.

These findings indicate that the parameter regime $0 < \mu < 1/4$ is of particular interest since the trivial steady state acquires quite different stability properties depending on whether we consider (1.1) on the real line or on bounded intervals. To reconcile these facts, we show that, even though the trivial rest state is asymptotically stable on each bounded interval, it is not uniformly stable: while for each $\epsilon > 0$, there is a $\delta > 0$ so that solutions starting in a δ-neighborhood of the origin stay in an ϵ-neighborhood of the origin for all positive times, we prove that δ cannot be chosen uniformly in ℓ. In other words, the larger ℓ, the more sensitive solutions become. A different measure for the sensitivity of $u = 0$ with respect to perturbations is the diameter of a maximal centered ball in its basin of attraction, i.e. the length of the shortest line from $u = 0$ to the boundary of its basin of attraction. We demonstrate that this diameter is exponentially small in ℓ for $\kappa = 1$ and bounded away from zero uniformly in ℓ for $\kappa = -1$. These results suggest that, for all practical purposes, there is no well-defined and clear-cut threshold of instability when considering large intervals. We encountered this phenomenon in a numerical study of spiral waves that break up on large planar disks [12], an investigation that motivated us to study these issues for the simple model problem (1.1).

The non-uniformity of stability with respect to the diameter of the domain is caused by the convective nature of the instability of the trivial steady state. A convective instability is characterized as follows: Perturbations of the underlying pattern grow in amplitude in a translation-invariant norm but decay pointwise at any fixed location in space. Note that this notion depends on the reference frame. When the domain is large but bounded, solutions experience transient growth but eventually decay to zero.

Convective instabilities of patterns are a common phenomenon in a variety of spatially extended systems. Such instabilities have been observed in fluid experiments [8, 9], flow-driven chemical reactions [1, 2, 10] but also in chemical experiments without convection [3, 16, 17]. Transport phenomena also play an important role for the dynamics of defects, such as pace makers, spiral waves and sinks, in excitable and oscillatory media (see [13] and references therein).

The pointwise decay associated with convective instabilities has been explained on a linear level for spatially homogeneous or periodic background states using Laplace transform [4, 5]. The key observation is that the complex contour integrals of the inverse Laplace transform for localized initial data can be deformed analytically across the essential spectrum into the stable complex half plane, since the resolvent of the linearized operator can be continued *pointwise* as an analytic function across the essential spectrum (even though the Greens function will typically grow exponentially in space). The contour of integration can be pushed to the left until it hits branch points of the resolvent which correspond to double roots of the linear dispersion relation. The situation for spatially inhomogeneous background patterns is more complicated since the resolvent may have poles in addition to branch points. The analysis of convective instabilities on the linear and the nonlinear level is inherently complicated by the fact that the relevant linear operators are non-normal (and, in particular, not self-adjoint). Further beyond the onset of instability, a convective instability may become an absolute instability: solutions grow both in amplitude and pointwise. Typically, branch points of the linear dispersion relation that cross the imaginary axis are responsible for absolute instabilities on unbounded domains.

The plan of this paper is as follows. In Section 2, we introduce various notions of stability to capture different effects of domain truncation. In particular, we show in Section 3 that asymptotic stability properties are not uniform in the diameter of the domain, neither for the linear nor the nonlinear problem. Instead, we propose to measure the effects of increasing domains by monitoring the boundary of the basin of attraction. In Section 4, we show that the basin boundary depends crucially on the nonlinearity. Lastly, we give in Section 5 a complete description of the amplitude scalings close to the threshold to an absolute instability.

Acknowledgments B. Sandstede was partially supported by the Alfred P Sloan Foundation and by the NSF through grant DMS-0203854. A. Scheel was partially supported by the NSF through grant DMS-0203301. We acknowledge helpful discussions with Peter Polacik.

2 Different notions of stability

Consider the scalar reaction-diffusion equation

$$u_t = u_{xx} + u_x + \mu u + \kappa u^3 \tag{2.1}$$

for $x \in \Omega \subset \mathbb{R}$. We consider four different domains, namely $\Omega = \mathbb{R}$, $\mathbb{R}^{\pm}$, and $(0, \ell)$ with ℓ large. On the boundaries of $\mathbb{R}^{\pm}$ and $(0, \ell)$, we impose Dirichlet boundary conditions $u(0, t) = u(\ell, t) = 0$.

Equation (2.1) generates a local semiflow on the phase space $X = H_0^1(\Omega)$, and we denote the solution with initial value $u_0 \in X$ by $u(t; u_0) \in X$. As a general rule, regularity properties of solutions will not play a major role in this article. In particular, working with Sobolev spaces with higher regularity or with spaces of continuous functions will not change the results. The results will change, however, when exponential weights are used.

We distinguish the *supercritical* case $\kappa = -1$ and the *subcritical* case $\kappa = 1$, where the term critical refers to the parameter value $\mu = 0$ as the onset of linear growth for the linear spatially homogeneous problem $u_t = \mu u$. The terms supercritical and subcritical refer to the existence of equilibria for the nonlinear spatially homogeneous problem $u_t = \mu u + \kappa u^3$ above ($\mu > 0$) or below ($\mu < 0$) the critical value $\mu = 0$, respectively.

Clearly, (2.1) admits the trivial stationary solution $u(t; 0) \equiv 0$. The goal of this work is to clarify the stability properties of this equilibrium for the different domains introduced above. We introduce various definitions of stability that capture different aspects.

Definition 2.1 (Stability) *We say that the trivial equilibrium $u = 0$ of (2.1) is* stable *in X if, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that, for any initial condition u_0 with $\|u_0\|_X < \delta$, the solution*

$u(t;u_0)$ of (2.1) remains in an ε-neighborhood of $u=0$ for all times $t \geq 0$ so that $\|u(t;u_0)\|_X < \varepsilon$. If, in addition, there exists an $\varepsilon_0 > 0$ such that $u(t;u_0) \to 0$ in X for each u_0 with $\|u_0\| < \varepsilon_0$, then we say that the trivial solution is asymptotically stable. *If the trivial solution is not stable, we say that it is unstable.*

We are particularly interested in studying (2.1) on large domains, i.e. for $\ell \ll 1$. The next definition is an attempt to characterize stability properties for the "infinite-domain" limit.

Definition 2.2 (Uniform Stability) *On $\Omega = (0, \ell)$, we say that the trivial solution $u = 0$ of (2.1) is* uniformly stable *if, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that, for any size $\ell > 0$ of the domain and any initial condition u_0 with $\|u_0\|_X < \delta$, the associated solution $u(t;u_0)$ of (2.1) satisfies $\|u(t;u_0)\|_X < \varepsilon$ for all times $t \geq 0$. If, in addition, there is an $\varepsilon_0 > 0$ such that $u(t;u_0) \to 0$ in X for all u_0 with $\|u_0\| < \varepsilon_0$ and all $\ell > 0$, then we say that the trivial solution is* uniformly asymptotically stable.

Sometimes, asymptotic stability is defined without requiring stability. Thus, from a slightly different viewpoint, we may be interested in the set of solutions that decay to zero as $t \to \infty$.

Definition 2.3 (Basin of Attraction) *The* basin of attraction $\mathcal{B}$ *of the trivial solution $u = 0$ is the set of those initial data $u_0 \in X$ for which $u(t;u_0) \to 0$ in X as $t \to \infty$. We define the* instability threshold

$$\delta^{\mathrm{u}}(\ell) := \mathrm{dist}_X(\{0\}, \partial\mathcal{B}) := \inf\{\|u_0\|_X;\ u_0 \notin \mathcal{B}\}$$

to be the distance of the boundary $\partial\mathcal{B}$ of the basin of attraction $\mathcal{B}$ to the origin.

The main result of this paper is as follows.

Theorem 1 *Fix μ with $0 < \mu < 1/4$. The trivial solution $u = 0$ is unstable on $\mathbb{R}$ and $\mathbb{R}^{\pm}$, while it is asymptotically stable, but not uniformly stable, on $(0, \ell)$ for each $\ell > 0$. In addition, the following is true. For $\kappa = -1$, we have $\delta^{\mathrm{u}}(\ell) \geq \delta^{\mathrm{u}}_{\infty} > 0$. For $\kappa = 1$, on the other hand, and each $\varepsilon > 0m$ there are constants $c_1, c_2 > 0$ such that*

$$c_1 \mathrm{e}^{-(\nu+\varepsilon)\ell} \leq \delta^{\mathrm{u}}(\ell) \leq c_2 \mathrm{e}^{-(\mu-\varepsilon)\ell}$$

where $\nu = [1 - \sqrt{1-4\mu}]/2$.

3 Spectra, linear, and nonlinear stability

We focus first on the linear equation

$$u_t = u_{xx} + u_x + \mu u \tag{3.1}$$

which we also write in the abstract form $u_t = \mathcal{L}u$ where

$$\mathcal{L}: \quad H^2(\Omega) \cap H^1_0(\Omega) \longrightarrow L^2(\Omega), \qquad u \longmapsto Lu = u_{xx} + u_x + \mu u.$$

The long-time dynamics of solutions to (3.1) can be determined by calculating the spectrum of $\mathcal{L}$.

Lemma 3.1 *The spectrum of the linear operator $\mathcal{L}$ is given by*

$$\begin{aligned}
\operatorname{spec}(\mathcal{L}) &= \left\{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda = -|\operatorname{Im}\lambda|^2 + \mu\right\} && \text{on } \Omega = \mathbb{R} \\
\operatorname{spec}(\mathcal{L}) &= \left\{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda \le -|\operatorname{Im}\lambda|^2 + \mu\right\} && \text{on } \Omega = \mathbb{R}^\pm \\
\operatorname{spec}(\mathcal{L}) &= \left\{\mu - \frac{1}{4} - \left(\frac{\pi k}{\ell}\right)^2;\ k = 1, 2, \ldots\right\} && \text{on } \Omega = (0, \ell).
\end{aligned}$$

On $\Omega = \mathbb{R}^\pm$, the operator $\mathcal{L} - \lambda$ is Fredholm with index $i = \pm 1$ in the region $\operatorname{Re}\lambda < -|\operatorname{Im}\lambda|^2 + \mu$.

Proof. The spectra can, of course, be computed explicitly on $(0, \ell)$ and on $\mathbb{R}$ using Fourier series or Fourier transform. It is also straightforward to see that the spectrum of $\mathcal{L}$ on $\mathbb{R}^+$ contains the spectrum of $\mathcal{L}$ on $\mathbb{R}$. On the other hand, whenever λ is not in the spectrum on $\mathbb{R}$, we find solutions to $[\mathcal{L} - \lambda]u = h$ using the variation-of-constants formula, while it is also easy to see that $[\mathcal{L} - \lambda]$ has a one-dimensional null space on $\mathbb{R}^+$ whenever $\operatorname{Re}\lambda < -|\operatorname{Im}\lambda|^2 + \mu$. ∎

Note that, as $\ell \to \infty$, $\operatorname{spec}_{(0,\ell)}(\mathcal{L})$ converges to the interval $(-\infty, \mu - 1/4]$ in the symmetric Hausdorff distance on any compact part of the complex plane. Invoking a spectral mapping theorem, the spectral result stated in Lemma 3.1 gives immediately linear stability and instability results for (3.1).

Lemma 3.2 *On $\Omega = \mathbb{R}, \mathbb{R}^\pm$, the trivial solution of (3.1) is unstable for $\mu > 0$ and asymptotically stable for $\mu < 0$. On $\Omega = (0, \ell)$, the trivial solution of (3.1) is asymptotically stable for all $\ell > 0$ and all $\mu < 1/4$, while it is unstable for each $\mu > 1/4$ provided ℓ is sufficiently large (namely, when $\ell > \pi[\mu - 1/4]^{-1}$).*

A more refined argument shows that stability is *not* uniform for $0 < \mu < 1/4$.

Lemma 3.3 *Fix μ so that $0 < \mu < 1/4$, then the trivial solution is not uniformly stable on $(0, \ell)$. More precisely, pick any fix μ with $0 < \mu < 1/4$. Fix $\varepsilon > 0$ with $0 < \varepsilon < \mu$ and an integer m with $m > \sqrt{2/\mu}$. Then there is a constant $C(m) > 0$ such that, for any $\ell > 3m$, there exists a smooth positive initial condition $u_0(x)$ with support contained in the interval $[\ell - m, \ell]$ so that $\|u_0\|_{L^\infty} \le C(m)\mathrm{e}^{-(\mu-\varepsilon)\ell}$ and $u(x, \ell - 2m) \ge 1$ for all $x \in [m, 2m]$ where $u(x, t)$ satisfies (3.1) with initial condition u_0.*

Proof. Define

$$u_0(x) := \begin{cases} \rho_0 \sin\left(\frac{\pi}{m}(x - \ell + m)\right) & x \in (\ell - m, \ell) \\ 0 & \text{otherwise} \end{cases}$$

where

$$\rho_0 = \exp\left(-\left[\mu - \frac{2}{m^2}\right](\ell - 2m)\right).$$

We then denote by $w(x, t)$ the function

$$w(x, t) = \exp\left(\left[\mu - \frac{2}{m^2}\right] t\right) u_0(x + t)$$

where $0 \le t \le \ell - 3m$.

We claim that w is a subsolution to the linear equation (4.3). A straightforward computation shows that

$$w_t - w_{xx} - w_x - \mu w = -\frac{1}{m^2} w \le 0$$

whenever (x,t) is in the interior of the support of $\bar{w}$. Since any solution that is pointwise strictly larger than $\bar{w}$ will be non-zero for all positive times, it can only intersect $\bar{w}(x,t)$ at points in the interior of the support of $\bar{w}(x,t)$. This, however, cannot happen due to computation above.

Lastly, we note that

$$w(x, \ell - 2m) \ge \mathrm{e}^{\left(\mu - \frac{2}{m^2}\right)(\ell - 2m)} u_0(x + \ell - 2m). \tag{3.2}$$

In particular, we see that $w(x, \ell - 2m) \ge 1$ for $x \in (m, 2m)$ since we have

$$w(x, \ell - 2m) \ge \mathrm{e}^{(\mu - 2/m^2)(\ell - 2m)} u_0(x + \ell - 2m) = \sin\left(\frac{\pi}{m}(x - m)\right)$$

for such x. ■

The argument in the above proof relies heavily on comparison principles. In the appendix, we give a proof based on properties of the resolvent which can be easily generalized to more complicated equations.

The linear stability results stated above carry over to the nonlinear system.

Proposition 3.4 *On $\Omega = \mathbb{R}, \mathbb{R}^{\pm}$, the trivial solution is asymptotically stable for the nonlinear equation (2.1) when $\mu < 0$ and unstable when $\mu > 0$. On $\Omega = (0, \ell)$, the trivial solution of (2.1) is asymptotically stable for all $\ell > 0$ whenever $\mu < 1/4$, while it is unstable for $\mu > 1/4$. The trivial solution of (2.1) is not uniformly stable for $0 < \mu < 1/4$.*

Proof. The statements follow easily from the contraction mapping theorem (to prove stability) and by establishing lower growth estimates (to prove instability). It remains to prove that $u = 0$ is not uniformly stable for $0 < \mu < 1/4$. We argue by contradiction and choose the constant ε in Definition 2.1 so small that $\mu - 4\varepsilon^2 > 0$. Thus, as long as the solution $u(t; u_0)$ satisfies $\|u(t; u_0)\|_{L^\infty} \le 2\varepsilon$, we have that $\mu + \kappa u^2 \ge \mu - 4\varepsilon^2$, and we conclude that solutions of the linear equation

$$w_t = w_{xx} + w_x + (\mu - 4\varepsilon^2) w$$

are subsolutions to the nonlinear equation (2.1). In particular, Lemma 3.3 tells us that, for all sufficiently large ℓ, there are arbitrarily small initial data that will eventually be of size equal to 2ε. ■

4 The basin of attraction

The results reviewed so far demonstrate a crucial difference between large but bounded intervals $(0, \ell)$ and the unbounded domains $\mathbb{R}$ and $\mathbb{R}^{\pm}$ in the convectively unstable regime $0 < \mu < 1/4$:

the equilibrium $u = 0$ is unstable on unbounded domains but stable on any bounded interval. The instability becomes visible on bounded intervals when we check for uniform stability (see Definition 2.2). Indeed, over time intervals of length ℓ, we observe a long transient dynamics for the linear equation (3.1) during which perturbations grow in amplitude before they eventually begin to decay and ultimately converge to zero. In this section, we focus on the role of the nonlinearity during the transient. It turns out that subcritical nonlinearities enhance the growth during the transient and prevent the eventual decay of solutions towards zero.

Thus, we consider equation (2.1)

$$u_t = u_{xx} + u_x + \mu u + \kappa u^3 \tag{4.1}$$

on the interval $(0, \ell)$ in the convectively unstable regime $0 < \mu < 1/4$ with subcritical or supercritical nonlinearity. We denote by $\mathcal{B}_\ell$ the basin of attraction of the trivial solution $u = 0$. The following results give estimates for the instability threshold $\delta^{\mathrm{u}}(\ell)$ that we defined in Definition 2.3.

Proposition 4.1 (Supercritical basin boundaries) *Fix μ with $0 < \mu < 1/4$, and suppose that $\kappa = -1$, then $\delta^{\mathrm{u}}(\ell) \geq \delta^{\mathrm{u}}_\infty > 0$ for all $\ell > 0$. In fact, all solutions converge to zero in H^1.*

Proof. In the supercritical case, it is easy to verify that there are no equilibria other than $u = 0$. Given the gradient structure of the problem and a priori bounds from standard energy estimates, we conclude that $B_\ell(0) = X$. ∎

Proposition 4.2 (Subcritical basin boundaries) *Fix $0 < \mu < 1/4$, suppose that $\kappa = 1$, and let $\nu = [1 - \sqrt{1 - 4\mu}]/2$. Then there exists an $\varepsilon_0 > 0$ such that the following is true for each ε with $0 < \varepsilon < \varepsilon_0$. There are constants $c_1, c_2 > 0$ and $\ell_0 \ll 1$ such that*

$$c_1 \mathrm{e}^{-(\nu+\varepsilon)\ell} < \delta^{\mathrm{u}}(\ell) < c_2 \mathrm{e}^{-(\mu-\varepsilon)\ell} \tag{4.2}$$

for all $\ell \geq \ell_0$. In particular, $\delta^{\mathrm{u}}(\ell) \to 0$ as $\ell \to \infty$.

The constant ν arises as the smaller root of the quadratic equation $\rho^2 - \rho + \mu = 0$. In particular, we have $\mu < \nu$ for the relevant values of μ, so that (4.2) makes sense. The inequality $\mu < \nu$ indicates that the estimate (4.2) is not sharp. We believe that a sharp estimate would involve only the spatial eigenvalue ν, so that the exponent in the second inequality in (4.2) should be $\nu - \varepsilon$ instead of $\mu - \varepsilon$.

Proof. We begin by providing a lower bound for the stability threshold. For any solution $u(x, t)$ of (4.1), we set $v(x, t) = \mathrm{e}^{\rho x} u(x, t)$ so that $v(x, t)$ satisfies

$$v_t = v_{xx} + (1 - 2\rho) v_x + (\rho^2 - \rho + \mu) v + \mathrm{e}^{-2\rho x} v^3.$$

Note that the linear driving $\rho^2 - \rho + \mu$ is negative if we evaluate it at $\rho = \nu - \varepsilon$. The maximum principle then implies that $|v(x, t)| \leq \sqrt{-\rho^2 + \rho - \mu}$ for all $t \geq 0$ if this inequality is true at $t = 0$ (here, we simply estimate the exponential term in the nonlinearity by 1). Transforming back to the solution $u(x, t)$ establishes the lower bound.

To prove the upper bound, we construct a non-negative initial condition with compact support in $(0,\ell)$ centered close to the boundary $x=\ell$, so that the associated solution blows up in finite time. We will show that the diameter of the support of this initial condition can be chosen independently of ℓ and that the supremum of the initial condition is of the order $\mathrm{O}(\mathrm{e}^{-(\mu-\varepsilon)\ell})$. Our strategy is as follows. We first estimate the solution from below by the solution to the linear equation on a time scale $\mathrm{O}(\ell)$. We then exploit the nonlinearity to show blowup on a time scale $\mathrm{O}(1)$.

First, for any non-negative initial condition $u(x,0)$, we note that the associated solution $u(x,t)$ is pointwise bounded from below by the positive solution $w(x,t)$ of the linear equation

$$w_t = w_{xx} + w_x + \mu w, \qquad w(x,0) = u(x,0), \tag{4.3}$$

since w will be a subsolution of the nonlinear equation.

By Lemma 3.3, we may find initial conditions $u_0(x)$ of amplitude $\mathrm{O}(\mathrm{e}^{-(\mu-\varepsilon)\ell})$ such that the solution $u(x,t)$ is larger than 1 on the interval $(2m,6m)$ at time $t=\mathrm{O}(\ell)$. We pick this function as our new initial condition and show that it blows up after time $t\le 1$.

Denote by $\chi:\mathbb{R}\to\mathbb{R}$ a smooth even cut-off function with the following properties:

$$\chi(\xi) := \begin{cases} 1 & \text{for} \quad 0\le\xi\le 1 \\ (2-\xi)^3 & \text{for} \quad 1.75\le\xi\le 2 \\ 0 & \text{for} \quad 2\le\xi \\ \chi(-\xi) & \text{for} \quad \xi\le 0 \end{cases}$$

so that $\chi'(\xi)\le 0$ for $\xi\ge 0$. We define $\chi_m(\xi) := \chi(\xi/m)$ for any constant $m>0$ and note that $\chi_m(\xi)$ vanishes for $|\xi|>2m$. Below, we shall choose $m\gg 1$ large and exploit the fact that $\chi_m' = \mathrm{O}(1/m)$ and $\chi_m'' = \mathrm{O}(1/m^2)$.

Suppose that $u(x,t)$ is a non-negative solution of (2.1) on $(0,\ell)$ with Dirichlet boundary conditions. We consider the localized function

$$\check{u}(x,t) := \chi_m(x+t+4m)u(x,t)$$

so that $\operatorname{supp}\check{u}\subset[m-t,2m-t]$. In particular, we may consider $\check{u}$ in a co-moving frame $\xi = x+t+2m-\ell$. The resulting function

$$w(\xi,t) := \check{u}(\xi-t-4m,t) = \chi_m(\xi)u(\xi-t-4m,t)$$

satisfies the equation

$$w_t = w_{\xi\xi} + \mu w + w^3 - \chi_m'' u - 2\chi_m' u_\xi + \left[\chi_m - \chi_m^3\right]u^3 \tag{4.4}$$

where χ_m and u are evaluated at ξ and $(\xi-t-4m,t)$, respectively. We consider (4.4) on the spatial domain $\xi\in(-2m,2m)$ and on the finite time interval $0\le t\le m$ so that u and w are well-defined for $\xi\in(-2m,2m)$. Note that

$$w(t,-2m) = w(t,2m) = 0, \qquad w_\xi(t,-2m) = w_\xi(t,2m) = 0 \tag{4.5}$$

since $\chi_m(\xi)$ and $\chi_m'(\xi)$ vanish at $\xi = \pm 2m$. We use the notation

$$[u] := \frac{1}{4m}\int_{-2m}^{2m} u(\xi)\,\mathrm{d}\xi, \qquad \langle u, v\rangle := \frac{1}{4m}\int_{-2m}^{2m} u(\xi)v(\xi)\,\mathrm{d}\xi.$$

Upon integrating (4.4) over ξ, we find

$$\begin{aligned} [w]_t &= [w_{\xi\xi}] + \mu[w] + [w^3] - \langle \chi_m'', u\rangle - 2\langle \chi_m', u_\xi\rangle + \langle \chi_m - \chi_m^3, u^3\rangle \\ &= \mu[w] + [w^3] - \langle \chi_m'', u\rangle - 2\langle \chi_m', u_\xi\rangle + \langle \chi_m - \chi_m^3, u^3\rangle \\ &= \mu[w] + [w^3] + \langle \chi_m'', u\rangle + \langle \chi_m - \chi_m^3, u^3\rangle. \end{aligned} \tag{4.6}$$

In the second equality, we exploited that

$$\int_{-2m}^{2m} w_{\xi\xi}(\xi)\,\mathrm{d}\xi = w_\xi\Big|_{2m}^{2m} = 0$$

due to (4.5). Similarly, in the third equality in (4.6), we used that

$$\langle \chi_m', u_\xi\rangle = -\langle \chi_m'', u\rangle$$

since $\chi_m(\xi)$ vanishes at $\xi = \pm 2m$. Next, we use Hölder's inequality ($\|fg\|_1 \leq \|f\|_p\,\|g\|_q$ for $\frac{1}{p} + \frac{1}{q} = 1$) to estimate

$$[w^3] = \int_{-2m}^{2m} \frac{w(\xi)^3}{4m}\,\mathrm{d}\xi \geq \left(\int_{-2m}^{2m} \frac{w(\xi)}{4m}\,\mathrm{d}\xi\right)^3 = [w]^3$$

(take $\varphi = 1/(4m)$, $f = w\varphi^{\frac{1}{3}}$, $g = \varphi^{\frac{2}{3}}$, $p = 3$ and $q = \frac{3}{2}$, and use $\int \varphi = 1$).

Thus, we conclude that

$$[w]_t \geq \mu[w] + [w]^3 + \langle \chi_m'', u\rangle + \langle \chi_m - \chi_m^3, u^3\rangle. \tag{4.7}$$

Next, we discuss the two remaining terms that involve u. Note that we can write $\chi_m''(\xi) = \mathrm{O}(1/m^2)\chi_m(\xi)$ as long as ξ is not close to $\pm 2m$. We conclude that

$$\left|\int_{-2m(1-\delta)}^{2m(1-\delta)} \chi_m'' u\right| \leq \mathrm{O}\left(\frac{1}{m^2}\right)[w].$$

In a neighborhood of the boundary, we use the explicit formula $\chi_m(\xi) = (\xi+2)^3/m^3$ for the cut-off function. Therefore, denoting by C various positive constants that are, however, independent of m, we obtain

$$\left|\int_{-2m}^{-2m(1-\delta)} \chi_m'' u\right| \leq \frac{C}{m^2}\int_{-2m}^{-2m(1-\delta)} \chi_m^{1/3} u \leq \frac{C}{m^2}\int_{-2m}^{-2m(1-\delta)} (1+\chi_m u^3) \leq \frac{C}{m} + \frac{C}{m^2}\int_{-2m}^{-2m(1-\delta)} \chi_m u^3.$$

Lastly, we note that

$$\begin{aligned} &-\frac{C}{m^2}\int_{-2m}^{-2m(1-\delta)} \chi_m u^3\,\mathrm{d}\xi + \int_{-2m}^{-2m(1-\delta)} (\chi_m - \chi_m^3)u^3\,\mathrm{d}\xi \\ &\quad = \int_{-2m}^{-2m(1-\delta)} \left[\chi_m\left(1 - \mathrm{O}\left(\frac{1}{m^2}\right)\right) - \chi_m^3\right] u^3\,\mathrm{d}\xi \;\geq\; 0 \end{aligned}$$

since $u \geq 0$ and $\chi(1-\mathrm{O}(1/m^2)) \geq \chi^3$ for $|\xi| \geq \xi_0 > 1$.

Combining the above estimates for the two terms in (4.7) that involve u, we find that

$$\langle \chi_m'', u\rangle + \langle \chi_m - \chi_m^3, u^3\rangle \geq -\mathrm{O}\left(\frac{1}{m}\right) - \mathrm{O}\left(\frac{1}{m^2}\right)[w]$$

and therefore

$$[w]_t \geq \mu[w] + [w]^3 + \langle \chi_m'', u\rangle + \langle \chi_m - \chi_m^3, u^3\rangle \geq \mu[w] + [w]^3 - \mathrm{O}\left(\frac{1}{m^2}\right)[w] - \mathrm{O}\left(\frac{1}{m}\right).$$

Writing $\rho = [w]$, we obtain the differential inequality

$$\rho_t \geq \frac{\mu}{2}\rho + \rho^3 - \frac{C}{m} \tag{4.8}$$

for m sufficiently large. By assumption, $\rho(0) \geq 1$, such that $\rho(t) = \infty$ for $t \geq 1$ for m sufficiently large. ∎

We note that (4.8) does not provide satisfactory estimates for small w. Only initial conditions with $\rho \gg 1/m \geq 1/\ell$ would give $\rho = \infty$ after a finite time.

5 Bifurcations at the onset to an absolute instability

As we vary the parameter μ, the trivial solution $u = 0$ destabilizes for $\mu = 0$ on $\Omega = \mathbb{R}$ and near $\mu = 1/4$ on $\Omega = (0, \ell)$. We focus here on an analysis of the instability at $\mu = 1/4$ on large but bounded intervals. Specifically, we are interested in patterns that bifurcate at this instability and in the dependence of their amplitude on the domain size ℓ. Due to the gradient-like structure of (2.1), bifurcating solutions will be steady states. Furthermore, using the maximum principle, it is not hard to see that there exists at most one positive solution for each fixed ℓ and μ. In fact, the unique positive steady-state bifurcates from $u = 0$ at $\mu_\mathrm{c} = 1/4 + \pi^2/\ell^2$. This bifurcation is subcritical for $\kappa = 1$ and supercritical for $\kappa = -1$.

Our goal is to derive expansions for the amplitude of the bifurcating positive steady state. Classical bifurcation theory predicts a growth in amplitude that scales like

$$|u| \sim A(\ell)\sqrt{\mu - \mu_\mathrm{c}}.$$

It seems difficult, however, to extract rigorous scaling laws for the dependence of the factor $A(\ell)$ on ℓ from such an analysis. Indeed, for large $\ell \gg 1$, eigenvalues form clusters and the norm of the associated spectral projections diverge. Still, as we shall see below, the scaling predicted by our analysis agrees with the formal results in [6, 7, 14, 15], although the range of validity predicted here is considerably larger than that mentioned in [6, 7, 14, 15].

We start with an analysis of the subcritical case $\kappa = 1$ and outline the differences in the supercritical case later. Throughout the rest of this section, we set

$$\mu = \sigma + \frac{1}{4}$$

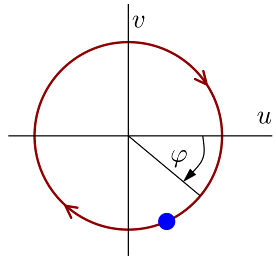

Figure 5.1: *The choice of polar coordinates used in the blowup and the dynamics on the circle $r = 0$ for $\sigma = 0$ are illustrated.*

and write the steady-state equation associated with (1.1) as the first-order differential equation

$$\begin{aligned} u_x &= v \\ v_x &= -v - \left(\sigma + \frac{1}{4}\right) u - u^3. \end{aligned} \tag{5.1}$$

The linearization about the equilibrium $(u, v) = 0$ changes its type from a node to a focus at $\sigma = 0$. We employ blow-up techniques to analyse the resulting bifurcation. Thus, we use polar coordinates

$$\tan\varphi = -\frac{v}{u}, \qquad r = \sqrt{u^2 + v^2},$$

which gives

$$\begin{aligned} r_x &= r\left[\left(\sigma - \frac{3}{4}\right)\cos\varphi\sin\varphi - \cos^2\varphi + r^2\cos^3\varphi\sin\varphi\right] \\ \varphi_x &= \left(\frac{1}{2}\cos\varphi - \sin\varphi\right)^2 + \sigma\cos^2\varphi + r^2\cos^4\varphi. \end{aligned} \tag{5.2}$$

Note that the set of all points (u, v) that satisfy Dirichlet boundary conditions and that lead to solutions in the half space $u > 0$ is given by $\varphi_-^* = -\pi/2$ at $x = 0$ and by $\varphi_+^* = \pi/2$ at $x = \ell$.

The circle $r = 0$, which corresponds to the equilibrium $(u, v) = 0$, is invariant under (5.2). For $\sigma = 0$, there is exactly one equilibrium given by $\tan\varphi_* = 1/2$ on the part of the circle in $u > 0$. At $\sigma = 0$, this equilibrium undergoes a saddle-node bifurcation, so that there are two equilibria for $\sigma < 0$, which correspond to the stable and the strong stable eigenspaces, and no equilibria for $\sigma > 0$. In fact, close to $\sigma = 0$ and $\varphi = \varphi_*$, we have the expansion

$$\varphi_x = \frac{5}{4}(\varphi - \varphi_*)^2 + \frac{4\sigma}{5} + \mathrm{O}\left(\sigma^2 + |\sigma||\varphi - \varphi_*| + (|\sigma| + |\varphi - \varphi_*|)^3\right). \tag{5.3}$$

Since the equilibria on the circle at $\sigma = 0$ are linearly stable in the direction normal to the circle, the circle is a normally hyperbolic, attracting invariant manifold. In particular, there exists a smooth strong stable foliation, so that we can bring (5.2) into the form

$$\begin{aligned} r_x &= r\,\mathcal{R}(r, \phi; \sigma) \\ \phi_x &= \left(\frac{1}{2}\cos\phi - \sin\phi\right)^2 + \sigma\cos^2\phi \end{aligned} \tag{5.4}$$

by a smooth change of coordinates.

Since the linearization of (5.2) at $r = 0$ is already of the form (5.4), the coordinate change is given by $\phi = \varphi + \mathrm{O}(r^2)$, so that the boundary conditions transform into

$$\begin{aligned}(r, \phi) &= (r, \varphi_-^* + b_- r^2 + \mathrm{O}(r^4)) \qquad \text{at } x = 0 \\ (r, \phi) &= (r, \varphi_+^* + b_+ r^2 + \mathrm{O}(r^4)) \qquad \text{at } x = \ell\end{aligned} \tag{5.5}$$

for certain numbers $b_\pm \in \mathbb{R}$. We claim that $b_\pm > 0$. Indeed, upon substituting $\phi = \varphi + b(\varphi) r^2 + \mathrm{O}(r^4)$ into (5.2), we obtain the equation

$$\left(\frac{1}{2}\cos\varphi - \sin\varphi\right)^2 b' = (1 + 3\cos\varphi\sin\varphi) b - \cos^4\varphi \tag{5.6}$$

for $b(\varphi)$. In particular, we see that $b(\varphi_*) = 16/55 > 0$. A qualitative analysis of (5.6) shows that $b_\pm = b(\varphi_\pm^*) > 0$ as claimed. We remark that the same analysis shows that $b_- < 0$ in the supercritical case.

Since $b_\pm > 0$, we can invert (5.5) and find that

$$r_- = \sqrt{\frac{\phi_- - \varphi_-^*}{b_-}}\,[1 + \mathrm{O}(\phi_- - \varphi_-^*)], \qquad r_+ = \sqrt{\frac{\phi_+ - \varphi_+^*}{b_+}}\,[1 + \mathrm{O}(\phi_+ - \varphi_+^*)]. \tag{5.7}$$

Suppose now that we have found a solution of (5.4) so that the ϕ-component is equal to ϕ_- at $x = 0$ and ϕ_+ at $x = \ell$. Expanding the time ℓ in terms of (ϕ_-, ϕ_+, μ) using equation (5.4) and the expansion (5.3), we obtain

$$\ell = \ell_0 + \frac{\pi}{\sqrt{\sigma}} - [\phi_- - \varphi_-^*] + \mathrm{O}(\sqrt{\sigma} + |\phi_+ - \varphi_+^*| + |\phi_- - \varphi^*|^2). \tag{5.8}$$

To find a solution to the boundary-value problem, we need that r evaluated at $x = 0$ and $x = \ell$ matches the boundary conditions. In particular, we need that $r(\ell) = \Phi_\ell^r(r(0), \phi_-)$ satisfies the second equation in (5.7), where Φ^r denotes the r-component of the flow associated with (5.4). The resulting condition reads

$$\sqrt{\frac{\phi_+ - \varphi_+^*}{b_+}}\,[1 + \mathrm{O}(\phi_+ - \varphi_+^*)] = \Phi_\ell^r\left(\sqrt{\frac{\phi_- - \varphi_-^*}{b_-}}\,[1 + \mathrm{O}(\phi_- - \varphi_-^*)], \phi_-\right).$$

Since the flow in the strong stable fibers is decaying exponentially with rate $-1/4$ (in fact, with rate $-1/2 + \varepsilon$ for any small $\varepsilon > 0$), we see that

$$\phi_+ - \varphi_+^* = \mathrm{O}\left(\mathrm{e}^{-\ell/4}\right).$$

Substituting this result into (5.8) gives the desired relation

$$\ell - \ell_0 - \frac{\pi}{\sqrt{\sigma}} = -[\phi_- - \varphi_-^*] + \mathrm{O}\left(\sqrt{\sigma} + \mathrm{e}^{-\ell/4} + |\phi_- - \varphi^*|^2\right) \tag{5.9}$$

between ℓ, σ and ϕ_-. We observe that the supremum norm of u is related via diffeomorphism to the value of r evaluated at $x = 0$ and that this relation is uniform in ℓ. Thus, we may measure the amplitude growth of the solution by expressing r as a function of $\phi_-(\ell, \sigma)$. We therefore set

$$\ell_1 := \ell - \ell_0 - \frac{\pi}{\sqrt{\sigma}}$$

and observe that ℓ_1 is close to zero when ϕ_- is close to φ_-^*. For $\phi_- = \varphi_-^*$, we see that $\ell_1 = \mathrm{O}(\sqrt{\sigma})$ which determines a unique parameter value

$$\sigma_{\mathrm{c}} = \frac{\pi^2}{\ell^2} + \mathrm{O}(\ell^{-3})$$

for the onset. It is convenient to set $\Delta\sigma := \sigma - \sigma_{\mathrm{c}}$ so that

$$\sigma = \Delta\sigma + \frac{\pi^2}{\ell^2} + \mathrm{O}(\ell^{-3}).$$

Substituting these results and definitions into (5.9), we obtain

$$\phi_- - \varphi_-^* = -\frac{2\ell^3\Delta\sigma}{\pi^2} + \mathrm{O}\left(\ell^5|\Delta\sigma|^2 + \ell\Delta\sigma\right)$$

and therefore, upon using (5.7),

$$r(0) = \sqrt{\frac{2}{\pi^2 b_-}}\,\ell^{3/2}\sqrt{-\Delta\sigma} + \mathrm{O}\left(\ell^3\Delta\sigma\right).$$

We emphasize that the expansion is valid uniformly in ℓ for small amplitudes (i.e. when $\ell^3\Delta\sigma$ is small).

The result proved above agrees with formal Lyapunov-Schmidt computations as carried out in [14, 15]. Indeed, the null space of the linearization of (2.1) on $(0,\ell)$ about $u=0$ at $\sigma = \pi^2/\ell^2$ is given by $p(x) = \mathrm{e}^{-x/2}\sin(\pi x/\ell)$, while the associated adjoint eigenfunction is $p_*(x) = \mathrm{e}^{x/2}\sin(\pi x/\ell)$. We parameterize the bifurcating solutions over the null space by writing $u(x) = Ap(x) + \mathrm{O}(A^2)$. Evaluating the nonlinearity at $u = Ap$ and projecting onto the null space using the spectral projection gives

$$\Delta\sigma\langle p_*, p\rangle_{L^2} + A^2\langle p_*, p^3\rangle_{L^2} = 0$$

which, after evaluating the integrals, shows that

$$A = \frac{\ell^{5/2}\sqrt{-\Delta\sigma}}{4\pi^2\sqrt{3}}.$$

Since

$$\sup_{x\in[0,\ell]} p(x) = \frac{2\pi}{\mathrm{e}\ell} + \mathrm{O}(\ell^{-2}),$$

we see that

$$\sup_{x\in[0,\ell]} u(x) = \frac{\ell^{3/2}\sqrt{-\Delta\sigma}}{2\pi\mathrm{e}\sqrt{3}} + \mathrm{O}\left(\sqrt{\ell|\Delta\sigma|} + \ell^{3/2}\Delta\sigma\right).$$

A An alternative proof of Lemma 3.3

We give proofs of Lemma 3.3 and Proposition 4.2 that do not use the maximum principle. In addition, the proof presented here gives the following sharper bounds on the instability threshold.

Proposition A.1 *Suppose that $\kappa = 1$, then there is a constant $C > 0$ such that $\delta^{\mathrm{u}}(\ell) < C\sqrt{\ell}\mathrm{e}^{-\mu\ell}$.*

To prove the proposition, we fix μ subject to $0 < \mu < 1/4$ and consider the nonlinear equation

$$u_t = u_{xx} + u_x + \mu u + u^3 \tag{A.1}$$

on $(0, \ell)$ with Dirichlet boundary conditions $u(0) = u(\ell) = 0$. Note that a non-negative solution $u(x, t)$ of (A.1) can be bounded pointwise from below by the solution $w(x, t)$ of the linear equation

$$w_t = w_{xx} + w_x + \mu w, \qquad w(x, 0) = u_0(x) \tag{A.2}$$

on $(0, \ell)$ again with zero Dirichlet boundary conditions. Below, we will choose specific smooth initial data $u_0(x)$ with compact support in $(0, \ell)$. Using Laplace transform, we see that the corresponding solution of (A.2) is given explicitly by

$$w(x, t) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \int_0^\ell \mathrm{e}^{\lambda t} G_\lambda(x, y) u_0(y)\, \mathrm{d}y\, \mathrm{d}\lambda, \qquad \Gamma = \{-k^2 + \mathrm{i}k + \mu;\ k \in \mathbb{R}\}, \tag{A.3}$$

where G_λ is the Green's function of

$$w_{xx} + w_x + \mu w = \lambda w, \qquad w(0) = w(\ell) = 0$$

and where we integrate along the contour Γ which is the essential spectrum of the problem posed on $\mathbb{R}$. Note that the integrals in (A.3) converge absolutely for any fixed $t > 0$ and each bounded and continuous function $u_0(x)$.

We are interested in the limit $\ell \to \infty$. Therefore, we denote by $G_\lambda^\infty(x, y)$ the Green's function of

$$w_{xx} + w_x + \mu w = \lambda w, \qquad x \in \mathbb{R}.$$

Taking the boundary conditions into account, we see that there is a function $R_\lambda(x, y; \ell)$ such that

$$G_\lambda(x, y) = G_\lambda^\infty(x, y)[1 + R_\lambda(x, y; \ell)]. \tag{A.4}$$

In fact, using, for instance, exponential dichotomies and inclination lemmas, it is not difficult to see (see, for example, [11, Eqns. (4.11) and (4.14) in §4.4]) that there is a constant $C > 0$ that does not depend on $\lambda \in \Gamma$ such that

$$|R_\lambda(x, y; \ell)| \leq C\mathrm{e}^{-\min\{x, \ell - x\}/2}\, \mathrm{e}^{-\min\{y, \ell - y\}/2} \tag{A.5}$$

for all $x, y \in (0, \ell)$ and $\lambda \in \Gamma$. The exponential rate $-1/2$ in the estimate for R_λ is a consequence of the fact that the spatial roots ρ_j of the characteristic polynomial $\rho^2 + \rho + \mu - \lambda = 0$ add up to -1 since the convection term w_x has a factor of one.

Substituting (A.4) into (A.3) gives

$$\begin{aligned} w(x, t) &= \frac{1}{2\pi \mathrm{i}} \int_\Gamma \int_{\operatorname{supp} u_0} \mathrm{e}^{\lambda t} G_\lambda^\infty(x, y)[1 + R_\lambda(x, y; \ell)] u_0(y)\, \mathrm{d}y\, \mathrm{d}\lambda \\ &= \frac{1}{2\pi} \int_\mathbb{R} \int_{\operatorname{supp} u_0} \mathrm{e}^{(-k^2 + \mathrm{i}k)t} \mathrm{e}^{-(\mathrm{i}k + \frac{1}{2} + \mu)|x - y| - \frac{x - y}{2}} \left[1 + R_{-k^2 + \mathrm{i}k + \mu}(x, y; \ell)\right] u_0(y)\, \mathrm{d}y\, \mathrm{d}k. \end{aligned}$$

In the next step, we transform into the co-moving frame $\xi = x + t$. We therefore set $w(x,t) = \breve{w}(x+t,t)$ so that

$$\begin{aligned}\breve{w}(\xi,t) &= \frac{1}{2\pi\mathrm{i}}\int_{\breve{\Gamma}}\int_{\operatorname{supp} u_0} \mathrm{e}^{\lambda t} G_\lambda^\infty(\xi-t,y)[1+R_\lambda(\xi-t,y;\ell)]u_0(y)\,\mathrm{d}y\,\mathrm{d}\lambda \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\operatorname{supp} u_0} \mathrm{e}^{(-k^2+\mu)t}\mathrm{e}^{-\mathrm{i}k(\xi-y)}[1+R_{-k^2+\mathrm{i}k+\mu}(\xi-t,y;\ell)]u_0(y)\,\mathrm{d}y\,\mathrm{d}k,\end{aligned}$$

where the first integral is evaluated along $\breve{\Gamma} = \{\lambda = -k^2+\mu;\ k\in\mathbb{R}\}$. We now evaluate the solution $\breve{w}(\xi,t)$ at $t = T = \ell - 3m$, where m is large but bounded with respect to ℓ. We also choose as initial condition the function

$$u_0(y) = \begin{cases} 1 & \text{for} \quad y \in (\ell-2m, \ell-m) \\ 0 & \text{for} \quad y \in \mathbb{R}\setminus(\ell-2m,\ell-m). \end{cases}$$

As a consequence, we conclude from (A.5) that the error term $R_\lambda(\xi-t,y;\ell)$ is exponentially small and, in fact, bounded by $\mathrm{O}(\mathrm{e}^{-m/2})$. Choose $\varepsilon \ll 1/m^2$ and write

$$\begin{aligned}\breve{w}(\xi,T) &= \int_{\mathbb{R}}\int_{\ell-2m}^{\ell-m} \mathrm{e}^{(\mu-k^2)T}\mathrm{e}^{-\mathrm{i}k(\xi-y)}[1+R_{-k^2+\mathrm{i}k+\mu}(\xi-T,y;\ell)]\,\mathrm{d}y\,\mathrm{d}k \\ &= \left(\int_{|k|\le\varepsilon} + \int_{|k|\ge\varepsilon}\right)\int_{\ell-2m}^{\ell-m} \mathrm{e}^{(\mu-k^2)T}\mathrm{e}^{-\mathrm{i}k(\xi-y)}[1+R_{-k^2+\mathrm{i}k+\mu}(\xi-T,y;\ell)]\,\mathrm{d}y\,\mathrm{d}k\end{aligned}$$

where we also substituted the initial condition. We estimate these integrals separately. First, by construction, we have

$$\begin{aligned}&\left|\int_{|k|\ge\varepsilon}\int_{\ell-2m}^{\ell-m} \mathrm{e}^{(\mu-k^2)T}\mathrm{e}^{-\mathrm{i}k(\xi-y)}[1+R_{-k^2+\mathrm{i}k+\mu}(\xi-T,y;\ell)]\,\mathrm{d}y\,\mathrm{d}k\right| \\ &\quad\le\ C(\varepsilon,m)\mathrm{e}^{(\mu-\varepsilon^2)T} = C(\varepsilon,m)\mathrm{e}^{(\mu-\varepsilon^2)\ell}.\end{aligned}$$

The remaining integral can be computed as follows:

$$\begin{aligned}&\int_{|k|\le\varepsilon}\int_{\ell-2m}^{\ell-m} \mathrm{e}^{(\mu-k^2)T}\mathrm{e}^{-\mathrm{i}k(\xi-y)}[1+R_{-k^2+\mathrm{i}k+\mu}(\xi-T,y;\ell)]\,\mathrm{d}y\,\mathrm{d}k \\ &= \int_{|k|\le\varepsilon} \mathrm{e}^{(\mu-k^2)T}\mathrm{e}^{-\mathrm{i}k\xi}\left[\frac{\mathrm{e}^{\mathrm{i}k(\ell-m)}-\mathrm{e}^{\mathrm{i}k(\ell-2m)}}{\mathrm{i}k} + \mathrm{O}(\mathrm{e}^{-m/2}\right]\mathrm{d}k \\ &= \int_{|k|\le\varepsilon} \mathrm{e}^{(\mu-k^2)T}\left[\mathrm{e}^{-\mathrm{i}k\xi} + \mathrm{O}(k)\right]\mathrm{d}k \\ &= \mathrm{e}^{\mu T}\left[\frac{1+\mathrm{O}(\varepsilon)}{\sqrt{T}} + \mathrm{O}\left(\frac{1}{T}\right)\right],\end{aligned}$$

where we used that $\varepsilon \ll 1/m^2 \ll \mathrm{e}^{-m/2}$ and that the constants in $\mathrm{O}(k)$ are uniform in m and ε. Tracing our coordinate changes back to the original variables completes the proof of Proposition A.1.

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