Exponential dichotomies for solitary-wave solutions of semilinear elliptic equations on infinite cylinders

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Abstract

In applications, solitary-wave solutions of semilinear elliptic equations

$$\Delta u + g(u, \nabla u) = 0 \qquad (x, y) \in \mathbb{R} \times \Omega$$

in infinite cylinders frequently arise as travelling waves of parabolic equations. As such, their bifurcations are an interesting issue. Interpreting elliptic equations on infinite cylinders as dynamical systems in x has proved very useful. Still, there are major obstacles in obtaining, for instance, bifurcation results similar to those for ordinary differential equations. In this article, persistence and continuation of exponential dichotomies for linear elliptic equations is proved. With this technique at hands, Lyapunov-Schmidt reduction near solitary waves can be applied. As an example, existence of shift dynamics near solitary waves is shown if a perturbation $\mu\, h(x, u, \nabla u)$ periodic in x is added.

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1 Introduction

In this article, semilinear elliptic equations

$$u_{xx} + \Delta_y u + g(y, u, u_x, \nabla_y u) = 0 \qquad (x, y) \in \mathbb{R} \times \Omega, \tag{1.1}$$

in infinite cylinders $\mathbb{R} \times \Omega$ are investigated. Here, Ω is an open and bounded subset of $\mathbb{R}^n$, and boundary conditions on $\mathbb{R} \times \partial\Omega$ should be added. Solitary waves are localized solutions $u(x, y)$ of (1.1) satisfying

$$\lim_{|x| \to \infty} u(x, y) = 0$$

uniformly for $y \in \Omega$. In applications, they frequently arise as travelling waves $u(x - ct, y)$ for parabolic equations

$$u_t = u_{xx} + \Delta_y u + g(y, u, u_x, \nabla_y u) - cu_x \qquad (x, y) \in \mathbb{R} \times \Omega. \tag{1.2}$$

As such, their bifurcations to periodic waves or N-solitary waves resembling N copies of a primary solitary wave are interesting issues. Of importance is also the question of their stability with respect to the parabolic equation (1.2). Another aspect is the numerical computation of solitary-wave solutions since it is in general impossible to obtain explicit expressions. Typical applications include problems in structural mechanics like rods and struts, chemical kinetics, combustion, and nerve impulses, see, for instance, [30] and the comprehensive bibliography there. Existence of solitary waves or fronts has been proven for many equations of the form (1.1), see again [30, Section 1.6.6] for references. Thus, in this paper, we will assume that a solitary wave of (1.1) exists, and shall study its bifurcations.

In order to investigate elliptic equations in cylinders $\mathbb{R} \times \Omega$, it has proved very useful to consider them a dynamical system in the unbounded variable x. Properties like dissipativity, reversibility, Hamiltonian structure, and zero numbers have been exploited in order to describe bounded solutions of such equations, see, for example, [4, 8, 16, 19, 21, 28]. The main technique has been reduction to local center or global essential manifolds containing some or all bounded solutions of (1.1). For instance, Mielke derived bifurcation equations close to stationary [19] and periodic [7, Chapter 4] solutions on a center manifold.

However, the use of geometric reductions like local center or global essential manifolds is limited. Finite-dimensional essential or inertial manifolds are only C^1 smooth. Also, the reduction requires spectral gaps and works only for particular nonlinearities, see [20, 21]. On the other hand, finite-dimensional smooth local center manifolds exist only in the neighborhood of small solutions. Using analytical methods like Lyapunov-Schmidt reduction near solutions of (1.1) with large amplitudes resolves some of these problems.

Therefore, rather than studying the set of all bounded solutions of (1.1), we shall only investigate solutions close to solitary waves hoping to get a more detailed picture of the nearby dynamics. Interpreting the variable x as time, we write (1.1) as the first order system

$$\begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ -\Delta_y & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} 0 \\ g(y,u,v,\nabla_y u) \end{pmatrix}. \tag{1.3}$$

Here, for each fixed $x \in \mathbb{R}$, $(u,v)(x)$ is a function of $y \in \Omega$ contained in some function space depending on the boundary conditions on $\partial\Omega$. A solitary wave of (1.1) corresponds to a homoclinic orbit of (1.3), that is to a solution $(q(x), q_x(x))$ of (1.3) with $\lim_{|x|\to\infty}(q(x), q_x(x)) \to 0$ in the underlying function space.

There are two different techniques available for investigating homoclinic solutions. The first approach is to consider Poincaré maps. However, (1.3) is still ill-posed and will *not* generate a semiflow. Thus it is not even possible to define a Poincaré map. The second approach, which is adopted in this article, is entirely analytic and based on Lyapunov-Schmidt reductions. The heart of this technique are exponential dichotomies for the linearization of (1.3)

$$\begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ -\Delta_y - D_u g - D_{\nabla_y u} g \, \nabla_y & D_{u_x} g \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{1.4}$$

along the solitary wave $(q(x), q_x(x))$. Here, derivatives of g are evaluated at $(y, q, q_x, \nabla_y q)$. Exponential dichotomies are projections onto x-dependent stable and unstable subspaces, say $E^s(x)$ and $E^u(x)$, such that solutions $(u,v)(x)$ of (1.4) associated with initial values $(u,v)(x_0)$ in the stable space $E^s(x_0)$ exist for $x > x_0$ and decay exponentially for $x \to \infty$. In contrast, solutions $(u,v)(x)$ associated with initial values $(u,v)(x_0)$ in the unstable space $E^u(x_0)$ satisfy (1.4) in backward x-direction $x < x_0$ and decay exponentially for decreasing x. Existence of exponential dichotomies for ordinary, parabolic or functional differential equations is well known, see, for instance, [5, 14, 11]. However, the proofs known thus far rely on the existence of a semiflow. Even though in [25] a functional-analytic framework for the existence on time intervals $[\tau, \infty)$ for large τ has been developed, the global extension to the half line $\mathbb{R}^+$ has been carried out using semiflows. In the context of elliptic equations, stable and unstable subspaces will both be infinite-dimensional and the semiflow on the unstable subspace defined for backward x-direction cannot be inverted. Hence, (1.4) will *not* define a semiflow.

In this article, we present a proof of the existence of dichotomies for equation (1.4). The proof employs a functional-analytic framework combining ideas from [25] and [28]. In the former work, exponential dichotomies for parabolic equations have been investigated using only integral equations. In [28], an integral-equation based approach has been given for elliptic equations. We will derive an integral equation, see equation (3.1), satisfied by exponential dichotomies. In contrast to previous works on ordinary and parabolic differential equations, we cannot use semiflows or the Gronwall lemma for the reasons explained above. Also, the integrands arising in the integral formulation are not small preventing us from using contraction mapping principles. Instead, Fredholm's alternative is employed for proving existence of dichotomies on arbitrary subintervals of $\mathbb{R}^+$. The advantage of this approach is that it preserves the symmetry between stable and unstable subspaces in the definition of dichotomies and does not a priori distinguish a time direction.

As a result, all bounded solutions of the nonlinear equation (1.3) staying close to the solitary wave for all values of x are accessible using Lyapunov-Schmidt reduction. For illustration, and as a first application, Melnikov's method for intersections of stable and unstable manifolds is extended to semilinear elliptic equations. Main result is the embedding of a shift on N symbols, with positive topological entropy, into the dynamical system generated by the shift of bounded solutions close to the solitary wave, provided a small generic perturbation $\mu\, h(x, y, u, u_x, \nabla_y u)$ periodic in x is added to (1.1).

In a forthcoming paper, we will give other applications. In particular, bifurcations to periodic waves as well as to N-solitary waves close to a primary solitary wave will be investigated using techniques developed in [17] and [25]. Moreover, algorithms for the numerical computation of homoclinic or heteroclinic orbits of elliptic equations introduced in [12, 13] will be justified by stability and convergence proofs.

We hope that the methods introduced here can be used to investigate stability of solitary waves with respect to the parabolic equation (1.2) using an extension of the Evans function.

Also, it may be possible to use this method to study elliptic equations for $\Omega = \mathbb{R}^n$ provided the solitary wave is localized in the x and y variable, see the remark at the end of Section 2.1. Note that in this case essential manifolds will not exist due to the presence of continuous spectrum.

This article is organized as follows. In Section 2, the main results on existence of exponential dichotomies for abstract linear equations are presented. They are proved in Section 3. Smoothing properties for abstract linear and nonlinear equations are addressed in Section 4. In Section 5, the effect of small non-autonomous perturbations of an abstract autonomous equation is investigated. Finally, Section 6 is devoted to applications to semilinear elliptic equations, and an example on the infinite cylinder $\mathbb{R} \times (0,\pi)^n$ is presented.

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2 Exponential Dichotomies

2.1 A class of abstract differential equations

Let X be a reflexive Banach space, and $A : D(A) \subset X \to X$ be a closed, possibly unbounded operator such that its domain $D(A)$ is dense in X. Then $X^1 := D(A)$ is a Banach space when equipped with the norm $|u|_{X^1} = |u|_X + |Au|_X$. Let Z be some Banach space such that there are continuous embeddings

$$X^1 \hookrightarrow Z \hookrightarrow X.$$

Later, Z is chosen as an interpolation space between X^1 and X. Moreover, let $B \in C^0(J, L(Z,X))$ be a continuous family of operators where $J \subset \mathbb{R}$ is some closed interval. We will be mainly interested in $J = \mathbb{R}$, $J = \mathbb{R}^+$ or $J = \mathbb{R}^-$.

Consider the differential equation

$$\dot{x} = (A + B(t))\, x. \tag{2.1}$$

A function $x(t)$ defined on a closed interval $J \subseteq \mathbb{R}$ is called a solution of (2.1) if

(i) $x(\cdot) \in C^0(\text{int}\, J, X^1) \cap C^1(\text{int}\, J, X)$,

(ii) $x(\cdot) \in C^0(J, Z)$,

(iii) $x(\cdot)$ satisfies equation (2.1) on $\text{int}\, J$ with values in X.

We are particularly interested in solutions with some prescribed exponential behavior. Throughout, range and kernel of an operator L are denoted $R(L)$ and $N(L)$, respectively.

Definition *(Exponential Dichotomy)*
Equation (2.1) is said to possess an exponential dichotomy in Z on the interval $J \subset \mathbb{R}$ if there exists a family of projections $P(t)$ for $t \in J$ such that

$$P(t) \in L(Z),\ P^2(t) = P(t),\ P(\cdot)z \in C^0(J, Z) \text{ for any } z \in Z$$

and there exist constants $K, \eta > 0$ with the following properties.

- *Stability. For any $\tau \in J$ and $z \in Z$, there exists a unique solution $x^s(t; \tau, z)$ of (2.1) defined for $t \geq \tau$ in J with $x^s(\tau; \tau, z) = P(\tau)z$ and*

$$|x^s(t; \tau, z)|_Z \leq K\, e^{-\eta|t-\tau|}\, |z|_Z$$

for all $t \geq \tau$ with $t \in J$.

- *Instability. For any $\tau \in J$ and $z \in Z$, there exists a unique solution $x^u(t; \tau, z)$ of (2.1) defined for $t \leq \tau$ in J with $x^u(\tau; \tau, z) = (\text{id} - P(\tau))z$ and*

$$|x^u(t; \tau, z)|_Z \leq K\, e^{-\eta|t-\tau|}\, |z|_Z$$

for all $t \leq \tau$ with $t \in J$.

- *Invariance. The solutions $x^s(t; \tau, z)$ and $x^u(t; \tau, z)$ satisfy*

$$x^s(t; \tau, z) \in R(P(t)) \quad \text{for all} \quad t \geq \tau \text{ with } t, \tau \in J$$
$$x^u(t; \tau, z) \in N(P(t)) \quad \text{for all} \quad t \leq \tau \text{ with } t, \tau \in J.$$

In other words, if an exponential dichotomy exists, we can solve equation (2.1) for $t \geq \tau$ for any initial value $z \in R(P(\tau))$. The solution is then given by $x^s(t, \tau; z)$ with $x^s(\tau, \tau; z) = z$. In addition, the solution is decaying exponentially in t. Moreover, the stable subspaces

$R(P(t))$ satisfy $R(x^s(t,\tau;\cdot)) \subset R(P(t))$. An analogous statement holds for $x^u(t,\tau;z)$. Therefore, the spaces $R(P(t))$ can be thought of as the time-slices of the stable manifold of the linear non-autonomous equation (2.1), while $x^s(t,\tau;\cdot)$ is the evolution operator mapping the time-slice $R(P(\tau))$ into $R(P(t))$ for $t \geq \tau$.

First, we give sufficient conditions such that the equation

$$\dot{x} = Ax, \tag{2.2}$$

that is (2.1) with $B(t) = 0$, has an exponential dichotomy on $\mathbb{R}$ in X. These conditions are not necessary for the existence of dichotomies, but will be used later in deriving the main perturbation and continuation result.

(H1) *Suppose that there is a constant C such that*

$$\|(A - \mathrm{i}\mu)^{-1}\|_{L(X)} \leq \frac{C}{1+|\mu|}$$

for all $\mu \in \mathbb{R}$. Assume that there is a projection $P_- \in L(X)$ such that A^{-1} and P_- commute. Furthermore, there exists a $\delta > 0$ such that $\operatorname{Re}\lambda < -\delta$ *for any $\lambda \in \sigma(AP_-)$ and* $\operatorname{Re}\lambda > \delta$ *for any $\lambda \in \sigma(A(\mathrm{id} - P_-))$.*

Sufficient conditions for the existence of the projection P_- have been given in [3] and [10]. We also refer to the explicit construction of the projections for semilinear elliptic equations in Section 6.1.

Define $P_+ = \mathrm{id} - P_-$ and $A_- = -P_-A$, $A_+ = P_+A$, and let $X_- = R(P_-)$ and $X_+ = R(P_+)$. By Hypothesis (H1), the operators A_- and A_+ are sectorial with their spectrum contained in the right half plane. Thus, they generate analytic semigroups

$$\begin{aligned} e^{A_+t} &= \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_+} e^{\lambda t}(\lambda - A)^{-1}\,d\lambda, \qquad && t<0 \\ e^{-A_-t} &= \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_-} e^{\lambda t}(\lambda - A)^{-1}\,d\lambda, \qquad && t>0 \end{aligned}$$

on X_+ and X_-, respectively. Here, the curve Γ_+ is asymptotic to $re^{\pm\mathrm{i}\varphi}$ as $r \to \infty$ for some fixed $\varphi \in (0, \frac{\pi}{2})$, and $\Gamma_- = -\Gamma_+$. We should point out that the semigroups e^{A_+t} and e^{-A_-t} are contained in $L(X_+)$ and $L(X_-)$, respectively. However, the products $e^{A_+t}P_+$ and $e^{-A_-t}P_-$ are defined on X. With the constant δ appearing in Hypothesis (H1), $e^{A_+t}P_+$ and $e^{-A_-t}P_-$ satisfy the estimate

$$\|e^{-A_-t}P_-\|_{L(X)} + \|e^{-A_+t}P_+\|_{L(X)} \leq C\,e^{-\delta t}$$

for some constant C and all $t \geq 0$.

Finally, we define the interpolation spaces $X_+^\alpha = D(A_+^\alpha)$ and $X_-^\alpha = D(A_-^\alpha)$ for $\alpha \geq 0$, see [14] or [31], and set $X^\alpha = X_+^\alpha \times X_-^\alpha$. The projection P_- obtained in Lemma 2.1 is then in $L(X^\alpha)$ for any $\alpha < 1$, and the semigroups $e^{-A_+ t}$ and $e^{-A_- t}$ satisfy

$$\|e^{-A_- t} P_-\|_{L(X,X^\alpha)} + \|e^{-A_+ t} P_+\|_{L(X,X^\alpha)} \leq C \max(1, t^{-\alpha}) e^{-\delta t}$$

for some constant C and all $t > 0$.

We summarize the above discussion in the following lemma.

Lemma 2.1 *Assume that Hypothesis (H1) is met. Equation (2.2) has then an exponential dichotomy on* $\mathbb{R}$ *in* X*. The projections* $P(t) = P_- \in L(X)$ *do not depend on* t *and commute with* A *on* $D(A)$*. Moreover,* $-P_- A$ *and* $(\mathrm{id} - P_-)A$ *are sectorial operators such that their domains are dense in* $R(P_-)$ *and* $N(P_-)$*, respectively.*

From now on, we consider the intervals $J = \mathbb{R}$, $J = \mathbb{R}^+$, or $J = \mathbb{R}^-$. The perturbation $B(t)$ appearing in (2.1) should satisfy the following hypothesis. The constant $\epsilon > 0$ appearing in (H2) is small and will be specified in the statement of the main theorem below.

(H2) *There exist* $\alpha \in [0, 1)$, $\vartheta > 0$, $t_* \geq 0$, *and* $S, K \in C^{0,\vartheta}(J, L(X^\alpha, X))$ *with* $B(t) = S(t) + K(t)$ *such that* $\|S(t)\|_{L(X^\alpha,X)} \leq \epsilon$ *for* $t \in J$, *and* $K(t) = 0$ *for all* $t \in J$ *with* $|t| \geq t_*$.

Hypothesis (H2) requires that $B(t)$ is small for all sufficiently large $|t|$. Such an assumption is needed as can be seen in the case that $B(t) = B$ is independent of t and Hypothesis (H1) is met for the operator A. Indeed, the perturbed equation $\dot{x} = (A + B)x$ has then an exponential dichotomy on $\mathbb{R}^+$ or $\mathbb{R}^-$ if, and only if, the spectrum of $A + B$ is bounded away from the imaginary axis which can only be guaranteed if $\|B\|_{L(X^\alpha,X)}$ is small.

As mentioned in the introduction, some compactness properties will be needed later on. We assume that either A has compact resolvent:

(H3) *Suppose that the inverse* A^{-1} *is a compact operator in* $L(X)$*.*

or else the operators $K(t)$ appearing in (H2) are compact:

(H4) *Suppose that there exists a Banach space* $Y \subset X$ *with compact inclusion such that* $K \in C^{0,\vartheta}(J, L(X^\alpha, Y))$*. In addition, the restriction of* A *to* Y *is a closed operator* A :

$D(A) \subset Y \to Y$ with domain dense in Y which satisfies Hypothesis (H1) with X replaced by Y.

Hypothesis (H4) may be useful when considering semilinear elliptic equations on $\mathbb{R} \times \mathbb{R}^n$ with localized solutions $u(x,y)$ such that $|u(x,y)| \leq Ce^{-\theta|y|}$ for some $\theta > 0$ uniformly in x. Then B is a differential operator with coefficients decaying exponentially in y, and Y can be chosen as a function space with exponential weights.

Finally, we assume forward and backward uniqueness of solutions of equation (2.1) on the interval J. This hypothesis seems to be necessary for the continuation of exponential dichotomies from a strict subinterval $\tilde{J}$ of J to J. For instance, backward uniqueness of solutions has been used in the context of parabolic or functional differential equations, see [14] and [11], respectively. There, forward uniqueness is met automatically. For elliptic equations, however, we also have to require forward uniqueness. Of course, for ordinary differential equations, forward and backward uniqueness are always satisfied.

(H5) *The only bounded solution $x(t)$ of (2.1) or its adjoint equation on the interval J with $x(0) = 0$ is the trivial solution $x(t) = 0$.*

Here, the adjoint equation is given by

$$\dot{\xi} = -(A^* + B(t)^*)\,\xi, \qquad \xi \in X^*. \tag{2.3}$$

Note that the adjoint operators A^* and $B(t)^*$ considered with range in X^* satisfy (H2), (H3) and (H4) whenever A and $B(t)$ do since X is reflexive, see [23, Section 1.10], [14, Section 7.3], and [15, Chapter III].

2.2 Perturbation and continuation of exponential dichotomies

The following theorem, which is the main result of this paper, is stated for the interval $J = \mathbb{R}^+$.

Theorem 1 *Suppose that Hypothesis (H1) is satisfied. Let $J = \mathbb{R}^+$. Choose η such that $0 \leq \eta < \delta$ where δ appears in Hypothesis (H1). There are then constants $\epsilon_0 > 0$ and $C > 0$ with the following properties. Assume that Hypotheses (H2), (H5) and either (H3) or (H4)* *are met for some $\epsilon \leq \epsilon_0$. Equation (2.1) has then an exponential dichotomy in X^α on the interval $J = \mathrm{IR}^+$ with rate η.*

Furthermore, the projections $P(t)$ are Hölder continuous in $t \in J = \mathrm{IR}^+$ with values in $L(X^\alpha)$. The range E^s of $P(0)$ is uniquely determined and satisfies

$$z \in E^s = R(P(0)) \qquad \Longrightarrow \qquad z = P_- z + P_+(S_0 + K_0)z$$

for some operators S_0 and K_0 in $L(X^\alpha)$ with $\|S_0\|_{L(X^\alpha)} \leq C\epsilon$ and K_0 compact. For any closed complement E^u of E^s there exists a unique exponential dichotomy with $R(P(0)) = E^s$ and $N(P(0)) = E^u$. In particular, closed complements of E^s exist.

An analogous theorem is true for the interval $J = \mathrm{IR}^-$.

It is straightforward to generalize Theorem 1 in that perturbations of the non-autonomous equation (2.1) instead of the autonomous equation (2.2) are considered. In that case, we have to require that the solutions $x^s(t;\tau,z)$ and $x^u(t;\tau,z)$ of (2.1) map X^α into $X^{\alpha+\theta}$ for some positive θ and are Hölder continuous between these spaces. We will not state a result but refer the reader to Section 4 where the necessary regularity properties are proved.

Theorem 1 shows that, up to factoring a finite-dimensional subspace of the stable subspace E^s, the range $R(P(0)) = E^s$ is close to the space $R(P_-)$. Hence, dimensions can be counted on account of the compactness assumptions (H3) or (H4).

Corollary 1 *Suppose that A and $B(t)$ satisfy the assumptions of Theorem 1 on both intervals, $J = \mathrm{IR}^+$ and $J = \mathrm{IR}^-$. Denote the projections of the associated exponential dichotomies on IR^+ and IR^- by $P(t)$ and $Q(t)$, respectively. The intersection $R(P(0)) \cap R(Q(0))$ is then finite-dimensional.*

If $J = \mathrm{IR}^+$ and the perturbation $B(t)$ tends to zero as $t \to \infty$, we expect the projection $P(t)$ of the exponential dichotomy on IR^+ to converge to the spectral projection P_-. This is made precise in the following corollary.

Corollary 2 *Suppose that A and $B(t)$ satisfy the assumptions of Theorem 1 on the interval $J = \mathrm{IR}^+$ and, in addition,*

$$\|B(t)\|_{L(X^\alpha,X)} \leq \hat{C} e^{-\theta t} \qquad\qquad t \geq 0$$

for some constants $\hat{C}, \theta > 0$. The rate η appearing in Theorem 1 can then be chosen in the range $0 \leq \eta \leq \delta$ and we have

$$\|P(t) - P_-\|_{L(X^\alpha)} \leq \tilde{C}(e^{-2\delta t} + e^{-\theta t}) \qquad t \geq 0$$

for some constant $\tilde{C} > 0$. An analogous statement is true on the interval $J = \mathbb{R}^-$.

Finally, we state a theorem characterizing equations having exponential dichotomies on the real line $\mathbb{R}$.

Theorem 2 *Suppose that the assumptions of Theorem 1 hold for both intervals $J = \mathbb{R}^+$ and $J = \mathbb{R}^-$. Then, $x(\cdot) = 0$ is the only bounded solution of equation (2.1) on $t \in \mathbb{R}$ if and only if equation (2.1) has an exponential dichotomy on $\mathbb{R}$.*

3 Proofs of the results in Section 2.2

We start with the proof of Theorem 1 which will occupy most of this section. The outline of its proof is as follows.

First, we give a mild formulation of the problem, an integral equation which is satisfied by the evolution operators $x^s(t, \tau; z)$ and $x^u(t, \tau; z)$. It is then shown that strong and mild formulation are equivalent. Using the mild integral equation, we construct the subspace $E^s = R(P(0))$ consisting of bounded solutions of (2.1) on $\mathbb{R}^+$ using Fredholm's alternative. Then, for a fixed choice of E^u, it is shown that the mild integral equation has a unique solution $(x^s(\cdot, \tau), x^u(\cdot, \tau))$ for any fixed $\tau \geq 0$ satisfying $x^u(0, \tau) \in E^u$. Finally, we verify that these solutions are strongly continuous in τ and that they satisfy the semigroup properties.

3.1 The integral formulation

We write $x^s(t; \tau, z) = x^s(t, \tau)$ and $x^u(t; \tau, z) = x^u(t, \tau)$ whenever confusion is impossible.

The following mild formulation of equation (2.1) is the key.

$$
\begin{aligned}
e^{-A_-(t-\tau)}P_-z &= x^s(t,\tau) + e^{-A_-t}P_-x^u(0,\tau) + \int_t^\infty e^{A_+(t-\sigma)}P_+B(\sigma)x^s(\sigma,\tau)\,d\sigma \\
&\quad - \int_\tau^t e^{-A_-(t-\sigma)}P_-B(\sigma)x^s(\sigma,\tau)\,d\sigma \\
&\quad + \int_0^\tau e^{-A_-(t-\sigma)}P_-B(\sigma)x^u(\sigma,\tau)\,d\sigma \\
e^{A_+(t-\tau)}P_+z &= x^u(t,\tau) - e^{-A_-t}P_-x^u(0,\tau) - \int_\tau^t e^{A_+(t-\sigma)}P_+B(\sigma)x^u(\sigma,\tau)\,d\sigma \\
&\quad + \int_t^0 e^{-A_-(t-\sigma)}P_-B(\sigma)x^u(\sigma,\tau)\,d\sigma \\
&\quad - \int_\tau^\infty e^{A_+(t-\sigma)}P_+B(\sigma)x^s(\sigma,\tau)\,d\sigma.
\end{aligned}
\tag{3.1}
$$

Here, $t \geq \tau \geq 0$ in the first and $\tau \geq t \geq 0$ in the second equation of (3.1). The pair (x^s, x^u) is written $x := (x^s, x^u)$. We will see that solutions of (3.1) are in fact the evolution operators arising in the definition of exponential dichotomies. In particular, we will prove that the projections of the exponential dichotomy are given by $P(t)z = x^s(t;t,z)$ and $(\mathrm{id} - P(t))z = x^u(t;t,z)$ for solutions $x^s(t;\tau,z)$ and $x^u(t;\tau,z)$ of (3.1). The operator $x^u(0;0,\cdot)$ is determined by the choice of the complement E^u.

Notice that the integrands appearing in (3.1) are not small since B might have large norm. Therefore, it is not possible to use the contraction mapping theorem for solving equation (3.1).

We have to show that the strong and the mild formulation are equivalent.

Lemma 3.1 *Suppose that $x = (x^s, x^u)$ satisfies equation (3.1) for some $z \in X^\alpha$. Then, $x^s(\cdot,\tau)$ and $x^u(\cdot,\tau)$ satisfy (2.1) on the intervals $J = [\tau,\infty)$ and $J = [0,\tau]$, respectively. Conversely, any two solutions $x^1(\cdot)$, $x^2(\cdot)$ of (2.1) on $J_1 = [\tau,\infty)$ and $J_2 = [0,\tau]$ are solutions of (3.1) with $x^s(t,\tau) = x^1(t)$, $x^u(t,\tau) = x^2(t)$ and $z = x^1(\tau) + x^2(\tau)$.*

Proof. Suppose $x = (x^s, x^u)$ satisfies equation (3.1). Then, by [14, Lemma 3.5.1], the integral operators are continuously differentiable in t since the family $B(t)$ is Hölder continuous. Thus, for $t \neq \tau$, we can differentiate with respect to t and obtain that

$$
\begin{aligned}
\dot{x}^s(t,\tau) &= (A + B(t))x^s(t,\tau) \qquad && t > \tau \\
\dot{x}^u(t,\tau) &= (A + B(t))x^u(t,\tau) \qquad && t < \tau.
\end{aligned}
$$

Therefore, $Ax^s(t,\tau)$ and $Ax^u(t,\tau)$ are continuous, too, and $x^s(t,\tau)$ and $x^u(t,\tau)$ are solutions.

Conversely, suppose that $x^1(t)$ and $x^2(t)$ satisfy (2.1). As $x^i(\cdot)$ are bounded for $i = 1, 2$, they are solutions of

$$\begin{aligned}
x^1(t) &= e^{-A_-(t-\tau)}P_- x^1(\tau) + \int_\tau^t e^{-A_-(t-\sigma)}P_- B(\sigma)x^1(\sigma)d\sigma \\
&\quad - \int_t^\infty e^{A_+(t-\sigma)}P_+ B(\sigma)x^1(\sigma)d\sigma \\
x^2(t) &= e^{-A_- t}P_- x^2(0) + e^{A_+(t-\tau)}P_+ x^2(\tau) + \int_\tau^t e^{A_+(t-\sigma)}P_+ B(\sigma)x^2(\sigma)d\sigma \\
&\quad + \int_0^t e^{-A_-(t-\sigma)}P_- B(\sigma)x^2(\sigma)d\sigma,
\end{aligned}$$

by integration. Setting $z = x^1(\tau) + x^2(\tau)$, we obtain equation (3.1). ■

3.2 Construction of the stable eigenspace

Here, we will determine those initial values for which we can solve (2.1) for $t \in \mathrm{IR}^+$ such that the associated solution is bounded on IR^+. Therefore, we set $\tau = 0$ in (3.1) and obtain

$$\begin{aligned}
e^{-A_- t}P_- z &= x^s(t) + e^{-A_- t}P_- x^u(0) + \int_t^\infty e^{A_+(t-\sigma)}P_+ B(\sigma)x^s(\sigma)\, d\sigma \\
&\quad - \int_0^t e^{-A_-(t-\sigma)}P_- B(\sigma)x^s(\sigma)\, d\sigma \\
P_+ z &= P_+ x^u(0) - \int_0^\infty e^{-A_+\sigma}P_+ B(\sigma)x^s(\sigma)\, d\sigma
\end{aligned}$$

for $t \geq 0$. Note that we have omitted the argument $\tau = 0$ in x^s and x^u. Since we are interested in the initial values with $x^s(0; z) = z$, we set $x^u(0) = 0$ and obtain the equation

$$\begin{aligned}
e^{-A_- t}P_- z &= x^s(t) + \int_t^\infty e^{A_+(t-\sigma)}P_+ B(\sigma)x^s(\sigma)\, d\sigma \\
&\quad - \int_0^t e^{-A_-(t-\sigma)}P_- B(\sigma)x^s(\sigma)\, d\sigma \\
P_+ z &= - \int_0^\infty e^{-A_+\sigma}P_+ B(\sigma)x^s(\sigma)\, d\sigma.
\end{aligned} \tag{3.2}$$

We will solve this equation in the following spaces. For a fixed choice of $\eta \in [0, \delta)$, and for fixed $\tau \geq 0$, let

$$\begin{aligned}
\mathcal{X}_\tau^s &= \{x \in C^0([\tau, \infty), X^\alpha);\ |x|_{\mathcal{X}_\tau^s} := \sup_{t \geq \tau} e^{\eta|t-\tau|}|x(t)|_{X^\alpha} < \infty\} \\
\mathcal{X}_\tau^u &= \{x \in C^0([0, \tau], X^\alpha);\ |x|_{\mathcal{X}_\tau^u} := \sup_{0 \leq t \leq \tau} e^{\eta|t-\tau|}|x(t)|_{X^\alpha} < \infty\}
\end{aligned} \tag{3.3}$$

equipped with the norms $|\cdot|_{\mathcal{X}_\tau^s}$ and $|\cdot|_{\mathcal{X}_\tau^u}$, respectively, and set $\mathcal{X}_\tau = \mathcal{X}_\tau^s \times \mathcal{X}_\tau^u$.

For fixed $z \in X^\alpha$, we shall then solve

$$\tilde{\varphi}_0 z = \tilde{T}_0 x^s \tag{3.4}$$

for $x^s \in \mathcal{X}_0^s$, where

$$(\tilde{T}_0 x^s)(t) \;=\; x^s(t) - \int_0^t e^{-A_-(t-\sigma)} P_- B(\sigma) x^s(\sigma)\, d\sigma + \int_t^\infty e^{A_+(t-\sigma)} P_+ B(\sigma) x^s(\sigma)\, d\sigma$$

and $(\tilde{\varphi}_0 z)(t) = e^{-A_- t} P_- z$ for $t \geq 0$. Thus, equation (3.4) coincides with the first equation in (3.2). It is straightforward to verify that $\tilde{\varphi}_0 : X^\alpha \to \mathcal{X}_0^s$ is bounded. We show next that $\tilde{T}_0$ is Fredholm with index zero on $\mathcal{X}_0^s$.

Lemma 3.2 *The operator $\tilde{T}_0 \in L(\mathcal{X}_0^s)$ is Fredholm with index zero.*

Proof. It is straightforward to show that $\tilde{T}_0$ is a bounded operator from $\mathcal{X}_0^s$ into itself. The operator $\tilde{T}_0$ is of the form $\tilde{T}_0 = \mathrm{id} + I_1 + I_2$, where I_1 and I_2 are the integral operators

$$\begin{aligned}(I_1 x^s)(t) &= -\int_0^t e^{-A_-(t-\sigma)} P_- B(\sigma) x^s(\sigma)\, d\sigma \\ (I_2 x^s)(t) &= \int_t^\infty e^{A_+(t-\sigma)} P_+ B(\sigma) x^s(\sigma)\, d\sigma.\end{aligned}$$

We have to show that $\tilde{T}_0 = \mathrm{id} + I_1 + I_2$ is Fredholm with index zero. It suffices to show that the operators I_j can be written as $I_j = S_j + K_j$ for $j = 1, 2$ such that S_j has norm less than $\frac{1}{4}$ and K_j is compact for $j = 1, 2$. Indeed, the operator $\mathrm{id} + S_1 + S_2$ is then invertible, and hence Fredholm with index zero. Adding a compact operator preserves this property.

For any $t^* \geq 0$, we may decompose $I_1 = S_1 + K_1$ according to

$$\begin{aligned}(K_1 x^s)(t) &= \begin{cases} -\int_0^t e^{-A_-(t-\sigma)} P_- B(\sigma) x^s(\sigma)\, d\sigma & \text{for } t \leq t^* \\ -e^{-A_-(t-t^*)} \int_0^{t^*} e^{-A_-(t^*-\sigma)} P_- B(\sigma) x^s(\sigma)\, d\sigma & \text{for } t \geq t^*, \end{cases} \\ (S_1 x^s)(t) &= \begin{cases} 0 & \text{for } t \leq t^* \\ -\int_{t^*}^t e^{-A_-(t-\sigma)} P_- B(\sigma) x^s(\sigma)\, d\sigma & \text{for } t \geq t^*. \end{cases}\end{aligned}$$

Since $S_1 x^s$ and $K_1 x^s$ are continuous at $t = t^*$, they map $\mathcal{X}_0^s$ into itself. Moreover, for large t^*, we have

$$\|S_1\|_{L(\mathcal{X}_0^s)} \leq C \sup_{t \geq t^*} \|B(t)\|_{L(X^\alpha, X)} \leq C\epsilon$$

by Hypothesis (H2). It remains to prove that K_1 is compact. We restrict $K_1 x^s$ to the interval $[0, t^*]$. The proof for compactness of K_1 then depends on whether Hypothesis (H3) or (H4) is satisfied.

First, assume that Hypothesis (H3) is met. It follows that K_1 maps $\mathcal{X}_0^s$ continuously into $C^{0,\kappa}([0, t^*], X^{\alpha+\kappa})$ for some small $\kappa > 0$, see [14, Lemma 3.5.1]. Since A has compact resolvent, the inclusion $X^{\alpha+\kappa} \hookrightarrow X^\alpha$ is compact. Thus, by Arzéla's theorem, the space $C^{0,\kappa}([0,t^*],X^{\alpha+\kappa})$ is compactly embedded into $C^0([0,t^*],X^\alpha)$.

Next, assume that Hypothesis (H4) is met. The proof is then similar to the one above. Note that $B(t) = S(t) + K(t)$ with S small. Subsume the part of K_1 associated with the operator $S(t)$ into S_1. The remaining term of K_1 associated with $K(t)$ is compact. Indeed, it maps $\mathcal{X}_0^s$ continuously into $C^{0,\kappa}([0,t^*],Y^\alpha)$ by applying the arguments given so far to the restriction of A to Y. Finally, $C^{0,\kappa}([0,t^*],Y^\alpha)$ is compactly embedded in $C^0([0,t^*],X^\alpha)$.

Thus, K_1 is a compact operator since it is the composition of the above restriction to $[0,t_*]$ with the bounded multiplication operator associated with

$$\begin{array}{cll} \mathrm{id} & \text{for} & 0 \le t \le t^* \\ e^{-A_-(t-t^*)}P_- & \text{for} & t^* \le t. \end{array}$$

The proof for I_2 is similar. ■

We denote the stable subspace at $t=0$ by

$$E^s := (\tilde{T}_0^{-1}(R(\tilde{\varphi}_0)))(0) = \{z \in X^\alpha;\ \exists x^s \in \mathcal{X}_0^s \text{ with } x^s(0) = z \text{ and } \tilde{T}_0 x^s = \tilde{\varphi}_0 z\}. \tag{3.5}$$

In other words, E^s consists of all initial values yielding bounded solutions on $\mathbb{R}^+$. Note that E^s is closed since $\tilde{T}_0$ is Fredholm, see Lemma 3.2, and $R(\tilde{\varphi}_0)$ is closed.

Lemma 3.3 *The equality*

$$\dim N(P_-|_{E^s}) = \dim N(\tilde{T}_0) = \operatorname{codim} R(\tilde{T}_0) = \operatorname{codim}_{X_-^\alpha} P_- E^s = k^s$$

holds for some $k^s < \infty$.

Proof. We start by showing the first equality. The mapping

$$\begin{array}{rcl} N(\tilde{T}_0) & \mapsto & N(P_-|_{E^s}) \\ x^s(\cdot) & \mapsto & x^s(0) \end{array}$$

is well defined, continuous and one-to-one by the uniqueness assumption (H5). It is also onto by construction of E^s. This proves $\dim N(P_-|_{E^s}) = \dim N(\tilde{T}_0) = k < \infty$.

Next, we have $\dim N(\tilde{T}_0) = \operatorname{codim} R(\tilde{T}_0)$ since $\tilde{T}_0$ is Fredholm with index zero.

In order to show the last equality, choose a complement V_- of P_-E^s in X_-^α. By construction, for any $z \in V_-$, the map $t \to e^{-A_-t}P_-z$ is not contained in $R(\tilde{T}_0)$. Thus the mapping $z \in V_- \to e^{-A_-\cdot}P_-z \in \mathcal{X}_0^s$ maps the complement V_- of P_-E^s in X_-^α one-to-one into a complement of $R(\tilde{T}_0)$ in $\mathcal{X}_0^s$. This implies $\operatorname{codim}_{X_-^\alpha} P_-E^s \leq \operatorname{codim} R(\tilde{T}_0) = k$.

We use the adjoint equation

$$\dot{\xi} = -(A^* + B(t)^*)\xi, \qquad \xi \in (X^*)^\alpha \tag{3.6}$$

to show equality. Note that results obtained so far apply to the adjoint equation as well, see the comments in Section 2.1. It is easy to see that

$$\frac{d}{dt}\langle \xi(t), x(t) \rangle = 0$$

for arbitrary solutions $\xi(t)$ and $x(t)$ of (3.6) and (2.1), respectively, where $\langle \cdot, \cdot \rangle$ denotes the dual pairing. Since all bounded solutions x^s satisfy the estimate

$$|x^s(t)|_{X^\alpha} \leq Ce^{-\eta t}|x^s(0)|_{X^\alpha},$$

any bounded solution of the adjoint equation has to annihilate E^s at $t = 0$. Call E_*^s the subspace of $(X^*)^\alpha$ consisting of initial values $\xi(0)$ of bounded solutions for (3.6). Next, we apply the arguments obtained thus far to the adjoint equation. The configuration space $(X^*)^\alpha$ can be written as $(X^*)_+^\alpha \times (X^*)_-^\alpha$. Therefore, using the arguments given so far, the stable subspace satisfies

$$\infty > \dim N\left(P_+^*|_{E_*^s}\right) = k^* \geq \operatorname{codim}_{(X^*)_+^\alpha} P_+^* E_*^s.$$

Hence, using that E_*^s annihilates E^s, we obtain

$$\begin{aligned} k^* &= \dim N\left(P_+^*|_{E_*^s}\right) \leq \dim N\left(P_+^*|_{\text{Annih.}(E^s)}\right) \\ &= \dim\left\{(\xi_-, 0) \in (X^*)_-^\alpha \times (X^*)_+^\alpha;\ \langle \xi_-, z_- \rangle = 0\ \forall z_- \in P_-E^s\right\} \\ &= \operatorname{codim}_{X_-^\alpha}(P_-E^s) \leq k. \end{aligned}$$

Repeating the same argument for the adjoint system and using reflexivity of X, yields

$$k^{**} = \dim N\left(P_-^{**}|_{E_{**}^s}\right) = k = \dim N\left(P_-|_{E^s}\right)$$

and

$$k = k^{**} \leq \operatorname{codim}_{(X^*)_+^\alpha} P_+^* E_*^s \leq k^* \leq k,$$

where the strict inequality holds if and only if $\dim N(P_-|_{E^s}) > \operatorname{codim}_{X_-^\alpha}(P_-E^s)$. ■

3.3 Existence of $x^s(\cdot,\tau;z)$ and $x^s(\cdot,\tau;z)$ for fixed τ

In the next step, we construct solutions $x^s(\cdot,\tau;z)$ and $x^s(\cdot,\tau;z)$ for fixed τ. For this purpose, we have to incorporate a fixed complement E^u of the stable subspace E^s into the functional-analytic setting. Therefore, choose any closed complement E^u of E^s in X^α subject to

$$\operatorname{codim}_{X_+^\alpha} P_+E^u = \dim N(P_+|_{E^u}) = k^u < \infty. \tag{3.7}$$

To accomplish this, choose, for instance, closed complements E_-^u of P_-E^s in X_-^α and E_+^u of $N(P_-|_{E^s})$ in X_+^α. Note that these complements exist since P_-E^s has finite codimension in X_-^α and $N(P_-|_{E^s})$ is finite-dimensional, see Lemma 3.3. The space $E_-^u \times E_+^u \subset X_-^\alpha \times X_+^\alpha$ is then a complement of E^s in X^α satisfying the above condition with $k^u = k^s$, since

$$\dim N(P_-|_{E^s}) = \operatorname{codim}_{X_-^\alpha} P_-E^s = k^s$$

by Lemma 3.3. Other complements will be considered later.

For any closed subspace $E \subset X^\alpha$, we define the closed subspace

$$\mathcal{X}_\tau^E = \{(x^s,x^u) \in \mathcal{X}_\tau^s \times \mathcal{X}_\tau^u;\, x^u(0) \in E\}$$

of $\mathcal{X}_\tau^s \times \mathcal{X}_\tau^u$.

For fixed $\tau \geq 0$, the right hand side of equation (3.1) defines an operator denoted T_τ

$$\begin{aligned}
(T_\tau x)^s(t) \quad &:= \quad x^s(t) + e^{-A_-t}P_-x^u(0) + \int_t^\infty e^{A_+(t-\sigma)}P_+B(\sigma)x^s(\sigma)\,d\sigma \\
&\qquad - \int_\tau^t e^{-A_-(t-\sigma)}P_-B(\sigma)x^s(\sigma)\,d\sigma + \int_0^\tau e^{-A_-(t-\sigma)}P_-B(\sigma)x^u(\sigma)\,d\sigma \\
(T_\tau x)^u(t) \quad &:= \quad x^u(t) - e^{-A_-t}P_-x^u(0) - \int_\tau^t e^{A_+(t-\sigma)}P_+B(\sigma)x^u(\sigma)\,d\sigma \\
&\qquad + \int_t^0 e^{-A_-(t-\sigma)}P_-B(\sigma)x^u(\sigma)\,d\sigma - \int_\tau^\infty e^{A_+(t-\sigma)}P_+B(\sigma)x^s(\sigma)\,d\sigma,
\end{aligned} \tag{3.8}$$

with $t \geq \tau$ in the first, and $\tau \geq t \geq 0$ in the second equation. Similarly, the left hand side of (3.1) defines a bounded operator $\varphi_\tau : X^\alpha \to \mathcal{X}_\tau^{X_+}$ by

$$\begin{aligned}
(\varphi_\tau z)^s(t) \quad &= \quad e^{-A_-(t-\tau)}P_-z \qquad\qquad t \geq \tau \geq 0 \\
(\varphi_\tau z)^u(t) \quad &= \quad e^{A_+(t-\tau)}P_+z \qquad\qquad \tau \geq t \geq 0,
\end{aligned} \tag{3.9}$$

with bound independent of τ.

Proposition 1 *For any fixed $\tau \geq 0$, the operator T_τ defined by (3.8) is an isomorphism when considered as a map $T_\tau : \mathcal{X}_\tau^{E^u} \longrightarrow \mathcal{X}_\tau^{X_+}$.*

Proof. First, notice that T_τ is well-defined and bounded independently of τ. Indeed, T_τ is bounded as an operator from $\mathcal{X}_\tau^s \times \mathcal{X}_\tau^u$ into itself and its bound does not depend on τ. Also, for any choice of E^u, the range of T_τ is included in $\mathcal{X}_\tau^{X_+}$, so T_τ is well-defined. Indeed, the only term appearing in the equation for x^u in (3.1) which does not belong to X^+ is the integral

$$\int_t^0 e^{-A_-(t-\sigma)} P_- B(\sigma) x^u(\sigma)\, d\sigma.$$

However, this term vanishes at $t = 0$.

We claim that

(i) $N(T_\tau) = \{0\}$ and

(ii) T_τ is Fredholm with index zero for $B = 0$.

By arguments similar to those given in Lemma 3.2, we conclude from (ii) that T_τ is Fredholm with index zero for any perturbation B satisfying Hypothesis (H2) for ϵ small enough. Note that ϵ can be chosen independent of τ since it depends only on the norm of P_- and the decay rates δ and η. The first assertion then shows that T_τ is one-to-one and thus, using the second assertion (ii), onto. Therefore, by the closed graph theorem, T_τ is continuously invertible.

With a slight abuse of notation, but for the sake of clarity, we write elements $(x^s(\cdot), x^u(\cdot)) \in \mathcal{X}_\tau$ as $(x^s(\cdot,\tau), x^u(\cdot,\tau))$ indicating the domain of definition.

We first prove (i). Suppose that $T_\tau(x^s, x^u) = 0$ for some $(x^s, x^u) \in \mathcal{X}_\tau^{E^u}$. This implies $x^u(\tau,\tau) = -x^s(\tau,\tau)$ by adding the two equations in (3.1). Thus, the function

$$\tilde{x}^s(t,0) := \begin{cases} x^u(t,\tau) & \text{for} \quad 0 \le t \le \tau \\ -x^s(t,\tau) & \text{for} \quad \tau \le t \le \infty \end{cases} \tag{3.10}$$

is continuous. Using the definition (3.9) of φ, we claim that $\tilde{x}^s(t,0)$ satisfies

$$T_0(\tilde{x}^s, 0) = \varphi_0(\tilde{x}^s(0,0)) = \varphi_0(x^u(0,\tau)), \tag{3.11}$$

that is,

$$\begin{aligned} e^{-A_- t} P_- x^u(0,\tau) &= \tilde{x}^s(t,0) + \int_t^\infty e^{A_+(t-\sigma)} P_+ B(\sigma) \tilde{x}^s(\sigma,0)\, d\sigma \\ &\quad - \int_0^t e^{-A_-(t-\sigma)} P_- B(\sigma) \tilde{x}^s(\sigma,0)\, d\sigma \qquad t \ge 0 \\ P_+ x^u(0,\tau) &= -\int_0^\infty e^{-A_+\sigma} P_+ B(\sigma) \tilde{x}^s(\sigma,0)\, d\sigma \qquad t = 0. \end{aligned} \tag{3.12}$$

By assumption, (x^s, x^u) satisfies (3.1) with $z = 0$, that is

$$
\begin{aligned}
0 &= x^s(t,\tau) + e^{-A_- t} P_- x^u(0,\tau) + \int_t^\infty e^{A_+(t-\sigma)} P_+ B(\sigma) x^s(\sigma,\tau)\, d\sigma \\
&\quad - \int_\tau^t e^{-A_-(t-\sigma)} P_- B(\sigma) x^s(\sigma,\tau)\, d\sigma + \int_0^\tau e^{-A_-(t-\sigma)} P_- B(\sigma) x^u(\sigma,\tau)\, d\sigma \\
0 &= x^u(t,\tau) - e^{-A_- t} P_- x^u(0,\tau) - \int_\tau^t e^{A_+(t-\sigma)} P_+ B(\sigma) x^u(\sigma,\tau)\, d\sigma \\
&\quad + \int_t^0 e^{-A_-(t-\sigma)} P_- B(\sigma) x^u(\sigma,\tau)\, d\sigma - \int_\tau^\infty e^{A_+(t-\sigma)} P_+ B(\sigma) x^s(\sigma,\tau)\, d\sigma
\end{aligned}
\tag{3.13}
$$

for $t \geq \tau$ and $t \leq \tau$, respectively. Using (3.10) and distinguishing the cases $t \leq \tau$ and $t \geq \tau$, it is seen that (3.12) and (3.13) are identical.

Thus $\tilde{x}^s(t,0)$ satisfies (3.11). However, $\tilde{x}^s(0,0) = x^u(0,\tau) \in E^u$ and, at the same time, belongs to E^s as it is a bounded solution of (3.1) at $\tau = 0$. Therefore $\tilde{x}^s(0,0) = 0$ vanishes since $E^u \cap E^s = \{0\}$. By the uniqueness hypothesis (H5), we conclude $\tilde{x}^s(t,0) = 0$ for all $t \geq 0$, which proves (i).

It remains to prove (ii). For $B = 0$, the equation $T_\tau(x^s, x^u) = (g^s, g^u) \in \mathcal{X}_\tau^{X_+}$ reads

$$
\begin{aligned}
P_+ x^s(t,\tau) &= P_+ g^s(t,\tau), & P_- x^s(t,\tau) &= P_- g^s(t,\tau) - e^{-A_- t} P_- x^u(0,\tau) \\
P_+ x^u(t,\tau) &= P_+ g^u(t,\tau), & P_- x^u(t,\tau) &= e^{-A_- t} P_- x^u(0,\tau).
\end{aligned}
\tag{3.14}
$$

First, suppose that $g = (g^s, g^u) = 0$. Then, for any $x^u(0,\tau) \in E^u$ satisfying $x^u(0,\tau) \in N(P_+|_{E^u})$, we get a unique solution of (3.14) in $\mathcal{X}_\tau^{E^u}$. Note that $\dim N(P_+|_{E^u}) = k^u$. On the other hand, we can solve for any g provided $P_+ g^u(0,\tau) \in P_+ E^u$ which defines a subspace of $\mathcal{X}_\tau^{X_+}$ of codimension k^u. This proves (ii) and thus the proposition. ■

3.4 Proof of Theorem 1

Finally, we show the assertions of Theorem 1. We consider a similar set-up as in the previous section.

Similar to (3.3), we define the function spaces

$$
\begin{aligned}
\mathcal{X}^s &= \{x \in C^0(D^s, X^\alpha);\ |x|_{\mathcal{X}^s} := \sup_{(t,\tau)\in D^s} e^{\eta|t-\tau|} |x(t,\tau)|_{X^\alpha} < \infty\} \\
\mathcal{X}^u &= \{x \in C^0(D^u, X^\alpha);\ |x|_{\mathcal{X}^u} := \sup_{(t,\tau)\in D^u} e^{\eta|t-\tau|} |x(t,\tau)|_{X^\alpha} < \infty\}
\end{aligned}
$$

with

$$
D^s = \{(t,\tau);\ t \geq \tau \geq 0\} \qquad \text{and} \qquad D^u = \{(t,\tau);\ \tau \geq t \geq 0\},
$$

and set

$$
\mathcal{X}^E = \{(x^s, x^u) \in \mathcal{X}^s \times \mathcal{X}^u;\ x^u(0,\tau) \in E \text{ for all } \tau \geq 0\}
$$

for any closed subspace E of X^α. As before, the left hand side of (3.1) defines a bounded operator $\varphi : X^\alpha \to \mathcal{X}^{X_+}$ by

$$\begin{aligned} (\varphi z)^s(t,\tau) &= e^{-A_-(t-\tau)}P_- z && (t,\tau) \in D^s \\ (\varphi z)^u(t,\tau) &= e^{A_+(t-\tau)}P_+ z && (t,\tau) \in D^u. \end{aligned}$$

Let T be the operator defined by the right hand side of (3.1). We shall solve $Tx = \varphi z$. We claim that $T : \mathcal{X}^{E^u} \to \mathcal{X}^{X_+}$ is an isomorphism. Notice that T is well-defined, see the proof of Proposition 1, and continuous.

Assuming that $x \in N(T)$, we get $x(\cdot,\tau) \in N(T_\tau)$ for any $\tau \geq 0$ whence $x(\cdot,\tau) = 0$ by Proposition 1. Thus $N(T) = \{0\}$.

It is more difficult to prove that T is onto. Due to Proposition 1, there exists a unique family $x(\cdot,\tau)$ satisfying $T_\tau x(\cdot,\tau) = \varphi_\tau z$ for any fixed τ. This family satisfies $Tx = \varphi$ provided $x(\cdot,\cdot) \in \mathcal{X}^{E^u}$. In particular, we have to show that $x(\cdot,\tau)$ is continuous in τ and decays exponentially uniformly in τ. Denoting the unique solution (x^s, x^u) of $T_\tau(x^s,x^u) = \varphi_\tau z$ by $(x^s(t;\tau,z), x^u(t;\tau,z))$, we will prove the following.

(i) Invariance and semigroup properties.

$$\begin{aligned} &x^s(t;\sigma,x^s(\sigma;\tau,z)) = x^s(t;\tau,z) && t \geq \sigma \geq \tau \\ &x^s(t;\sigma,x^u(\sigma;\tau,z)) = 0 && \sigma \leq t,\tau \\ &x^u(t;\sigma,x^u(\sigma;\tau,z)) = x^u(t;\tau,z) && t \leq \sigma \leq \tau \\ &x^u(t;\sigma,x^s(\sigma;\tau,z)) = 0 && \sigma \geq t,\tau. \end{aligned}$$

(ii) Continuity.

$x^s(\cdot;\cdot,z)$ and $x^u(\cdot;\cdot,z)$ are continuous.

(iii) Exponential decay.

$$\begin{aligned} &|x^s(t;\tau,z)|_{X^\alpha} \leq Ce^{-\eta|t-\tau|}\,|z|_{X^\alpha} && t \geq \tau \\ &|x^u(t;\tau,z)|_{X^\alpha} \leq Ce^{-\eta|t-\tau|}\,|z|_{X^\alpha} && t \leq \tau. \end{aligned}$$

First consider (i). Let $\sigma \geq \tau$, and define $\hat{z} := x^s(\sigma;\tau,z)$ and

$$\begin{aligned} y^s(t) &:= x^s(t;\sigma,\hat{z}) = x^s(t;\sigma,x^s(\sigma;\tau,z)) && t \geq \sigma \\ y^u(t) &:= x^u(t;\sigma,\hat{z}) = x^u(t;\sigma,x^s(\sigma;\tau,z)) && t \leq \sigma. \end{aligned} \tag{3.15}$$

By definition, $(y^s,y^u) = (x^s,x^u)(\cdot;\sigma,\hat{z})$ satisfies $T_\sigma(y^s,y^u) = \varphi_\sigma\hat{z}$, that is,

$$\begin{aligned} e^{-A_-(t-\sigma)}P_-\hat{z} &= (T_\sigma(y^s,y^u))^s(t) && t \geq \sigma \\ e^{A_+(t-\sigma)}P_+\hat{z} &= (T_\sigma(y^s,y^u))^u(t) && t \leq \sigma, \end{aligned} \tag{3.16}$$

where $(T_\sigma y)^s$ and $(T_\sigma y)^u$ are the components of $T_\sigma y$ in $\mathcal{X}^s_\sigma = \mathcal{X}^s_\sigma \times \mathcal{X}^u_\sigma$.

On the other hand, using the definition $\hat{z} = x^s(\sigma; \tau, z)$, we obtain

$$\begin{aligned}
\hat{z} \;=\; & e^{-A_-(\sigma-\tau)}P_- z - e^{-A_-\sigma}P_- x^u(0;\tau,z) - \int_0^\tau e^{-A_-(\sigma-\rho)}P_- B(\rho)x^u(\rho;\tau,z)\,d\rho \qquad (3.17)\\
& - \int_\sigma^\infty e^{A_+(\sigma-\rho)}P_+ B(\rho)x^s(\rho;\tau,z)\,d\rho + \int_\tau^\sigma e^{-A_-(\sigma-\rho)}P_- B(\rho)x^s(\rho;\tau,z)\,d\rho.
\end{aligned}$$

Substituting (3.17) into (3.16) yields

$$\begin{aligned}
e^{-A_-(t-\tau)}P_- z \;&=\; \int_0^\tau e^{-A_-(t-\rho)}P_- B(\rho)x^u(\rho;\tau,z)\,d\rho \\
&\quad - \int_\tau^\sigma e^{-A_-(t-\rho)}P_- B(\rho)x^s(\rho;\tau,z)\,d\rho \\
&\quad + e^{-A_- t}P_- x^u(0;\tau,z) + (T_\sigma(y^s,y^u))^s(t) \\
0 \;&=\; \int_\sigma^\infty e^{A_+(t-\rho)}P_+ B(\rho)x^s(\rho;\tau,z)\,d\rho + (T_\sigma(y^s,y^u))^u(t),
\end{aligned} \qquad (3.18)$$

for $t \geq \sigma$ and $t \leq \sigma$, respectively. Regarding (y^s, y^u) as unknowns, we can uniquely solve (3.18) since T_σ is invertible. Thus the unique solution (y^s, y^u) is given by (3.15). On the other hand, it is straightforward to calculate that

$$\begin{aligned}
y^s(t) &= x^s(t;\tau,z) \qquad t \geq \sigma \\
y^u(t) &= 0 \qquad t \leq \sigma
\end{aligned}$$

satisfies (3.18) as well, proving two of the four identities in (i). The remaining two are proved in a similar way, see also [25].

Next, we prove (ii). This is achieved by comparing the solutions $x(\cdot, \tau + h)$ and $x(\cdot, \tau)$ for small h. First, we take $h > 0$ and fix $z \in X^\alpha$ with $|z|_{X^\alpha} = 1$. The case $h < 0$ is proved similarly. Define

$$\begin{aligned}
y^s_h(t) &= \begin{cases} x^s(t, \tau+h) & t \geq \tau + h \\ z - x^u(t, \tau+h) & \tau + h \geq t \geq \tau \end{cases} \\
y^u_h(t) &= \quad x^u(t, \tau+h) \qquad\qquad t \leq \tau.
\end{aligned}$$

Then, $y_h \in \mathcal{X}^{E^u}_\tau$ since y^s_h is continuous at $t = \tau + h$. With an abuse of notation, we will denote the norms $|\cdot|_{\mathcal{X}^E_\tau}$ by $\|\cdot\|$ in this paragraph. We claim that the estimate

$$\|T_\tau y_h - T_\tau x(\cdot,\tau)\| \leq \mathrm{o}(1)\,(1 + \|y_h\|) \qquad (3.19)$$

holds for some function $\mathrm{o}(1)$ satisfying $\mathrm{o}(1) \to 0$ as h tends to zero. Assume for the moment that (3.19) is true. Since the inverse of T_τ is continuous, we then have

$$\|y_h - x(\cdot,\tau)\| \leq C_1\|T_\tau y_h - T_\tau x(\cdot,\tau)\| \leq \mathrm{o}(1)(1 + \|y_h\|) \leq \mathrm{o}(1)(1 + \|y_h - x(\cdot,\tau)\| + \|x(\cdot,\tau)\|)$$

for some constant $C_1 > 0$ independent of h which we subsume into the $\mathrm{o}(1)$ term. Therefore, we conclude that $\|y_h - x(\cdot,\tau)\| = \mathrm{o}(1) \to 0$ as h tends to zero. Thus, in order to prove (ii), it suffices to prove (3.19).

Note that, by definition, $T_{\tau+h}x(\cdot,\tau+h) = \varphi_{\tau+h}$. We compare $T_\tau y_h$ with $T_{\tau+h}x(\cdot,\tau+h)$. Consider $t \le \tau$ first. Using equation (3.1) and the definition of y_h, we obtain

$$
\begin{aligned}
(T_\tau y_h)^u(t) &= (T_{\tau+h}x(\cdot,\tau+h))^u(t) - \int_\tau^{\tau+h} e^{A_+(t-\sigma)}P_+B(\sigma)x^u(\sigma,\tau+h)\,d\sigma \\
&\quad - \int_\tau^{\tau+h} e^{A_+(t-\sigma)}P_+B(\sigma)(z - x^u(\sigma,\tau+h))\,d\sigma \\
&= e^{A_+(t-\tau-h)}P_+z + \mathrm{o}(1)\,\mathrm{O}(e^{-\eta|t-\tau|})\;(1+\|y_h\|),
\end{aligned}
$$

since the arguments in the integrals are bounded by $\|x(\cdot,\tau+h)\|$ which is bounded by $1+\|y_h\|$. Next, consider $t \ge \tau+h$. Then

$$
\begin{aligned}
(T_\tau y_h)^s(t) &= (T_{\tau+h}x(\cdot,\tau+h))^s(t) - \int_\tau^{\tau+h} e^{-A_-(t-\sigma)}P_-B(\sigma)(z - x^u(\sigma,\tau+h))\,d\sigma \\
&\quad + \int_{\tau+h}^{\tau} e^{-A_-(t-\sigma)}P_-B(\sigma)x^u(\sigma,\tau+h)\,d\sigma \\
&= e^{-A_-(t-\tau-h)}P_-z + \mathrm{o}(1)\,\mathrm{O}(e^{-\eta|t-\tau|})\,(1+\|y_h\|)
\end{aligned}
$$

holds. It remains to consider $\tau \le t \le \tau+h$.

$$
\begin{aligned}
(T_\tau y_h)^s(t) &= z - (T_{\tau+h}x(\cdot,\tau+h))^u(t) - \int_\tau^{t} e^{-A_-(t-\sigma)}P_-B(\sigma)(z - x^u(\sigma,\tau+h))\,d\sigma \\
&\quad + \int_t^{\tau+h} e^{A_+(t-\sigma)}P_+B(\sigma)z\,d\sigma + \int_t^{\tau} e^{-A_-(t-\sigma)}P_-B(\sigma)x^u(\sigma,\tau+h)\,d\sigma \\
&= z - e^{A_+(t-\tau-h)}P_+z + \mathrm{o}(1)\,\mathrm{O}(e^{-\eta|t-\tau|})\,(1+\|y_h\|).
\end{aligned}
$$

Summarizing the above inequalities and using $T_\tau x(\cdot,\tau) = \varphi_\tau$, we obtain

$$
\begin{aligned}
(T_\tau y_h)^s(t) \quad &- \quad (T_\tau x(\cdot,\tau))^s(t) = \\
&\begin{cases} e^{A_+(t-\tau)}(e^{-A_+h}P_+ - P_+)z + R^s(t) & t \ge \tau+h \\ z - e^{A_+(t-\tau-h)}P_+z - e^{A_-(t-\tau)}P_-z + R^s(t) & \tau+h \ge t \ge \tau \end{cases} \\
(T_\tau y_h)^u(t) \quad &- \quad (T_\tau x(\cdot,\tau))^u(t) = e^{-A_-(t-\tau-h)}(P_- - e^{-A_-h}P_-)z + R^u(t) \qquad t \le \tau
\end{aligned}
$$

for some remainder term with norm $\|R\| = \mathrm{o}(1)\,(1+\|y_h\|)$. This completes the proof of inequality (3.19).

It remains to show (iii). In order to prove uniform exponential decay for x^s, it suffices to consider $t,\tau \ge t^*$ for some t^* large. Indeed, as $x^s(t;\tau,z) = x^s(t;t^*,x^s(t^*;\tau,z))$ for $t > t^* > \tau$, we can employ boundedness of $x^s(t;\tau,z)$ on $t,\tau \le t^*$ and obtain the result in full generality. Up to this point, we have investigated the operator T on the interval $[0,\infty)$.

However, we may as well restrict to $[t^*, \infty)$. On this smaller interval, T is continuously invertible as $T = \mathrm{id} + I$ for some integral operator I which is small in norm on $[t^*, \infty)$ as B is small, see the proof of Lemma 3.2 or [25]. Thus the operators $x^s(t; \tau, \cdot)$ have uniform exponential bounds for $t \geq \tau \geq t^*$. The arguments for x^u are similar. Note that, by calculating the norm of I, the constant ϵ_0 determining the largest admissible norm of $B(t)$ on $[t^*, \infty)$ depends only on the choice of the exponent η.

Thus, T is onto and therefore continuously invertible. Finally, we construct the exponential dichotomy. Let

$$P(t)z = x^s(t; t, z).$$

By the semigroup property (i), $P(t)$ is a projection. Moreover, $P(t)$ is bounded as T^{-1} is. The invariance properties of $R(P(t))$ and $N(P(t))$ follow immediately from the invariance property (i). The uniform exponential bounds can be obtained from the uniform bounds on x^s and x^u.

Until now, we have only considered complements E^u which meet (3.7). Exponential dichotomies actually exist for any complement E^u of E^s and not just for the ones satisfying (3.7). Indeed, let x^s and x^u be the evolution operators for some complement satisfying (3.7) and denote the associated projections by $P(t)$. Choose an arbitrary complement $\tilde{E}^u$ of E^s and let $L : R(\mathrm{id} - P(0)) \to R(P(0))$ be a bounded operator such that $\mathrm{graph}\, L = \tilde{E}^u$. Define

$$\begin{array}{rcll} \tilde{P}(t) & := & P(t) - x^s(t; 0, \cdot)\, L\, x^u(0; t, \cdot) & t \geq 0 \\ \tilde{x}^s(t; \tau, \cdot) & := & x^s(t; \tau, \cdot)\, \tilde{P}(\tau) & t \geq \tau \geq 0 \\ \tilde{x}^u(t; \tau, \cdot) & := & (\mathrm{id} - \tilde{P}(t))\, x^u(t; \tau, \cdot)\, (\mathrm{id} - P(\tau)) & \tau \geq t \geq 0, \end{array} \tag{3.20}$$

then $\tilde{x}$ is an exponential dichotomy of (2.1) such that $R(\tilde{P}(0)) = \mathrm{graph}\, L$, see [25]. Note that we still have $R(\tilde{P}(0)) = E^s$ with E^s defined in (3.5).

Finally, by inspecting (3.1) and (3.20), we have

$$z \in E^s \quad \Longrightarrow \quad z = P_- z - \int_0^\infty e^{-A_+ \sigma} P_+ B(\sigma) x^s(\sigma; 0, z)\, d\sigma$$

as $x^u(0; 0, z) = (\mathrm{id} - P(0))z = 0$. It has been proved in Lemma 3.2 that the integral operator is the sum of a compact operator and an operator with norm less than $C\epsilon$ for some constant C independent of ϵ.

This completes the proof of Theorem 1.

3.5 Proof of the corollaries and Theorem 2

Proof of Corollary 1. The corollary follows easily from the characterization of the stable subspaces in Theorem 1. ■

Proof of Corollary 2. We prove the corollary for complements E^u satisfying (3.7). Using the expression (3.20), it is straightforward to show the statements of the corollary for arbitrary complements.

It is straightforward to verify that the right hand side of the integral equation (3.1) is well-defined and an isomorphism from $\mathcal{X}^{E^u}$ to $\mathcal{X}^{X_+}$ even for $\eta = \delta$ provided $B(t)$ decays exponentially as $t \to \infty$. This proves the claim concerning the choice of η.

The projection $P(t)$ satisfies

$$\begin{aligned} P(t)z &= P_- z - e^{-A_- t} P_- x^u(0;t,z) - \int_0^t e^{-A_-(t-\sigma)} P_- B(\sigma) x^u(\sigma;t,z)\, d\sigma \\ &\quad + \int_t^\infty e^{-A_+(t-\sigma)} P_+ B(\sigma) x^s(\sigma;t,z)\, d\sigma. \end{aligned} \tag{3.21}$$

We will prove the corollary using the assumption that $B(t)$ decays exponentially with rate θ. Using (3.21) and Theorem 1, we have

$$\begin{aligned} |P(t)z - P_- z|_{X^\alpha} &\leq |e^{-A_- t} P_- x^u(0;t,z)|_{X^\alpha} + \Big| \int_0^t e^{-A_-(t-\sigma)} P_- B(\sigma) x^u(\sigma;t,z)\, d\sigma \Big|_{X^\alpha} \\ &\quad + \Big| \int_t^\infty e^{-A_+(t-\sigma)} P_+ B(\sigma) x^s(\sigma;t,z)\, d\sigma \Big|_{X^\alpha} \\ &\leq C e^{-(\delta+\eta)t} |z|_{X^\alpha} + C\hat{C} \Big| \int_0^t (1 + (t-\sigma)^{-\alpha}) e^{-\delta(t-\sigma)}\, e^{-\theta\sigma}\, e^{-\eta(t-\sigma)}\, d\sigma \Big|\, |z|_{X^\alpha} \\ &\quad + C\hat{C} \Big| \int_t^\infty (1 + (t-\sigma)^{-\alpha}) e^{-\delta(\sigma-t)}\, e^{-\theta\sigma}\, e^{-\eta(\sigma-t)}\, d\sigma \Big|\, |z|_{X^\alpha} \\ &\leq \tilde{C}(e^{-(\delta+\eta)t} + e^{-\theta t})\, |z|_{X^\alpha}, \end{aligned}$$

which proves the corollary. ■

Proof of Theorem 2. If (2.1) has an exponential dichotomy $P(t)$ on $\mathbb{R}$, any bounded solution $x(t)$ satisfies $(\mathrm{id} - P(0))x(0) = 0$, since $x(t)$ is bounded for $t \geq 0$. Similarly, $P(0)x(0) = 0$ on account of boundedness of $x(t)$ for $t \leq 0$. Therefore, $x(0) = 0$, which implies $x(\cdot) = 0$ by the uniqueness hypothesis (H5).

Assume conversely, that $x(\cdot) = 0$ is the only bounded solution of (2.1) on $\mathbb{R}$. The mild formulation (3.1) can be written as

$$\begin{aligned} T^- x &= \varphi^- \xi && t \in \mathbb{R}^+ \\ T^+ x &= \varphi^+ \xi && t \in \mathbb{R}^-. \end{aligned}$$

Here, T^+ and φ^+ denote right and left hand side of (3.1), respectively, for $t \in \mathbb{R}^+$, while T^- and φ^- correspond to the mild formulation on $J = \mathbb{R}^-$. We denote the associated projections of the exponential dichotomies by $P(t)$ and $Q(t)$ defined for $t \in \mathbb{R}^+$ and $t \in \mathbb{R}^-$, respectively. We have $R(P(0)) \cap R(\mathrm{id} - Q(0)) = \{0\}$, since, by assumption, equation (2.1) has no bounded non-trivial solution on $\mathbb{R}$. Therefore, $R(\mathrm{id} - Q(0))$ is a complement of $R(P(0))$ whence we can construct an exponential dichotomy on $\mathbb{R}^+$ with associated projection $\tilde{P}(t)$ such that $R(\tilde{P}(0)) = R(P(0))$ and $N(\tilde{P}(0)) = R(\mathrm{id} - Q(0))$. By the same token, an exponential dichotomy exists for $t \in \mathbb{R}^-$ such that the associated projection at $t = 0$ is again given by $\tilde{P}(0)$. Thus, the projections are continuous at $t = 0$, whence we obtain an exponential dichotomy on $\mathbb{R}$. ■

4 Regularity and nonlinear equations

From now on, we will use the notation

$$\begin{aligned} \Phi^s(t,\tau)z &:= x^s(t;\tau,z), && t \geq \tau \\ \Phi^u(t,\tau)z &:= x^u(t;\tau,z), && t \leq \tau, \end{aligned}$$

where $z \in X^\alpha$ and $t, \tau \in J$. Indeed, in the last section, we considered the solutions $x^s(t;\tau,z)$ and $x^u(t;\tau,z)$ for *fixed* $z \in X^\alpha$. Here, however, z will vary. We therefore emphasize the operator-point-of-view and choose a notation which is closer to semigroup theory.

In this section, we will verify some additional properties for the families $\Phi^s(t,\tau)$ and $\Phi^u(t,\tau)$ of evolution operators where $t, \tau \in J$ with $t \geq \tau$ and $t \leq \tau$, respectively. The statements are similar to the parabolic case, where the ranges $R(\Phi^u(t,\tau))$ are finite-dimensional for $t \leq \tau$, see [14, Theorem 7.1.3]. However, the Gronwall-type lemma which is the main tool in Henry's proof is not available in the present setting.

Theorem 3 *Assume that A and $B(t)$ satisfy the conditions of Theorem 1 with $J = \mathbb{R}^+$. The evolution operators $\Phi^s(t,\tau)$ with $t, \tau \in J$ and $t \geq \tau$ then have the following properties.*

(i) For $t \geq \tau$, $\Phi^s(t,\tau)$ has a bounded extension to X satisfying $\Phi^s(t,t) = P(t)$ and $\Phi^s(t,\sigma)\Phi^s(\sigma,\tau)z = \Phi^s(t,\tau)z$ for all $t \geq \sigma \geq \tau$ and any $z \in X$.

(ii) For fixed $0 \leq \beta < 1$, $\Phi^s(t,\tau)$, $t \geq \tau$ is strongly continuous in (t,τ) with values in $L(X^\beta)$.

(iii) For any $0 \leq \gamma, \beta < 1$, there is a constant $C > 0$ such that $\Phi^s(t,\tau) \in L(X^\gamma, X^\beta)$ for $t > \tau$ and

$$\|\Phi^s(t,\tau)\|_{L(X^\gamma,X^\beta)} \leq C \max(1, (t-\tau)^{\gamma-\beta})\, e^{-\eta(t-\tau)}.$$

Analogous properties hold for $\Phi^u(t,\tau)$ with $t,\tau \in J$ and $t \leq \tau$.

Proof. As mentioned above, the assertion of the theorem is similar to [14, Theorem 7.1.3]. However, the weak integral formulation (3.1) involves integrals over intervals $[0,t]$ and $[t,\infty)$. Moreover, these integrals are not small. We therefore cannot use the Gronwall lemma but have to adopt a different strategy. For the sake of clarity, we take the exponential weight $\eta = 0$.

First, we prove (i) and (ii). Note that the claims are true if $\beta \geq \alpha$ by applying Theorem 1 to the space X^β. Thus, we would like to solve the equation $Tx = \varphi z$ for $z \in X^\beta$ with $\beta < \alpha$. However, φz is continuous with values in X^α only for $t \neq \tau$, but satisfies an estimate

$$|(\varphi z)^s(t,\tau)|_{X^\alpha} = |e^{-A_-(t-\tau)}P_- z|_{X^\alpha} \leq C|t-\tau|^{\beta-\alpha}|z|_{X^\beta},$$

as $t \to \tau$, and similarly for $(\varphi z)^u(t)$.

The key idea is to subtract the part coming from the autonomous equation, that is the operator φz, from the solution $x(t,\tau)$. So, define

$$y^1(t;\tau,z) = x(t;\tau,z) - (\varphi z)(t-\tau).$$

The new unknown y^1 satisfies the equation $Ty^1 = \varphi^1 z$ where φ^1 is given by

$$\varphi^1 z = (\mathrm{id} - T)\varphi z.$$

Again, the crucial point is continuity of φ^1 as $t \to s$. We claim that φ^1 is continuous with values in X^γ for any $\gamma < 1 - \alpha + \beta$, and satisfies the slightly better estimate

$$|(\varphi^1 z)^s(t,\tau)|_{X^\alpha} \leq C|t-\tau|^{\beta-\alpha+(1-\alpha)}|z|_{X^\beta},$$

as $t \to \tau$, and similarly for $(\varphi^1 z)^u$. Assuming that the claim has been proved, we may proceed by induction. Let

$$y^k = x - \sum_{i=0}^{k-1} (\mathrm{id} - T)^i \varphi z$$

which satisfies the equation

$$Ty^k = (\mathrm{id} - T)^k \varphi z. \tag{4.1}$$

By the same arguments as in the first step, we see that the right hand side of this equation is continuous for $z \in X^\beta$ with values in X^α provided $k(1-\alpha) > \alpha - \beta$.

So, we have split the solution x in a well-behaving, continuous part y^k and explicitly given discontinuous parts $(\mathrm{id} - T)^i \varphi z$, which behave better than φz. Choosing k large enough, we can solve equation (4.1) as its right hand side is continuous with values in X^α.

From this observation, (i) and (ii) follow immediately. Indeed, the explicit part

$$\sum_{i=0}^{k-1} (\mathrm{id} - T)^i \varphi z$$

extends to X^β for any $\beta < \alpha$. Therefore, it suffices to prove the smoothing property for the operators $(\mathrm{id} - T)^i$.

The function $\varphi^1 z = (\mathrm{id} - T)\varphi z$ is given by

$$\begin{aligned}
(\varphi^1 z)^s(t,\tau) = &- \int_t^\infty e^{A_+(t-\sigma)} P_+ B(\sigma) e^{-A_-(\sigma-\tau)} P_- z \, d\sigma \\
&+ \int_\tau^t e^{-A_-(t-\sigma)} P_- B(\sigma) e^{-A_-(\sigma-\tau)} P_- z \, d\sigma \\
&- \int_0^\tau e^{-A_-(t-\sigma)} P_- B(\sigma) e^{-A_+(\sigma-\tau)} P_+ z \, d\sigma, \qquad t \geq \tau \\
(\varphi^1 z)^u(t,\tau) = &\int_\tau^t e^{A_+(t-\sigma)} P_+ B(\sigma) e^{-A_+(\sigma-\tau)} P_+ z \, d\sigma \\
&- \int_t^0 e^{-A_-(t-\sigma)} P_- B(\sigma) e^{-A_+(\sigma-\tau)} P_+ z \, d\sigma \\
&+ \int_\tau^\infty e^{A_+(t-\sigma)} P_+ B(\sigma) e^{-A_-(\sigma-\tau)} P_- z \, d\sigma, \qquad t \leq \tau,
\end{aligned}$$

see (3.1), as the exponential terms disappear due to the definition of φz. Note that this property is preserved under the iteration $(\mathrm{id} - T)^k$ for the same reason as in the proof of Proposition 1.

First, consider the integral

$$(I_1 g)(t,\tau) = \int_t^\infty e^{A_+(t-\sigma)} P_+ B(\sigma) g(\sigma,\tau) \, d\sigma$$

where $g(t,\tau)$ is continuous for $t>\tau$ with values in X^α satisfying

$$|g(t,\tau)|_{X^\alpha} \le C|t-\tau|^{-\theta}$$

as $t\to\tau$ for some $\theta>0$. Notice that I_1 is continuous for $t>\tau$ with values in X^α. We estimate

$$\begin{aligned}
|(I_1 g)(\tau+h,\tau)|_{X^\alpha} &\le \Big|\int_{\tau+h}^{\infty} e^{A_+(\tau+h-\sigma)}P_+B(\sigma)g(\sigma,\tau)\,d\sigma\Big|_{X^\alpha} \\
&\le C\Big|\int_{\tau+h}^{\infty} e^{\delta(\tau+h-\sigma)}\,|\tau+h-\sigma|^{-\alpha}\,|\tau-\sigma|^{-\theta}\,d\sigma\Big| \\
&\le \hat{C}h^{1-\alpha-\theta}
\end{aligned}$$

as $h\to 0$ for some constants C and $\hat{C}$ independent of h. Thus, as claimed, the exponent θ is decreased by $1-\alpha$. The calculations for the other integral operators are similar, and we will omit them.

The proof of (iii) is completely analogous to the above and we will omit it, too. ∎

Theorem 1 and 3 are used for obtaining existence of solutions of inhomogeneous linear equations

$$\dot{x} = (A+B(t))x + f(t) \qquad\qquad f\in C^{0,\vartheta}(\mathbb{R}^+,X),\ \vartheta>0$$

as well as nonlinear equations

$$\dot{x} = (A+B(t))x + G(t,x) \qquad\qquad G\in C^{1,1}(\mathbb{R}^+\times X^\alpha,X)$$

with $G(t,0)=DG(t,0)=0$. The associated weak formulation is given by

$$\begin{aligned}
e^{-A_-(t-\tau)}P_-z &= x^s(t,\tau) + e^{-A_-t}P_-x^u(0,\tau) \\
&\quad + \int_t^\infty e^{A_+(t-\sigma)}P_+\Big(B(\sigma)x^s(\sigma,\tau)+F(\sigma,x^s(\sigma,\tau))\Big)d\sigma \\
&\quad - \int_\tau^t e^{-A_-(t-\sigma)}P_-\Big(B(\sigma)x^s(\sigma,\tau)+F(\sigma,x^s(\sigma,\tau))\Big)d\sigma \\
&\quad + \int_0^\tau e^{-A_-(t-\sigma)}P_-\Big(B(\sigma)x^u(\sigma,\tau)+F(\sigma,x^u(\sigma,\tau))\Big)d\sigma \\
e^{A_+(t-\tau)}P_+z &= x^u(t,\tau) - e^{-A_-t}P_-x^u(0,\tau) \\
&\quad - \int_\tau^t e^{A_+(t-\sigma)}P_+\Big(B(\sigma)x^u(\sigma,\tau)+F(\sigma,x^u(\sigma,\tau))\Big)d\sigma \\
&\quad + \int_t^0 e^{-A_-(t-\sigma)}P_-\Big(B(\sigma)x^u(\sigma,\tau)+F(\sigma,x^u(\sigma,\tau))\Big)d\sigma \\
&\quad - \int_\tau^\infty e^{A_+(t-\sigma)}P_+\Big(B(\sigma)x^s(\sigma,\tau)+F(\sigma,x^s(\sigma,\tau))\Big)d\sigma,
\end{aligned} \tag{4.2}$$

where F is replaced by either f or G. In the former case, using Theorem 1 and 3, existence is easily obtained, see [14, Theorem 7.1.4]. In the latter case, the right hand side of (4.2) defines a differentiable map from $\mathcal{X}^{E^u}$ to $\mathcal{X}^{X_+}$ with $\eta = 0$. Also, the linear part is invertible as T is. Thus, we may employ an implicit function theorem and obtain solution operators $\Phi^s(t;\tau,z)$ and $\Phi^u(t;\tau,z)$ for $t \geq \tau$ and $0 \leq t \leq \tau$, respectively, defined for small $z \in X^\alpha$ and depending smoothly on z.

5 Transverse homoclinic orbits in periodically perturbed equations

In this section, we extend the Melnikov theory, see, for instance, [18] or [22], for intersections of stable and unstable manifolds to the general class of differential equations investigated in the previous sections. Except for the proof of Theorem 4, we can closely follow the presentation in [22], and will only indicate the changes necessary to adapt the proofs given there to the situation studied here. We refer to [2] and [24] for proofs for parabolic equations.

Throughout this section, we assume that X is a reflexive Banach space, and A is a closed operator on X with compact resolvent satisfying Hypothesis (H1) stated in Section 2. Consider the following small non-autonomous perturbation of an autonomous nonlinear equation

$$\dot{x} = Ax + G(x) + \mu H(t,x,\mu) \qquad (x,\mu) \in X^\alpha \times \mathbb{R} \tag{5.1}$$

for some fixed $\alpha \in [0,1)$. Suppose that $G \in C^{1,1}(X^\alpha, X)$ with $G(0) = 0$ and $DG(0) = 0$. The perturbation H belongs to $C^1(\mathbb{R} \times X^\alpha \times \mathbb{R}, X)$ such that, in addition,

$$t \to D_t H(t,x,\mu) \qquad \text{and} \qquad x \to D_x H(t,x,\mu)$$

are locally Hölder and Lipschitz continuous, respectively, in the operator norm. Furthermore, H is periodic in t with period p, that is $H(t+p,\cdot,\cdot) = H(t,\cdot,\cdot)$ for all $t \in \mathbb{R}$.

(H6) *Assume that A meets Hypothesis (H1) and has compact resolvent. Suppose that equation (5.1) has a homoclinic orbit for $\mu = 0$, that is a solution $q(t) \in C^1(\mathbb{R}, X^\alpha) \cap C^0(\mathbb{R}, X^1)$ with $q(t) \to 0$ as $t \to \pm\infty$. We assume that the operator $DG(q(t))$ satisfies Hypothesis (H5). Finally, assume that $\dot{q}(t)$ is the only bounded solution (up to constant* *multiples) of the variational equation*

$$\dot{x} = Ax + DG(q(t))x \tag{5.2}$$

along $q(t)$.

Note that Hypothesis (H2) is met for the variational equation for any $\epsilon > 0$ since $q(t) \to 0$. Hypothesis (H3) is also satisfied since the resolvent $A^{-1} \in L(X)$ of A is compact. With these assumptions at hand, equation (5.2) and its adjoint equation

$$\dot{y} = -(A^* + DG(q(t))^*)y \tag{5.3}$$

have exponential dichotomies on the intervals $\mathbb{R}^+$ and $\mathbb{R}^-$ by Theorem 1. Moreover, the results of Section 4 apply to the nonlinear equation (5.1), and all bounded solutions close to the homoclinic orbit are given by (4.2).

It is then a consequence of Hypothesis (H6) that the adjoint equation (5.3) has a unique, up to scalar multiples, bounded solution $\psi(t)$. The proof is similar to the one given in [22]. We define the Melnikov integral

$$M(\beta) = \int_{-\infty}^{\infty} \langle \psi(t), H(t-\beta, q(t), 0) \rangle \, dt \tag{5.4}$$

for $\beta \in S^1 = [0,p]/\sim$. Note that M is C^1 in β. The next theorem characterizes transverse intersections of the stable and unstable manifold of zero (more precisely, of the unique hyperbolic p-periodic orbit μ-close to zero).

Theorem 4 *Assume that Hypothesis (H6) is met. If there is a number $\beta_0 \in S^1$ such that $M(\beta_0) = 0$ and $M'(\beta_0) \neq 0$, then there exist positive constants μ_0 and δ_0 such that equation (5.2) has a unique solution $x(t,\mu)$ for any μ with $0 < |\mu| < \mu_0$ satisfying*

$$\sup_{t \in \mathbb{R}} |x(t,\mu) - q(t+\beta_0)|_{X^\alpha} \leq \delta_0.$$

In fact,

$$\sup_{t \in \mathbb{R}} |x(t,\mu) - q(t+\beta_0)|_{X^\alpha} = \mathrm{O}(\mu)$$

as $\mu \to 0$ and the variational equation

$$\dot{y} = (A + DG(x(t,\mu)) + \mu D_x H(t, x(t,\mu), \mu))y \tag{5.5}$$

has an exponential dichotomy on $\mathbb{R}$.

Proof. First, we prove the existence of $x(t,\mu)$. We introduce a new variable z by

$$x(t) = q(t+\beta) + z(t+\beta) \qquad\qquad \beta \in \mathbb{R},$$

and write equation (5.1) in the form

$$\dot{z} = Az + DG(q(t))z + F(t,z,\mu,\beta). \tag{5.6}$$

with

$$F(t,z,\mu,\beta) = G(q(t)+z) - G(q(t)) - DG(q(t))z + \mu H(t-\beta, q(t)+z,\mu).$$

On account of Theorem 1 and the hypotheses made, we know that the linear part of equation (5.6), that is equation (5.2), has an exponential dichotomy on $\mathbb{R}^+$ and $\mathbb{R}^-$, respectively. As in Section 4 and Theorem 3, we denote the solution operators of (5.2) by $\Phi_1^s(t,\tau)$ and $\Phi_1^u(t,\tau)$ for $t \geq \tau \in \mathbb{R}^+$ and $\tau \geq t \in \mathbb{R}^+$, respectively, and by $\Phi_2^u(t,\tau)$ and $\Phi_2^s(t,\tau)$ for $t \leq \tau \in \mathbb{R}^-$ and $\tau \leq t \in \mathbb{R}^-$, respectively. We decompose the subspaces of bounded solutions for $t \to \pm\infty$ according to

$$R(\Phi_1^s(0,0)) = Y_1 \oplus \operatorname{span}\dot{q}(0) \qquad \text{and} \qquad R(\Phi_2^u(0,0)) = Y_2 \oplus \operatorname{span}\dot{q}(0).$$

Solutions of the nonlinear equation (5.6) are bounded on $\mathbb{R}^+$ and $\mathbb{R}^-$, respectively, if and only if there exist $\xi_1 \in Y_1$ and $\xi_2 \in Y_2$ such that

$$\begin{aligned}
z_1(t) &= \Phi_1^s(t,0)\xi_1 + \int_0^t \Phi_1^s(t,\tau)F(\tau,z_1(\tau),\mu,\beta)\,d\tau \\
&\quad - \int_t^\infty \Phi_1^u(t,\tau)F(\tau,z_1(\tau),\mu,\beta)\,d\tau \qquad\qquad \text{for } t \in \mathbb{R}^+ \\
z_2(t) &= \Phi_2^u(t,0)\xi_2 + \int_0^t \Phi_2^u(t,\tau)F(\tau,z_2(\tau),\mu,\beta)\,d\tau \\
&\quad + \int_{-\infty}^t \Phi_2^s(t,\tau)F(\tau,z_2(\tau),\mu,\beta)\,d\tau \qquad\qquad \text{for } t \in \mathbb{R}^-,
\end{aligned}$$

respectively. Thus, for any $\xi_1 \in Y_1$ and $\xi_2 \in Y_2$ near zero, we get bounded solutions $z_1(t;\xi_1,\beta,\mu)$ and $z_2(t;\xi_2,\beta,\mu)$ of equation (5.6) for $t \in \mathbb{R}^+$ and $t \in \mathbb{R}^-$, respectively, by the implicit function theorem, see Theorem 3. The maps $(\xi_1,\beta,\mu) \to z_1(t;\xi_1,\beta,\mu)$ and $(\xi_2,\beta,\mu) \to z_2(t;\xi_2,\beta,\mu)$ are C^1. Next, for any small μ, we seek $\xi = \xi_1 + \xi_2 \in Y_1 \oplus Y_2$ and $\beta \in S^1$ such that $z_1(0;\xi,\beta,\mu) = z_2(0;\xi,\beta,\mu)$. This is equivalent to solving the equation

$$\begin{aligned}
(\Phi_1^s(0,0) - \Phi_2^u(0,0))\xi &= \int_{-\infty}^0 \Phi_2^s(0,\tau)F(\tau,z_2(\tau,\xi,\beta,\mu),\mu,\beta)\,d\tau \\
&\quad + \int_0^\infty \Phi_1^u(0,\tau)F(\tau,z_1(\tau,\xi,\beta,\mu),\mu,\beta)\,d\tau.
\end{aligned} \tag{5.7}$$

According to the proof of Theorem 1, $L = \Phi_1^s(0,0) - \Phi_2^u(0,0) \in L(X^\alpha)$ is a Fredholm operator with index zero, null space $N(L) = \operatorname{span} \dot q(0)$ and range $R(L) = \{\eta \in X^\alpha;\ \langle \psi(0), \eta\rangle = 0\}$. Therefore, using Lyapunov-Schmidt reduction, it follows that equation (5.7) is solvable near $\beta = \beta_0$ if and only if

$$\begin{aligned} \int_{-\infty}^{\infty} \langle \psi(t), H(t-\beta_0, q(t), 0)\rangle \, dt &= 0 \\ \int_{-\infty}^{\infty} \langle \psi(t), D_\beta H(t-\beta_0, q(t), 0)\rangle \, dt &\neq 0 \end{aligned}$$

for some $\beta_0 \in S^1$. The solution is given by $x(t,\mu) = q(t+\beta(\mu)) + z(t+\beta(\mu),\mu)$ with $\beta(\cdot) \in C^1((-\mu_0,\mu_0), \mathbb{R})$ and $\beta(0) = \beta_0$. This proves the first part of the theorem.

It remains to show that equation (5.5) has an exponential dichotomy on $\mathbb{R}$. On account of Theorem 1, equation (5.5) has an exponential dichotomy on $\mathbb{R}^+$ and $\mathbb{R}^-$, respectively, for any small μ.

For a bounded solution $y(t)$ of equation (5.5), we set $y(t) = \dot x(t,\mu) + w(t)$ such that

$$\begin{aligned} \dot w &= \Big(A + DG(x(t,\mu)) + \mu D_x H(t, x(t,\mu),\mu)\Big) w - \mu D_t H(t, x(t,\mu),\mu) \\ &= \Big(A + DG(q(t,\mu))\Big) w + \Big(DG(x(t,\mu)) - DG(q(t,\mu)) + \\ &\quad \mu D_x H(t, x(t,\mu),\mu)\Big) w - \mu D_t H(t, x(t,\mu),\mu) \\ &= \Big(A + DG(q(t,\mu))\Big) w + \mathrm{O}(\mu) w - \mu D_t H(t, x(t,\mu),\mu). \end{aligned} \tag{5.8}$$

Lyapunov-Schmidt reduction shows that this equation has a bounded solution if and only if

$$\begin{aligned} \tilde M(\mu) &:= \int_{-\infty}^{\infty} \Big\langle \psi(t+\beta(\mu)), \Big(DG(x(t,\mu)) - DG(q(t+\beta(\mu))) + \\ &\quad \mu D_x H(t, x(t,\mu),\mu)\Big) w(t,\mu) - \mu D_t H(t, x(t,\mu),\mu)\Big\rangle \, dt \\ &= 0, \end{aligned}$$

where $w(t,\mu) = \mathrm{O}(\mu)$ satisfies the invertible part of (5.8). Therefore,

$$\begin{aligned} \tilde M(\mu) &= -\mu \int_{-\infty}^{\infty} \langle \psi(t), D_t H(t-\beta_0, q(t),\mu)\rangle \, dt + \\ &\int_{-\infty}^{\infty} \Big\langle \psi(t+\beta(\mu)), \Big(DG(x(t,\mu)) - DG(q(t+\beta(\mu))) + \mu D_x H(t, x(t,\mu),\mu)\Big) w(t,\mu) - \\ &\mu \Big(D_t H(t, x(t,\mu),\mu) - D_t H(t, q(t+\beta_0),\mu)\Big)\Big\rangle \, dt. \end{aligned}$$

The first integral is $M'(\beta_0)$ which we keep. The other integral is of order $\mathrm{o}(\mu)$. Indeed, $DG(x)$ is Lipschitz continuous in x, $w(t,\mu) = \mathrm{O}(\mu)$, and $x(t,\mu) - q(t+\beta(\mu)) = z(t,\mu) =$

$\mathrm{O}(\mu)$, whence the term involving w is of order $\mathrm{O}(\mu^2)$. The difference $D_tH(t, x(t, \mu), \mu) - D_tH(t, q(t+\beta_0), \mu) = \mathrm{o}(1)$ converges to zero as μ tends to zero since β is C^1 and $D_tH(t, x, \mu)$ is continuous in x. Thus, we have

$$\tilde{M}(\mu) = -M'(\beta_0)\mu + \mathrm{o}(\mu),$$

which is non-zero since $M'(\beta_0) \neq 0$. An application of Theorem 2 then shows that equation (5.5) has an exponential dichotomy on $\mathbb{R}$. ■

We proceed by proving the shadowing lemma, see also [2] for a proof for the parabolic case. We consider the slightly more general nonlinear equation

$$\dot{x} = Ax + F(t, x) \tag{5.9}$$

with $F \in BC^1(\mathbb{R} \times X^\alpha, X)$ for some $\alpha \in [0, 1)$ and $D_xF(t, \cdot)$ being Lipschitz. Note that F is not necessarily periodic in t.

Theorem 5 *Assume that A satisfies Hypothesis (H1) and has compact resolvent. Furthermore, suppose that equation (5.9) has solutions $u_{-n_1}(t)$, $u_k(t)$, and $u_{n_2}(t)$ for $-n_1 < k < n_2$ defined on the intervals $I_{-n_1} = (-\infty, t_{-n_1}]$, $I_k = [t_{k-1}, t_k]$, and $I_{n_2} = [t_{n_2}, \infty)$ for $-n_1 < k < n_2$, respectively, such that*

(i) the variational equation

$$\dot{y} = (A + D_xF(t, u_k(t)))y$$

has an exponential dichotomy on I_k with projections $P_k(t)$, exponent δ and bound K for $-n_1 \leq k \leq n_2$. Also, Hypotheses (H2) and (H5) are met for the variational equation.

(ii) $|t_k - t_{k-1}| \geq \delta^{-1} \ln 3K$.

Then, there exists a positive constant ϵ_0 such that the following holds. For any ϵ with $0 < \epsilon < \epsilon_0$ there exists a constant $\nu(\epsilon) > 0$ such that, if in addition

(iii) $|u_{k-1}(t_{k-1}) - u_k(t_{k-1})|_{X^\alpha} \leq \nu(\epsilon)$, and

(iv) $\|P_{k-1}(t_{k-1}) - P_k(t_{k-1})\|_{L(X^\alpha)} \leq \nu(\epsilon)$,

are met, equation (5.9) has a unique bounded solution $x(t)$ on $\mathbb{R}$ satisfying

$$|x(t) - u_k(t)|_{X^\alpha} < \epsilon$$

for $t \in I_k$ and $-n_1 \leq k \leq n_2$.

Proof. We define a function $u(t)$ for $t \in \mathbb{R}$ by $u(t) = u_k(t)$ for $t \in I_k$. Then, $u(t)$ is Hölder continuous except at the points t_k. For any fixed $\gamma > 0$, there is a function $\theta(t) \in L^\infty(\mathbb{R}, X)$ with $\sup_{t\in\mathbb{R}} |\theta(t)|_X < \gamma$ such that $F(u(t),t) + \theta(t)$ is Hölder continuous on $\mathbb{R}$. We approximate $u(t)$ by the unique bounded solution $z(t)$ of the equation

$$\dot{z} = Az + F(u(t),t) + \theta(t).$$

Since the equation $\dot{z} = Az$ has an exponential dichotomy on $\mathbb{R}$, the above equation has a unique solution. We have the estimate

$$|u(t) - z(t)|_{X^\alpha} \leq C(\gamma + \nu)$$

for some constant $C > 0$. Thus, for ν and γ sufficiently small, and due to Hypothesis (ii),

$$\dot{y} = (A + D_x F(t, z(t)))y$$

has an exponential dichotomy on $\mathbb{R}$, see [22] for the details.

Finally, we introduce new coordinates $x(t) = z(t) + w(t)$ and write equation (5.9) in the form

$$\begin{aligned} \dot{w} \;&=\; (A + D_x F(t,z(t)))w + F(t, z(t) + w) - F(t, z(t)) - D_x F(t, z(t))w \\ &\quad + F(t, z(t)) - F(t, u(t)) - \theta(t). \end{aligned}$$

For γ and ν small, we thus obtain a unique solution of equation (5.9) employing an implicit function theorem. ■

We now define the Bernoulli shift. Let N be a positive integer and

$$\mathcal{S}_N = \{(a_k)_{k\in\mathbb{Z}};\ a_k \in \{0, ..., N-1\} \text{ for all } k \in \mathbb{Z}\}$$

with the product topology. The shift $\sigma : \mathcal{S}_N \to \mathcal{S}_N$, defined by $(\sigma(a))_k = a_{k+1}$, is a homeomorphism.

Corollary 3 *Assume that the hypotheses of Theorem 5 are met and that, in addition, $F(t,x)$ is periodic in t with period p. Moreover, suppose that (5.9) has a bounded solution $v(t)$ and a T-periodic solution $u(t)$ such that* *(i) the variational equation*

$$\dot{y} = Ay + D_x F(t, v(t))y$$

has an exponential dichotomy on $\mathbb{R}$ *and*

(ii) $|v(t) - u(t)|_{X^\alpha} \to 0$ *as* $|t| \to \infty$.

Then there are $\epsilon_0 > 0$ *and functions* $M_N(\cdot)$ *for each* $N \in \mathbb{N}$ *such that, for given* ϵ *with* $0 < \epsilon \leq \epsilon_0$ *and* $m \geq M_N(\epsilon)$ *the following holds. For any* $a \in \mathcal{S}_N$*, equation (5.9) has a unique bounded solution* $x_a(t)$ *defined on* $\mathbb{R}$ *satisfying*

$$|x_a(t + (2k-1)mT) - v(t + a_k T)|_{X^\alpha} \leq \epsilon \tag{5.10}$$

for $t \in [-mT, mT]$ *and for all* $k \in \mathbb{Z}$*. The map* $\phi(a) = x_a(0)$ *is a homeomorphism onto a compact subset* Σ *of* X^α*. Furthermore,*

$$x_a(2mp) \in \Sigma$$
$$x_a(2mp) = x_{\sigma(a)}(0) = \phi(\sigma(a))$$

is true for any $a \in \mathcal{S}_N$*.*

Proof. The conditions of Theorem 5 are satisfied for $k \in [-n_0, n_0]$ and $n_0 \in \mathbb{N}$ if we define $u_k(t) = v(t + a_k T - (2k-1)mT)$ and $t_k = 2kmT$ for m large enough. Thus, for any n_0, we obtain a solution $x_{a_{n_0}}$ that satisfies inequality (5.10) for $k \in [-n_0, n_0]$. The sequence of solutions $\{x_{a_{n_0}}\}_{n_0 \in \mathbb{N}}$ is a Cauchy sequence on compact intervals and converges to the solution x_a. The remaining part of the proof is similar to the one given by Palmer [22, Corollary 3.6]. ■

We can interpret the statement of the corollary as follows. The solution $v(t)$ has N parts which correspond to the time segments

$$[-mT, mT],\ [(-m+1)T, (m+1)T], ..., [(-m+N-1)T, (m+N-1)T].$$

The solution $x_a(t)$ shadows one of these N parts of $v(t)$ in each time segment

$$[(2k-2)mT, 2kmT]$$

but switches randomly from one part to another.

6 An application to semilinear elliptic equations

In this section, we apply Melnikov's method as developed in the last section to semilinear elliptic equations. First, we have to relate the abstract equation investigated in the previous sections to elliptic equations. Then, elliptic equations on infinite cylinders are considered. We state conditions guaranteeing that the theory developed in the present paper applies. Finally, a concrete example on the infinite cylinder $\mathbb{R} \times (0, \pi)^n$ is presented.

6.1 Abstract elliptic equations

Let Y be a Hilbert space and $L : D(L) \subset Y \to Y$ a densely defined, strictly positive and self-adjoint operator. Moreover, denote the fractional power spaces associated with L by Y^α. In particular, $Y^1 = D(L)$. Finally, suppose that

$$g : Y^{\frac{1+\alpha}{2}} \times Y^{\frac{\alpha}{2}} \to Y$$

is a nonlinearity of class C^k for some $\alpha \in [0, 1)$ which we will fix from now on. We are interested in the abstract elliptic equation

$$u_{xx} - Lu = g(u, u_x) \qquad\qquad x \in \mathbb{R} \tag{6.1}$$

for $u \in Y^\alpha$.

Consider the operator

$$A = \begin{pmatrix} 0 & \mathrm{id} \\ L & 0 \end{pmatrix} : Y^1 \times Y^{\frac{1}{2}} \to Y^{\frac{1}{2}} \times Y, \tag{6.2}$$

then Hypothesis (H1) is met. In fact, the projections $P_\pm$ are given by

$$P_\pm = \frac{1}{2} \begin{pmatrix} \mathrm{id} & \pm L^{-\frac{1}{2}} \\ \pm L^{\frac{1}{2}} & \mathrm{id} \end{pmatrix} : Y^{\frac{1}{2}} \times Y \to Y^{\frac{1}{2}} \times Y,$$

and the operators $A_\pm$ by

$$A_\pm = \frac{1}{2} \begin{pmatrix} L^{\frac{1}{2}} & \pm \mathrm{id} \\ \pm L & L^{\frac{1}{2}} \end{pmatrix}.$$

The fractional powers are then given by

$$A_\pm^\alpha = \frac{1}{2} \begin{pmatrix} L^{\frac{\alpha}{2}} & \pm L^{\frac{\alpha-1}{2}} \\ \pm L^{\frac{1+\alpha}{2}} & L^{\frac{\alpha}{2}} \end{pmatrix}$$

with associated fractional power spaces $X^\alpha = Y^{\frac{1+\alpha}{2}} \times Y^{\frac{\alpha}{2}}$. Consider the equation

$$\frac{d}{dx} v = Av + G(v) \tag{6.3}$$

with $v = (u, u_x)$ and $G(v) = (0, g(v))$. Since $g : Y^{\frac{1+\alpha}{2}} \times Y^{\frac{\alpha}{2}} \to Y$ is C^k, we see that $G : X^\alpha \to X$ is C^k as well. Furthermore, it is straightforward to show that A has compact resolvent whenever L has.

Therefore, it suffices to verify the assumptions made on L and g stated at the beginning of this section in order to apply the results in Section 2 and 5 to equation (6.3) which is (6.1) written as a first order system in x. We emphasize that similar statements hold if (6.1) is of fourth order in x, and refer to a forthcoming paper for the details.

6.2 Semilinear elliptic equations on infinite cylinders

Consider a scalar semilinear elliptic equation

$$u_{xx} + \Delta_y u + \hat{g}(y, u, u_x, \nabla_y u) + \mu \hat{h}(x, y, u, u_x, \nabla_y u) = 0 \qquad (x, y) \in \mathbb{R} \times \Omega. \tag{6.4}$$

Here, μ is a small real parameter, h is periodic in x with period p and $\Omega \subset \mathbb{R}^n$ is an open bounded domain with smooth boundary. For the sake of simplicity, we consider Neumann boundary conditions

$$\partial_\nu u(x, y) = 0 \qquad\qquad (x, y) \in \mathbb{R} \times \partial\Omega \tag{6.5}$$

where ν denotes the outer normal of $\partial\Omega$. Let $Y = L^2(\Omega)$. Then $L = -\Delta_y + u$ is a self-adjoint and positive operator with compact resolvent and dense domain

$$Y^1 = D(L) = \{u \in H^2(\Omega);\ \partial_\nu u = 0 \text{ on } \partial\Omega\}$$

in $L^2(\Omega)$, see, for instance, [9]. Finally, we assume that the nonlinearities g and h defined by

$$\begin{aligned} (g(v_1, v_2))(y) &:= \hat{g}(y, v_1(y), v_2(y), (\nabla_y v_1)(y)) \\ (h(x, v_1, v_2))(y) &:= \hat{h}(x, y, v_1(y), v_2(y), (\nabla_y v_1)(y)) \end{aligned}$$

map the space $Y^{\frac{1+\alpha}{2}} \times Y^{\frac{\alpha}{2}}$ smoothly into $L^2(\Omega)$ for some $\alpha \in [0, 1)$. Depending on the dimension of Ω, this may require some nonlinear growth restrictions for which we refer to the literature, see, for instance, [1, Chapter 9], [29, Chapter II], and [27, Chapter 7]. We remark that the spaces chosen above always allow for linear dependence of $\hat{g}$ and $\hat{h}$ on the gradient u_x of u in the unbounded variable x. This is important when the elliptic equation describes travelling waves of parabolic equations travelling in the x-direction.

The uniqueness assumption (H5) is met under very weak conditions on equation (6.4). Indeed, Cordes [6, Satz 5] proved that any solution u of class C^2 satisfying

$$\begin{aligned} &u_{xx} + \Delta_y u + a(x,y)u_x + b(x,y)\nabla_y u + c(x,y)u = 0 && (x,y) \in \mathbb{R} \times \Omega \\ &u(0,y) = u_x(0,y) = 0 && y \in \Omega \end{aligned} \tag{6.6}$$

vanishes identically $u(x,y) = 0$ on $\mathbb{R} \times \Omega$ provided the coefficients a, b, and c are locally Lipschitz continuous.

Suppose that $q(x,y)$ is a homoclinic solution of (6.4) for $\mu = 0$ satisfying

$$\lim_{|x| \to \infty} q(x,y) = 0.$$

In addition, assume that $q_x(x,y)$ is the unique, up to scalar multiples, bounded solution of

$$\begin{aligned} &v_{xx} + \Delta_y v + D_{u_x}\hat{g}(y,q,q_x,\nabla_y q)v_x \\ &\quad + D_{\nabla_y u}\hat{g}(y,q,q_x,\nabla_y q)\nabla_y v + D_u \hat{g}(y,q,q_x,\nabla_y q)v = 0, \end{aligned} \tag{6.7}$$

which is of the form (6.6). Also, as $\lim_{|x|\to\infty} q(x,y) = 0$, the coefficients converge for $|x| \to \infty$ to functions depending only on y.

Thus, the theory developed in the previous sections applies. Indeed, using the results in Section 6.1, it is possible to write (6.4) as an evolution equation

$$\frac{d}{dx}v = Av + G(v) + \mu H(x,v) \tag{6.8}$$

where

$$A = \begin{pmatrix} 0 & \mathrm{id} \\ -\Delta_y + \mathrm{id} & 0 \end{pmatrix}$$

and

$$G(v)(y) = \begin{pmatrix} 0 \\ -g(v_1,v_2) - v_1 \end{pmatrix}, \qquad H(x,v)(y) = \begin{pmatrix} 0 \\ -\mu h(x,v_1,v_2) \end{pmatrix}.$$

The linearization

$$\frac{d}{dx}v = Av + DG(q,q_x)v$$

at the homoclinic solution satisfies Hypothesis (H5) whenever, for instance, Cordes' result applies to (6.7). Also, the smallness assumption (H2) is always satisfied based on the above remarks.

6.3 An example on an infinite cylinder

As an example, we take $\Omega = (0,\pi)^n$ and consider

$$u_{xx} + \gamma^2 \Delta_y u - u + u^2 + \mu(1 + h(y)) \cos x = 0 \qquad (x,y) \in \mathbb{R} \times (0,\pi)^n, \tag{6.9}$$

for $n \in \mathbb{N}$ with Neumann boundary conditions

$$\partial_y u(x,y) = 0 \qquad \text{for } (x,y) \in \mathbb{R} \times \partial\Omega.$$

Here, $\gamma \neq 0$, and $h(y)$ is a smooth function with zero mean, that is $\int_\Omega h(y)\, dy = 0$. Note that the nonlinearity is analytic for $\mu = 0$. Hence the uniqueness hypothesis (H5) is satisfied since any solution of either (6.9) or its linearization is analytic as well. Though the domain Ω is not smooth, equation (6.9) fits into the setting of the last section. Alternatively, the reader may consider the n-dimensional unit ball using spherical harmonics instead of the trigonometric expansion employed below.

We remark that the reduction to essential manifolds developed by Mielke [21] applies to equation (6.9) provided $n = 1$. However, as pointed out in the introduction, the resulting manifold will only be of class C^1. For $n > 1$, the results in [20] do not apply since they require that the nonlinearity is independent of x. Also, the example can be modified easily such that the spectral gaps are not arbitrarily large as required by any inertial-manifold reduction. Replace, for instance, Ω as defined above by $\prod_{j=1}^n (0, a_j\pi)$ with rationally independent constants $a_j > 0$.

Rewrite equation (6.9) according to

$$\begin{aligned} \frac{d}{dx} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -\gamma^2 \Delta_y + 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} 0 \\ v_1^2 + \mu(1 + h(y)) \cos x \end{pmatrix} \\ &= Av + G(v) + \mu H(x,v). \end{aligned}$$

Let $k \in \mathbb{N}_0^n$ be a multi-index and define $|k|^2 := \sum_{j=1}^n k_j^2$. Then, the eigenvalues of the linear operator A are given by

$$\lambda_k^\pm = \pm\sqrt{1 + \gamma^2 |k|^2} \qquad \text{for } k \in \mathbb{N}_0^n$$

with associated eigenfunctions

$$w_k^{\pm}(y) = \begin{pmatrix} 1 \\ \pm\sqrt{1+\gamma^2|k|^2} \end{pmatrix} \prod_{j=1}^{n} \cos k_j y_j \qquad \text{for } k \in \mathbb{N}_0^n.$$

In the invariant subspace $W_0 = \operatorname{span}\{w_0^+, w_0^-\}$, the homoclinic solution

$$(q(x), q_x(x)) = \Big(\frac{3}{2}\operatorname{sech}\frac{1}{2}x, -\frac{3}{4}\operatorname{sech}\frac{1}{2}x \tanh\frac{1}{2}x\Big)$$

of (6.9) is found for $\mu = 0$. Consider the variational equation

$$\frac{d}{dx}v = (A + DG(q(x)))v. \tag{6.10}$$

It turns out that the subspaces $W_k = \operatorname{span}\{w_k^+, w_k^-\}$ are invariant under the flow of (6.10) for $k \in \mathbb{N}_0^n$. In the subspace W_k, equation (6.10) reads

$$w_{xx} - (1 + \gamma^2|k|^2 - 2q(x))w = 0 \qquad x \in \mathbb{R}, \tag{6.11}$$

where $w(x)$ is the amplitude. We are interested in the set of bounded solutions to this equation. First consider the spectrum of the operator

$$Lw = w_{xx} - (1 - 2q(x))w \qquad x \in \mathbb{R}. \tag{6.12}$$

The spectrum of L is given by isolated simple eigenvalues $\lambda_0 = \frac{5}{4}$, $\lambda_1 = 0$, and $\lambda_2 = -\frac{3}{4}$ with eigenfunctions $\tilde{w}_0(x) = \operatorname{sech}^{\frac{3}{2}}(\frac{1}{2}x)$ and $\tilde{w}_1(x) = q_x(x)$. The remainder part $(-\infty, -1]$ of the spectrum is essential spectrum. See [26, Lemma 2.1] for the proofs.

Now suppose that

$$\gamma \neq \frac{\sqrt{5}}{2l} \qquad \text{for all } l \in \mathbb{N}. \tag{6.13}$$

Then the linearized equation (6.11) has non-trivial bounded solutions only for $k = 0$ and Hypothesis (H6) holds by non-degeneracy of the homoclinic orbit in the plane W_0. Therefore, Theorem 4 and Corollary 3 apply once (6.13) is met. Note that, in particular, (6.13) is met if $\gamma > \frac{\sqrt{5}}{2}$.

In passing, we remark that the subspace W_0 becomes normally hyperbolic for $\gamma \to \infty$. In this case, equation (6.9) is posed on a thin domain as can be readily seen by rescaling the y variable.

It remains to calculate the Melnikov integrals. The bounded solution of the adjoint equation

$$\frac{d}{dx}v = -(A^* + DG(q(x))^*)v$$

is given by

$$(-\psi_x(x), \psi(x)) = (-q_{xx}(x), q_x(x)).$$

Therefore, we obtain

$$\begin{aligned}
M(\beta) &= \int_{-\infty}^{\infty}\int_{\Omega} q_x(x)(1+h(y))\cos(x-\beta)\,dy\,dx \\
&= \pi^n \int_{-\infty}^{\infty} q(x)\sin(x-\beta)\,dx \\
&= \pi^n \int_{-\infty}^{\infty} \frac{3}{1+\cosh x}\sin(x-\beta)\,dx \\
&= \frac{6\pi^{n+1}}{\sinh\pi}\sin\beta.
\end{aligned}$$

For $\beta = 0$, we have $M(0) = 0$ and $M'(0) \neq 0$. Thus, the conclusions of Theorem 4 and Corollary 3 apply to this particular example.

Note that, for non-zero $h(y)$ and $\mu \neq 0$, the subspace W_0 is no longer invariant whence the solutions ensured by Corollary 3 do have non-trivial y-dependence. These solutions can be viewed as complicated equilibria $u(x,y)$ of the parabolic equation

$$u_t = u_{xx} + \gamma^2\Delta_y u - u + u^2 + \mu(1+h(y))\cos x \qquad (x,y) \in \mathrm{I\!R} \times (0,\pi)^n \tag{6.14}$$

on the cylinder $\mathrm{I\!R} \times (0,\pi)^n$. Moreover, for small c, the above results still hold if a term $\mu c u_x$ is added to (6.9). Then Corollary 3 ensures existence of many travelling-wave solutions $u(x - \mu ct, y)$ of (6.14) with non-trivial spatial dependence travelling with non-zero speed μc.

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