The saddle-node of nearly homogeneous wave trains in reaction-diffusion systems

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Abstract

We study the saddle-node bifurcation of a spatially homogeneous oscillation in a reaction-diffusion system posed on the real line. Beyond the stability of the primary homogeneous oscillations created in the bifurcation, we investigate existence and stability of wave trains with large wavelength that accompany the homogeneous oscillation. We find two different scenarios of possible bifurcation diagrams which we refer to as elliptic and hyperbolic. In both cases, we find all bifurcating wave trains and determine their stability on the unbounded real line. We confirm that the accompanying wave trains undergo a saddle-node bifurcation parallel to the saddle-node of the homogeneous oscillation, and we also show that the wave trains necessarily undergo sideband instabilities prior to the saddle-node.

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Key Words: saddle-node bifurcation, wave trains, homogeneous oscillation, stability, reaction diffusion systems

Acknowledgments This work was partially supported by the National Science Foundation through grant NSF DMS-0504271 (A. S.), and the Priority Program SPP 1095 of the German Research Foundation (J. R.). The authors are grateful to the referee for careful reading of the manuscript and many helpful suggestions.

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1 SADDLE-NODE OF PERIODIC ORBITS

We consider wave trains $u(kx - \omega t)$ near a temporally 2π-periodic solution $u_*(-\omega_* t)$, $\omega_* \neq 0$, to a reaction-diffusion system

$$u_t = Du_{xx} + f(u;\mu), \tag{1.1}$$

where $u \in \mathbb{R}^N$, $x \in \mathbb{R}$, $\mu \in \mathbb{R}$, $D = \operatorname{diag}(d_j) \geq 0$, and f is a smooth nonlinearity. We assume that u_* undergoes a saddle-node bifurcation at $\mu = 0$, that is, $\lambda = 0$ is geometrically simple and algebraically double as the only Floquet exponent in $\operatorname{Re}\lambda \geq 0$ of the linearized kinetics $u_t = f'(u_*(-\omega_* t);0)u$.

Homogeneous oscillations in reaction-diffusion systems are starting points for the understanding of wave trains $u(kx - \omega t)$, since we can typically find such solutions with $k \sim 0$ near a homogeneous oscillation. Under robust assumptions, for instance $D \sim d \cdot \mathrm{id}$, homogeneous oscillations can be stable for the reaction-diffusion system in large or unbounded domains. A perturbation argument then shows that there is a family of stable accompanying wave trains with $k \sim 0$, see e.g. [8].

Next to the Hopf bifurcation from equilibria *towards* periodic solutions, modeled by a complex Ginzburg-Landau equation, the saddle-node bifurcation of periodic solutions is the simplest bifurcation scenario, where spatio-temporal dynamics involving wave trains can be studied in a systematic fashion. The present paper is a first step towards such an understanding, describing spatio-temporally periodic wave trains as the building blocks. Spatially non-homogeneous wave trains are typically unstable in the neighborhood of a saddle-node [2, 8], and hence the saddle-node bifurcation is not expected to be observed experimentally. However, spatially homogeneous oscillations can be stable up to the saddle node, and are accompanied by non-homogeneous wave trains that may give rise to more complicated dynamics [1].

Our results extend the considerations in [2] by adding the wavenumber as a free parameter to the stability problem.

Our main results may be summarized as follows, see also Figure 1.1. Typically, wave trains come along curves, the *nonlinear dispersion curves*, in the plane of frequency ω and

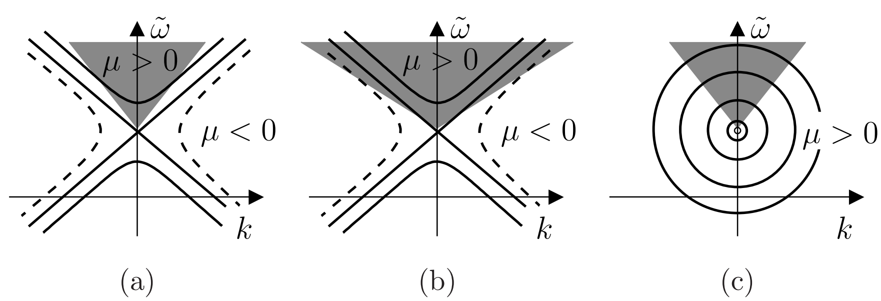


Figure 1.1: Sketch of stability sectors (shaded) and curves of existence for wave trains for fixed μ. (a) the first hyperbolic cases where wave trains at $\mu = 0$ are unstable, (b) second hyperbolic case where wave trains at $\mu = 0$ are stable. (c) the elliptic case.

wavenumber k. Varying the parameter μ, theses curves change. Our results characterize the location of these curves and stability in the $\omega - k$-parameter plane.

Theorem 1 (existence): Near the fold point, μ close to 0, dispersion curves are diffeomorphic to

- circles, $\mu \sim \omega^2 + k^2$ (*elliptic* case), or
- hyperbolas, $\mu \sim \omega^2 - k^2$ (*hyperbolic* case);

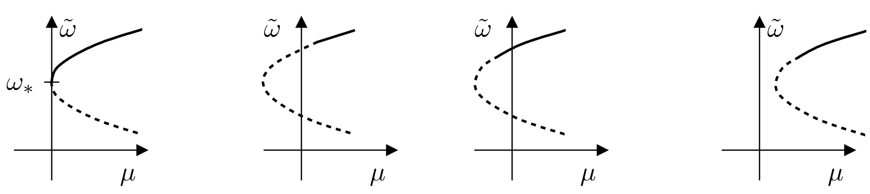
see Section 2 for a precise statement. In the elliptic case, only the homogeneous oscillation is left at $\mu = 0$ and no wave trains for $\mu < 0$. In the hyperbolic case, there exist two smooth curves of wave trains for $\mu = 0$, crossing transversely in the origin.

Theorem 2 (stability): Near the fold point wave trains are stable in a conical sector, containing a half line on the ω-axis, with opening angle less than π, that is, the sector does not contain the fold points ($\frac{\mathrm{d}k}{\mathrm{d}\omega} = 0$) of the dispersion curves of wave trains.

The assumptions in these theorems require a generic unfolding of the saddle-node in the kinetics and in the wavenumber, with typical square-root asymptotics of frequency and wavenumber $k, \omega - \omega_* \sim \pm\sqrt{\mu}$. For the stability result, we additionally require a quadratic tangency of the marginally stable spectrum at the fold point.

The boundary of the stability sector is always marked by a sideband instability, where perturbations with wavenumber $\gamma \sim 0$, but $\gamma \neq 0$ destabilize first; see for example [3].

One may view the bifurcation diagram as a one-parameter unfolding by fixing either μ or k. Fixing μ, the branches and instabilities are depicted in Figure 1.1. In all cases, the dispersion curves emanating from the stable homogeneous oscillations for $\omega \neq 0$ are stable. Only in the second hyperbolic case, the entire branches are stable, in the first hyperbolic and in the elliptic case, wave trains destabilize for small nonzero wavenumbers $k \sim \sqrt{\mu}$. In the second hyperbolic case, one can also find stable wave trains for $|k| \geq \mathrm{O}(\sqrt{\mu})$, $\mu < 0$, that is, when the homogeneous oscillations have disappeared. Therefore, in this case spatial pattern formation via (stable) wave trains is in fact a precursor for the homogeneous oscillation of the kinetics as μ increases from negative values.



(a) (b) (c) (d)

Figure 1.2: Sketches of the bifurcation and stability of wave trains for fixed k. Solid lines indicate stable wave trains, dashed lines unstable ones; the onset is always a sideband instability. (a) $k = 0$, the homogeneous oscillation. (b) first hyperbolic case for $k \neq 0$, (c) second hyperbolic case for $k \neq 0$. (d) elliptic case for $k \neq 0$.

A different view point on the result fixes the wavenumber k and follows wave trains in the parameter μ, as depicted in Figure 1.2. One finds existence for $\mu > \mu_*(k)$, $\mu_*(0) = 0$, $\mu_*(k) = \mu_*(-k)$, say, where a saddle-node bifurcation occurs. In the elliptic case, $\mu_*''(0) > 0$, so that wave trains with $k \neq 0$ disappear prior to the homogeneous saddle-node bifurcation when following them with decreasing μ. In the hyperbolic case, $\mu_*''(0) < 0$, wave trains with $k \neq 0$ still exist at the homogeneous saddle-node $\mu = 0$ and disappear only at $\mu_* < 0$. In all cases, wave trains destabilize at some $\mu_+(k)$ in a sideband instability before the saddle-node, $\mu_+(k) > \mu_*(k)$.

2 EXISTENCE OF WAVE TRAINS

In this section, we study the existence of wave trains near the saddle-node using Lyapunov-Schmidt reduction; wave trains $u(kx - \tilde{\omega}t)$ solve

$$k^2 D u'' + \tilde{\omega} u' + f(u;\mu) = 0, \quad u(\xi) = u(\xi + 2\pi). \tag{2.1}$$

At $\tilde{\omega} = \omega_* \neq 0$, $k = 0$, and $\mu = 0$, we assume the existence of a basic solution $u_*(\xi)$. The linearization at this solution

$$L_* u := \omega_* u' + f'(u_*;0)u : H^1_{\mathrm{per}}(0,2\pi) \to L^2(0,2\pi), \tag{2.2}$$

possesses a kernel containing $e_0 := u_*'$. Note that L_* has a compact resolvent, so that L_* is Fredholm of index zero, and $\lambda = 0$ is an isolated eigenvalue of finite multiplicity. For the saddle-node situation we make the generic assumption that $\lambda = 0$ is geometrically simple and algebraically double.

Hypothesis 1 (Fold) *The kernel of L_* is spanned by e_0, there is a generalized eigenvector e_1 with $L_* e_1 = e_0$, and no solution $u \in H^1_{\mathrm{per}}(0,2\pi)$ to $L_* u = e_1$.*

We define the (generalized) eigenvectors of the L^2-adjoint L_*^{ad} of L_* for $j = 0,1$ by e_j^*, so that

$$L_*^{\mathrm{ad}} e_0^* = 0,\ L_*^{\mathrm{ad}} e_1^* = e_0^*,\ (e_j^*, e_j) = 0,\ (e_j^*, e_{1-j}) = 1,$$

and let P denote the projection onto the generalized kernel, that is,

$$Pw = (e_0^*, w)e_1 + (e_1^*, w)e_0.$$

We denote $\partial_j f_* := \partial_j f(u_*;0)$ and define the coefficients

$$a_\mu = (\partial_\mu f_*, e_0^*), \quad a_k = -(D u_*'', e_0^*), \quad a_\omega = (e_1' - \frac{1}{2}\partial_{uu} f_* e_1^2, e_0^*). \tag{2.3}$$

Theorem 1 *Assume Hypothesis 1 and that the coefficients a_ω, a_k, and a_μ, defined in (2.3) do not vanish. Then, up to rescaling in ω, k, and μ, and a near-identity change of parameters k, ω, and μ, wave trains exist for*

$$\text{hyperbolic case, } a_\omega a_k < 0: \quad \omega^2 - k^2 = \mu$$
$$\text{elliptic case, } a_\omega a_k > 0: \quad \omega^2 + k^2 = \mu.$$

Proof. Exploiting Fredholm properties, we can write any vector $u \in H^1_{\text{per}}(0, 2\pi)$ uniquely as $u = u_* + \alpha e_0 + \beta e_1 + v$, with $Pv = 0$. We set $\alpha = 0$ in order to eliminate the trivial phase shift of solutions and write $\tilde{\omega} = \omega_* + \omega$. Expanding (2.1) and using $\omega_* u_*' + f_* = 0$ gives,

$$\begin{aligned} \mathcal{F}(\beta, v; k^2, \omega, \mu) := L_*(\beta e_1 + v) + \omega(u_* + \beta e_1 + v)' + \\ k^2 D(u_* + \beta e_1 + v)'' + \mu \partial_\mu f_* + \frac{1}{2}\partial_{uu} f_*(\beta e_1 + v)^2 + R &= 0, \end{aligned} \tag{2.4}$$

with remainder

$$R = \mathrm{O}(|\beta|^3 + |v|^3 + \mu^2 + |\mu|(|\beta| + |v|)),$$

in $H^1 := H^1_{\text{per}}(0, 2\pi)$, when $|v| = \|v\|_{H^1}$. In order to regularize the leading order derivative, we also consider the regularized equation

$$\mathcal{M}(k^2, \omega)\mathcal{F}(\beta, v; k^2, \omega, \mu) = 0, \tag{2.5}$$

where

$$\mathcal{M}(k^2, \omega) = \left(\frac{k^2}{\omega_*} D \frac{\mathrm{d}}{\mathrm{d}\xi} + 1 + \frac{\omega}{\omega_*}\right)^{-1} : H^1 \times \mathbb{R} \times \mathbb{R}_+ \mapsto L^2,$$

is continuously differentiable in k^2 and ω as a direct calculation in Fourier series shows.[3].

Since $\mathcal{M}\mathcal{F} = \omega_* u' + \mathcal{M} f(u)$, the left-hand side of (2.5) is C^1 in k^2 and ω, and smooth in the other variables. We first solve (2.5) projected on the kernel of P. Using $1 - \mathcal{M} = O(k^2 + |\omega|)$ and $(1 - P)u_*' = 0$, and $(1 - P)L_* e_1 = 0$ we obtain

$$\mathcal{N} v + (1 - P)\left[\mathcal{M}\left(k^2 D u_*'' + \mu \partial_\mu f_* + \frac{1}{2}\partial_{uu} f_*(\beta e_1 + v)^2 + R\right)\right] + O((|\beta| + |\omega|)k^2) = 0,$$

where $\mathcal{N} = (1 - P)(\omega_* v' + \mathcal{M} f_*' v) = (1 - P)(L_* + O(k^2 + |\omega|))v$ and the terms of order $O((|\beta| + |\omega|)(k^2 + |\omega|))$ are $(1 - P)(\beta(\omega_* e_1' + \mathcal{M} f_*' e_1) + \omega \mathcal{M} u_*')$. Since the linearization $\mathcal{N} : (1 - P)H^1 \to (1 - P)L^2$ of the left-hand side with respect to v at 0 is boundedly invertible for small k^2, we can solve with the implicit function theorem for v as

$$v = \mathrm{O}(\beta^2 + k^2 + \omega^2 + |\mu|).$$

We substitute the result into equation (2.5) projected on the generalized kernel, $\mathrm{Rg}\, P$. Since $P\mathcal{M} - \mathcal{M}P = \mathrm{O}(k^2 + |\omega|)$ and smoothness of e_j^* allows for integration by parts, we may equivalently solve (2.4), projected on $\mathrm{Rg}\, P$, which slightly simplifies our task of deriving the reduced equations.

[3]One can obtain higher order derivatives in k^2 by considering M as a map from H^m into H^{m-1}. Since the subsequent analysis can be performed in any space H^m, we see that u is C^1 with values in H^m, and the equation for the derivatives of u with respect to k^2 inductively show that u is actually of class C^m in k^2. Since our analysis does only require k^2-terms, we omit the details of this bootstrap argument.

In order to obtain the reduced equation in the direction of e_0, e_1, we first take the scalar product with e_1^*, use that $u_*' = e_0$, and expand, which gives

$$\beta + \omega + \mu(\partial_\mu f_*, e_1^*) + O\left(k^2 + \beta^2 + |\mu|^2 + \omega^2\right) = 0. \tag{2.6}$$

On the other hand, projecting with e_0^* and expanding gives

$$\begin{aligned}\omega\beta(e_1', e_0^*) + \beta^2 \frac{1}{2}(\partial_{uu} f_* e_1^2, e_0^*) + k^2(Du_*'', e_0^*) + \mu(\partial_\mu f_*, e_0^*)\\ +O\left(\omega^3 + k^3 + \beta^3 + \mu^2 + \beta|\mu|\right) = 0.\end{aligned} \tag{2.7}$$

We now solve the projection on the kernel, (2.6) for β with the implicit function theorem,

$$\beta = -\omega + \mathrm{O}(\omega^2 + |\mu| + k^2)$$

and substitute the result into the equation on the generalized kernel (2.7)

$$\omega^2(-e_1' + \frac{1}{2}\partial_{uu} f_* e_1^2, e_0^*) + k^2(Du_*'', e_0^*) + \mu(\partial_\mu f_*, e_0^*) + \mathrm{O}(|\omega|^3 + |\omega\mu| + |\omega|k^2 + \mu^2) = 0.$$

Using definition (2.3) for the coefficients a_j, this equation can be rewritten in the short form

$$a_\mu \mu = a_\omega \omega^2 + a_k k^2 + \mathrm{O}(|\omega|^3 + |\omega\mu| + |k|^3 + \mu^2). \tag{2.8}$$

Solving for μ with the implicit function theorem, we obtain

$$a_\mu \mu = a_\omega \omega^2 + a_k k^2 + \mathrm{O}(|\omega|^3 + |k|^3).$$

The function $\mu(\omega, k)$ possesses a non-degenerate critical point in the origin, when $a_j \neq 0$, so that we can use the Morse Lemma to find near-identity coordinates ω_1, k_1 where

$$\mu = \pm\omega_1^2 \pm k_1^2, \quad (\omega_1, k_1) = \psi_1(\omega, k), \quad \psi_1(z_1, z_2) = (z_1\sqrt{|a_\omega/a_\mu|}, z_2\sqrt{|a_k/a_\mu|}) + \mathrm{O}(z_1^2 + z_2^2).$$

Here, the $\pm$-signs corresponds to the signs of a_ω/a_μ and a_k/a_μ respectively. Also, the change of coordinates respects the symmetry $k \mapsto -k$ by uniqueness, and in particular leaves the ω-axis invariant. ■

Remark 2.1 *The coefficients a_j can be interpreted as follows. For $k = 0$, a_ω/a_μ determines the direction of branching in the saddle-node and thereby fixes the sign of μ where homogeneous oscillations exist. The sign of a_k/a_ω decides on the nature of the bifurcation for the wave trains: it distinguishes precisely between the elliptic and the hyperbolic case by characterizing whether wave trains with $k \neq 0$ exist at $\mu = 0$ or not, that is, if the curve of wave trains $\omega(\mu)$ with fixed $k \neq 0$ undergoes a saddle node* after *crossing $\mu = 0$ (hyperbolic) or* before *crossing $\mu = 0$ (elliptic).*

Remark 2.2 *The condition on the multiplicity of the eigenvalue $\lambda = 0$ for L_* in Hypothesis 1 translates into a condition for the Floquet multipliers of the linearization of the ODE (2.1) in the periodic orbit u_*. In fact, geometric and algebraic multiplicities of L_* and the* *trivial Floquet multiplier* 1 *coincide. Eigenfunctions to* L_* *coincide with periodic solutions to the ODE, hence geometric multiplicities coincide. A generalized Floquet eigenfunction solves* $\omega u_1' = f_*' u_1$, *with* $u_1(2\pi) = u_1(0) + 2\pi/\omega_* u_*(0)$. *The function* $e_1(\xi) := u_1(\xi) - \xi u_*(\xi)$ *is now easily checked to yield a generalized eigenfunction to* L_*. *Hence our condition on the algebraic multiplicity of* $\lambda = 0$ *can be reinterpreted as the same condition on the multiplicity of the trivial Floquet multiplier.*

Remark 2.3 *The unfoldings of the saddle-node actually occur as typical unfoldings of singularities of dispersion relations also for* $k_* \neq 0$. *Indeed, assume that for* $\tilde{\omega} = \omega_* + \omega$ *and* $\tilde{k} = k_* + k$, *we cannot continue the nonlinear dispersion curve with the implicit function theorem at* $k = \omega = 0$, $k_*\omega_* \neq 0$. *Typically, we can still reduce to a one-dimensional bifurcation problem, where linear terms in* ω *and* k *vanish. A typical unfolding in* μ *then gives a reduced equation*

$$a_\mu \mu = a_\omega \omega^2 + a_{\omega k}\omega k + a_k k^2 + \dots.$$

which can again be reduced to either the hyperbolic or the ellipitic case by the implicit function theorem and the Morse lemma. The hyperbolic case has been observed in [6, Figure 4] when dispersion curves of trigger waves and phase waves collide.

Of course, one could attempt to unfold higher codimension singularities of homogeneous wave trains in order to detect the emergence of this singularity for nonzero values of k. *A typical scenario would be the case when the reduced equation expands into*

$$a_1\mu_1 = a_\omega \omega^2 + a_k \mu_2 k^2 + a_4 k^4, \tag{2.9}$$

where elliptic or hyperbolic singularities bifurcate from the $k = 0$*-axis, depending on the sign of* $a_4 a_\omega$. *We can also interpret level sets* $\mu \equiv \text{const}$ *in (2.9) as level sets of the Hamiltonian* $H(\omega, k)$ *defined by the right-hand side of (2.9), so that dispersion curves or level lines are actually flow lines of the second-order pendulum* $\ddot{k} = -4a_\omega a_k \mu_2 k + 8a_4 a_\omega k^3$.

3 STABILITY OF THE BIFURCATING WAVE TRAINS

The spectrum near a wave train on the unbounded real line consists precisely of those λ for which there exists a $\nu = \mathrm{i}\gamma \in \mathrm{i}\mathbb{R}$, and a nontrivial solution to the boundary-value problem

$$\lambda w = D(k\partial_\xi + \nu)^2 w + \tilde{\omega}\partial_\xi w + f'(u;\mu)w, \quad w(0) = w(2\pi), \tag{3.1}$$

where u denotes the wave train solution from the preceding section to the wavenumber k with frequency $\tilde{\omega} = \omega_* + \omega$; see for instance [5, 9].

By assumption, at $k = 0$, $\mu = 0$, there are no solutions to this boundary-value problem in $\operatorname{Re}\lambda > 0$ and the only solution in $\operatorname{Re}\lambda = 0$ is given by $\nu = 0$, $\lambda = 0$, $w = u'$. By the assumptions on a simple saddle-node, this eigenvalue is algebraically double, see Remark 2.2.

We denote the operator on the right-hand side of (3.1) as $\mathcal{L}_{\nu,k,\mu}$. We consider $\mathcal{L}_{\nu,k,\mu}$ as a closed operator on L^2 with periodic boundary conditions. The domain of definition is H^1

or H^2, for $k = 0$ and $k \neq 0$, respectively. Using the regularization employed in the previous section, it readily follows that the resolvent is continuously differentiable in $k^2 = 0$. Since $\lambda = 0$ is an isolated eigenvalue of multiplicity two for $\nu = k = \mu = 0$, we can continue the corresponding eigenprojection $P = P_{0,0,0}$ as $P_{\nu,k,\mu}$ for nearby parameter values, so that P is continuously differentiable in k^2. In particular, $P_{\nu,k,\mu} : \mathrm{Rg}\, P \to \mathrm{Rg}\, P_{\nu,k,\mu}$ is an isomorphism, so that we obtain an operator

$$\hat{\mathcal{L}} := P^{-1}_{\nu,k,\mu} \mathcal{L}_{\nu,k,\mu} P_{\nu,k,\mu} : \mathrm{Rg}\, P \to \mathrm{Rg}\, P.$$

Recall the definition of the eigenvectors e_j and the adjoint eigenvectors e_j^* from §2. For our stability result we define and abbreviate

$$\begin{aligned}
\alpha(\omega, k, \nu) &:= \frac{1}{2}[(e_1^*, \hat{\mathcal{L}}e_0) + (e_0^*, \hat{\mathcal{L}}e_1)], \\
\beta(\omega, k, \nu) &:= \frac{1}{4}[(e_1^*, \hat{\mathcal{L}}e_0) - (e_0^*, \hat{\mathcal{L}}e_1)]^2 + (e_1^*, \hat{\mathcal{L}}e_1)(e_0^*, \hat{\mathcal{L}}e_0), \\
\alpha^0_{\nu\nu} &:= \partial_\nu^2 \alpha(0,0,0), \\
\alpha^0_\omega &:= \partial_\omega \alpha(0,0,0), \\
\beta^0_{\nu\nu} &:= \partial_\nu^2 \beta(0,0,0).
\end{aligned} \tag{3.2}$$

Theorem 2 *Assume that in the unfolding of a homogeneous saddle-node, the homogeneous oscillation is marginally stable at criticality,* $\alpha^0_{\nu\nu}, \beta^0_{\nu\nu} > 0$, $\alpha^0_\omega \neq 0$. *Then there exists a 'cone' given by*

$$\{(\omega, k) \mid -\alpha^0_\omega \omega > \omega_{\mathrm{marg}}(|k|)\}, \qquad \omega_{\mathrm{marg}}(|k|) = \left| \frac{a_k \alpha^0_\omega}{a_\omega \sqrt{\beta_{\nu\nu}}} \right| \cdot |k| + \mathrm{O}(k^2), \tag{3.3}$$

enclosing the stable homogeneous wave train, such that wave trains are stable inside the cone but unstable outside. The instability at the threshold $\omega_{\mathrm{marg}}(|k|)$ *is always a sideband instability.*

Proof. We expand the eigenvalue problem for $\hat{\mathcal{L}}$ at $\lambda = 0$ in coordinates, and retrieve a 2×2-matrix M, whose eigenvalues are precisely the critical eigenvalues of the wave train for values of ν close to zero:

$$M(\nu; \mu, k) = \begin{pmatrix} (e_1^*, \hat{\mathcal{L}}e_0) & (e_1^*, \hat{\mathcal{L}}e_1) \\ (e_0^*, \hat{\mathcal{L}}e_0) & (e_0^*, \hat{\mathcal{L}}e_1) \end{pmatrix}.$$

For $\mu = k = \nu = 0$, the matrix is given by the standard Jordan block, and in general, we can expand

$$M(\nu; \mu, k) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathrm{O}(|\nu| + |k| + |\mu|).$$

A difficulty with the (natural) parameterization of solutions by the parameters μ and k is the discontinuity of branches in the bifurcation point. This discontinuity is removed by considering ω and k as independent parameters, just like in the existence proof, and let $\mu = \mu(\omega, k)$ be a function of ω and k. We then have

$$M(\nu; \omega, k) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \mathrm{O}(|\nu| + |k| + |\omega|).$$

Up to a smooth change of coordinates, we may assume that M is in the normal form

$$M(\nu;\omega,k) = \begin{pmatrix} \alpha & 1 \\ \beta & \alpha \end{pmatrix},$$

and we readily compute that the two coefficients are as given in (3.2). Note that for the matrix M, the eigenvalues are given through

$$\lambda = \alpha \pm \sqrt{\beta}. \tag{3.4}$$

In order to analyze this still very general eigenvalue problem, we will exploit the following additional ingredients:

(i) at the fold points, we have double zero eigenvalues also for wave trains, $k \neq 0$;

(ii) $M(0;\omega,k)$ possesses a nontrivial kernel for all ω, k;

(iii) the homogeneous oscillations are marginally stable at the bifurcation point with quadratic tangency (otherwise, all wave trains are unstable nearby);

(iv) the slope of curves in the ω-k-existence diagram gives us linear group velocities, $\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}$, since linear and nonlinear group velocities coincide, which will allow us to find a simple expression for the relevant term of order νk.

We first exploit (i). By (2.8) wave trains exist for points in the ω-k-plane with

$$a_\mu \mu = a_\omega \omega^2 + a_k k^2 + a_1 \mu\omega + \dots,$$

so that for fixed parameter μ, wave trains undergo a saddle-node bifurcation when $\frac{\partial k}{\partial \omega} = 0$, that is, $\omega = -\frac{a_1}{2a_\omega}\mu + \mathrm{O}(\mu^2)$, or, in terms of k,

$$\omega = \omega_{\mathrm{sn}} k^2 + \mathrm{O}(k^4), \quad \omega_{\mathrm{sn}} = -\frac{a_1 a_k}{2a_\omega a_\mu}.$$

Along this curve, α *and* β vanish, and (3.1) depends only on k^2 at $\nu = 0$, so that

$$\begin{aligned} \alpha(\omega,k,0) &= \alpha_\omega(\omega,k^2) = \alpha_\omega^0 \cdot (\omega - \omega_{\mathrm{sn}} k^2) + \mathrm{O}(\omega^2 + k^4) \\ \beta(\omega,k,0) &= \beta_\omega(\omega,k^2) = \beta_\omega^0 \cdot (\omega - \omega_{\mathrm{sn}} k^2) + \mathrm{O}(\omega^2 + k^4) \end{aligned} \tag{3.5}$$

Now, (ii) implies $\beta = \alpha^2$ at $\nu = 0$, that is,

$$\beta_\omega(\omega,k^2) = \alpha_\omega^2(\omega,k^2) = (\alpha_\omega^0)^2(\omega - \omega_{\mathrm{sn}} k^2)^2 + \mathrm{O}(|\omega|^3 + k^4).$$

Assumption (iii) on stability at $\omega = k = 0$ together with the fact that (3.1) at $k = 0$ depends only on ν^2, imply that

$$\alpha(0,0,\nu) = \alpha_{\nu\nu}^0 \nu^2 + \mathrm{O}(\nu^4), \quad \beta(0,0,\nu) = \beta_{\nu\nu}^0 \nu^2 + \mathrm{O}(\nu^4), \quad \alpha_{\nu\nu}^0, \beta_{\nu\nu}^0 > 0, \tag{3.6}$$

where we excluded the boundary cases $\alpha_{\nu\nu}^0 \beta_{\nu\nu}^0 = 0$.

Before exploiting property (iv), we first infer the form of the stability criterion, and denote $\beta^0_{\nu k} := \partial_{\nu k}\beta(0,0,0)$ and $\alpha^0_{\nu k} := \partial_{\nu k}\alpha(0,0,0)$.

Since eigenvalues $\lambda(\mathrm{i}\gamma)$ in (3.4) to the 2×2-matrix form connected curves and eigenvalues are stable for $\nu = \mathrm{i}\gamma$ not small, stability is equivalent to the absence of purely imaginary eigenvalues $\lambda = \mathrm{i}\tau$. We therefore look for solutions of

$$-\tau^2 - 2\mathrm{i}\alpha\tau + \alpha^2 = \beta, \quad \alpha = \alpha_\omega + \mathrm{i}\gamma k \alpha^0_{\nu k} - \gamma^2\alpha^0_{\nu\nu} + h.o.t., \quad \beta = \alpha^2_\omega + \mathrm{i}\gamma k\beta^0_{\nu k} - \gamma^2\beta^0_{\nu\nu} + h.o.t. \tag{3.7}$$

The real part of this equation only contains terms in $\tau\gamma$ and even powers in τ, γ. It can be expanded into

$$-\tau^2 + \gamma^2\beta^0_{\nu\nu} = \mathrm{O}\left((|k|+|\omega|)(|\tau|^2+|\gamma|^2) + |\gamma|^4 + |\tau|^4\right),$$

which we can solve for γ using the Newton-Polygon in the analytic variables γ and τ as

$$\gamma = \pm\frac{\tau}{\sqrt{\beta^0_{\nu\nu}}} + \mathrm{O}(|k\tau| + |\omega\tau| + |\tau|^3). \tag{3.8}$$

The imaginary part of (3.7) can be expanded as

$$-2\alpha_\omega\tau + 2\alpha^0_{\nu\nu}\tau\gamma^2 + 2\alpha_\omega\alpha^0_{\nu k}k\gamma = \beta^0_{\nu k}k\gamma + \mathrm{O}\left((|\gamma|^3+|\tau|^3)(|k|+|\omega|) + |\gamma|^5 + |\tau|^5\right),$$

with $\alpha_\omega = \alpha_\omega(\omega, k^2)$. Substituting the expansion (3.8) for γ into this result and dividing by τ ($\tau = 0$ always gives a solution to the analytic equation by translational symmetry) gives

$$-2\alpha_\omega + 2\frac{\alpha^0_{\nu\nu}}{\beta^0_{\nu\nu}}\tau^2 = \pm\frac{\beta^0_{\nu k}}{\sqrt{\beta^0_{\nu\nu}}}k + \mathrm{O}\left((|k|+|\omega|)\tau^2 + \tau^4 + |k|^2 + |\omega|^2\right).$$

This equation can be solved via the implicit function theorem for τ^2, with expansion

$$\tau^2 = \frac{\beta^0_{\nu\nu}}{2\alpha^0_{\nu\nu}}\left(\pm\frac{\beta^0_{\nu k}}{\sqrt{\beta^0_{\nu\nu}}}k + 2\alpha^0_\omega\omega + \mathrm{O}(k^2+\omega^2)\right).$$

Imaginary eigenvalues exist when $\tau^2 \geq 0$. We therefore solve the equation $\tau^2 = 0$ for ω by the implicit function theorem and find that wave trains are stable for

$$-\alpha^0_\omega\omega > \left|\frac{\beta^0_{\nu k}}{2\sqrt{\beta^0_{\nu\nu}}}\right| \cdot |k| + \mathrm{O}(k^2). \tag{3.9}$$

From the construction, the onset of instability is at $\tau = 0$, hence it is a sideband instability [7, 8], which can be detected directly by solving

$$\partial_{\nu\nu}\lambda = 0, \qquad \text{with } \lambda = \alpha + \sqrt{\beta},$$

and expanding in k and ω.

Lastly, we use property (iv) to express $\beta^0_{\nu k}$ in terms of α^0_ω and a_k, a_ω from the existence result. We consider a line $\omega = a_c k$ in the bifurcation diagram, with $\mu = \mu(k)$ chosen appropriately. The existence condition (2.8) implies that on the one hand

$$\frac{\mathrm{d}\omega}{\mathrm{d}k} = -\frac{a_k}{a_\omega a_c} + \mathrm{O}(|k|). \tag{3.10}$$

On the other hand, the group velocity can also be computed from the *linearization*, see e.g. [4], as $c_{\mathrm{g}} = -\mathrm{d}\lambda/\mathrm{d}\nu|_{\nu=0}$. In our case, this is well defined only for $\mu \neq 0$. At $k = 0$, reflection symmetry implies $\mathrm{d}\lambda/\mathrm{d}\nu|_{\nu=0} = 0$. More precisely, (3.6) shows that the leading terms are order νk, hence c_{g} to leading order is given by evaluating

$$k\partial_{\nu k}M = k\begin{pmatrix} \partial_{\nu k}\alpha & 0 \\ \partial_{\nu k}\beta & \partial_{\nu k}\alpha \end{pmatrix}$$

on the kernel of M and projecting back onto the kernel. Kernel span(e) and cokernel span(e^*) at $\nu = 0$ are

$$e = \begin{pmatrix} 1 \\ -\alpha_\omega \end{pmatrix}, \quad e^* = \frac{1}{2\alpha_\omega}\begin{pmatrix} \alpha_\omega \\ -1 \end{pmatrix},$$

and we compute

$$c_{\mathrm{g}} = -\frac{k}{2\alpha_\omega}\begin{pmatrix} \alpha_\omega \\ -1 \end{pmatrix}^T \begin{pmatrix} \partial_{\nu k}\alpha & 0 \\ \partial_{\nu k}\beta & \partial_{\nu k}\alpha \end{pmatrix}\begin{pmatrix} 1 \\ -\alpha_\omega \end{pmatrix} = \frac{k\partial_{\nu k}\beta}{2\alpha_\omega} - k\partial_{\nu k}\alpha.$$

Evaluating in $\omega = a_c k$, substituting (3.5) and, using (iv), equating with (3.10) at $k = 0$ gives

$$-\frac{a_k}{a_\omega a_c} = \frac{k\beta^0_{\nu k}}{2\alpha^0_\omega a_c k} \quad \Leftrightarrow \quad \beta^0_{\nu k} = -\frac{2\alpha^0_\omega a_k}{a_\omega}.$$

Together with (3.9) we obtain the stability criterion (3.3) as claimed. ∎

Remark 3.4 *In the hyperbolic case, the width of the marginal stability cone is basically unrelated to the width of the sectors that contain a half line on the ω-axis and are bounded by the curves of wave trains at $\mu = 0$. In the following section, we give an example where the stability sector is larger and an example where it is smaller than the corresponding sector at $\mu = 0$. We also note that the stability sector does not contain the fold points $\frac{\mathrm{d}k}{\mathrm{d}\omega} = 0$ of the curves of wave trains: wave trains destabilize in a sideband instability prior to undergoing a saddle-node as predicted in [8].*

Remark 3.5 *It would be interesting to determine stability in multiple spatial dimensions $x \in \mathbb{R}^n$. In that case, the boundary value problem (3.1) has an additional parameter accounting for transverse perturbations $we^{\mathrm{i}\kappa y}$: $\lambda w = D[(k\partial_\xi + \nu)^2 - \kappa^2]w + f'(u;\mu)w$. While it is not difficult to compute the influence of small and large κ, we do not know if the stability sector may shrink, or even collapse to the $k = 0$ half line due to transverse instabilities of the wave trains prior to the longitudinal instabilities which we computed here.*

Remark 3.6 *A somewhat larger stability region is obtained when one weakens the notion of decay to a pointwise decay instead of decay in norm. This* convective stability *is determined by the location of critical branch points of the dispersion relation, where* $\det M = \partial_\nu(\det M) = 0$*, for complex ν and λ. A straightforward computation reveals that these branch points come in a complex conjugate pair and cross the imaginary axis when ω crosses a parabola $\omega = \omega_{\mathrm{abs}}k^2 + O(k^4)$. It would be interesting to relate the coefficient ω_{abs} to the coefficient ω_{sn}*

that locates the saddle-node bifurcation of wave trains. We do not know whether the saddle-node of the wave trains can be contained in the region of convective stability, or if, near the saddle-node, perturbations grow pointwise as well as in norm.

For the class of complex Ginzburg Landau equations (4.1) considered below we show in Remark 4.7 that the aforementioned branch points are always unstable near the saddle-nodes.

On large bounded domains with separated boundary conditions stability criteria are a little more subtle. Spectra of the linearization are discrete but converge to curves as the size of the domain goes to infinity. These curves were characterized in terms of the roots ν_j of the dispersion relation in [10] and called the absolute spectrum. *Curves of absolute spectrum terminate in the branch points that determine convective stability. Therefore the region where wave trains are stable in large but finite domains is contained in the region of convective stability and contains the region of stability from our main theorem. The region of stability in large domains may well differ from the region of convective stability, since the curves of absolute spectrum might cross the imaginary axis before the branch points cross. We have not attempted to resolve this somewhat more subtle and interesting question. However, in numerical computations for some examples of the form (4.1) the branch points were the most unstable points of the absolute spectrum.*

Note that all critical curves of absolute spectrum can be computed from the matrix M, since the absolute spectrum converges locally uniformly to the essential spectrum as $k \to 0$.

4 AN EXAMPLE

To illustrate the results above, we consider the complex Ginzburg-Landau equation

$$A_t = (1 + \mathrm{i}a)A_{xx} + Af(|A|^2; \mu) + \mathrm{i}Ag(|A|^2), \tag{4.1}$$

where we suppress μ whenever $\mu = 0$, and assume

$$\begin{aligned} f(1) = 0, \quad f'(1) = 0, \quad g(1) = 0, \\ \partial_\mu f(1) \neq 0, \quad f''(1) \neq 0, \quad g'(1) \neq 0. \end{aligned} \tag{4.2}$$

Wave trains here come as relative equilibria with respect to the gauge symmetry, $A = r\mathrm{e}^{\mathrm{i}(kx-\omega t)}$, with

$$-k^2 + f(r^2; \mu) = 0, \qquad -ak^2 + \omega + g(r^2) = 0. \tag{4.3}$$

Using (4.2), we may solve the second equation in (4.3) for r^2 by

$$r^2 = 1 - \frac{\omega}{g'(1)} + \frac{ak^2}{g'(1)} + \mathrm{O}(\omega^2 + k^4). \tag{4.4}$$

Substituting into the first equation, using that $\omega = ak^2$ at $r = 1$, and expressing the higher order terms through ω, k gives

$$-k^2 + \mu\partial_\mu f(1) + \frac{f''(1)}{2(g'(1))^2}\omega^2 + \mathrm{O}(|\omega|^3 + k^4) = 0. \tag{4.5}$$

In particular, we find the hyperbolic case if $f''(1) > 0$ and the elliptic case if $f''(1) < 0$.

Linear stability of $A_* = r_*(\omega, k)\mathrm{e}^{\mathrm{i}(\omega t - kx)}$, with r_* given by (4.4), leads to

$$\begin{aligned}
\lambda A &= (1+\mathrm{i}a)A_{xx} + \left(f_* + f'_*|A_*|^2 + \mathrm{i}g_* + \mathrm{i}g'_*|A_*|^2 + \mathrm{i}\omega\right)A + \left(f'_*A_*^2 + \mathrm{i}g'_*A_*^2\right)\bar{A},\\
\lambda \bar{A} &= (1-\mathrm{i}a)\bar{A}_{xx} + \left(f_* + f'_*|A_*|^2 - \mathrm{i}g_* - \mathrm{i}g'_*|A_*|^2 + \mathrm{i}\omega\right)\bar{A} + \left(f'_*A_*^2 - \mathrm{i}g'_*A_*^2\right)A,
\end{aligned}$$

where $f_* = f(r_*)$, $f'_* = f'(r_*)$, and $g'_* = g'(r_*)$. Writing $A = B\mathrm{e}^{\mathrm{i}kx}$ this reduces at $r_* = B\bar{B} = 1$ to

$$\begin{aligned}
\lambda B &= (1+\mathrm{i}a)(B_{xx} + 2\mathrm{i}kB_x) + (f'_* + \mathrm{i}g'_*)(B + \bar{B}),\\
\lambda \bar{B} &= (1-\mathrm{i}a)(\bar{B}_{xx} - 2\mathrm{i}k\bar{B}_x) + (f'_* - \mathrm{i}g'_*)(B + \bar{B}).
\end{aligned}$$

The ansatz $B = b\mathrm{e}^{\mathrm{i}\gamma x}$, $\bar{B} = \bar{b}\mathrm{e}^{\mathrm{i}\gamma x}$ gives the spectrum as eigenvalues to the γ-dependent family of 2×2-matrices

$$M(\gamma;\omega,k) = \begin{pmatrix} (1+\mathrm{i}a)(-\gamma^2 - 2k\gamma) + (f'_* + \mathrm{i}g'_*) & f'_* + \mathrm{i}g'_* \\ f'_* - \mathrm{i}g'_* & (1-\mathrm{i}a)(-\gamma^2 + 2k\gamma) + (f'_* - \mathrm{i}g'_*) \end{pmatrix}.$$

The eigenvalues are

$$\lambda_\pm = f'_* - 2\mathrm{i}ak\gamma - \gamma^2 \pm \sqrt{(f'_*)^2 - 4\mathrm{i}g'_*k\gamma + 4k^2\gamma^2 + 2g'_*a\gamma^2 + 4\mathrm{i}ak\gamma^3 - a^2\gamma^4}.$$

For stability at $k = 0$, we infer the necessary condition $g'_*a < 0$. Following the arguments in the general case, we have stability if $\gamma = 0$ is stable, i.e. $f'_* < 0$, and if the wave train is sideband stable [8], which can be detected by expanding λ_- to quadratic order in γ:

$$\lambda_- = 2\mathrm{i}\left(\frac{g'_*}{f'_*} - a\right)k\gamma + \left(-1 - \frac{2k^2}{f'_*} - \frac{2(g'_*)^2k^2}{(f'_*)^3} - \frac{g'_*a}{f'_*}\right)\gamma^2 + \mathrm{O}(|\gamma|^3).$$

From $f'_* < 0$ we find that the wave trains are sideband stable if, and only if,

$$2(g'_*)^2k^2 + (g'_*a + 2k^2 + f'_*)(f'_*)^2 < 0. \tag{4.6}$$

Expanding $f'_* = -f''(1)\omega/g'(1) + \mathrm{O}(k^2 + \omega^2)$, see (4.2) and (4.4), and employing the implicit function theorem for the critical curve where equality holds, we obtain the sharp stability condition

$$\omega^2 > \left|\frac{2(g'(1))^3}{a(f''(1))^2}\right| k^2 + \mathrm{O}(k^4). \tag{4.7}$$

The condition $f'_* < 0$ singles out a sign of ω as it can be expanded into

$$f''(1)g'(1)\omega > 0. \tag{4.8}$$

Conditions (4.7) and (4.8) together characterize the stability cone.

These results are plotted in Figure 4.1 (compare with Fig. 1.1) for the specific choices

$$f(y;\mu) = -\mu \pm (y-1)^2\,, \quad g(y) = y - 1. \tag{4.9}$$

In this case, the stability condition $g'_* a < 0$ is $a < 0$, and we compute from (4.3) that

$$\pm\mu = \omega^2 \mp k^2 + a^2 k^4 + 2|a|k^2\omega$$

for the nonlinear dispersion curves. Note that the upper choice of signs gives the hyperbolic case where $f(y;\mu) = (y-1)^2 - \mu$. The stability cone is given by $f'_* < 0$, i.e. here $f''(1)\omega > 0$, and

$$\omega^2 > \frac{k^2}{2|a|} - ak^2(ak^2 + 2\omega) - \frac{1}{a}(2k^2 \pm 2ak^2 \mp 2\omega)(ak^2 - \omega)^2,$$

according to (4.6). Hence, in the hyperbolic case, the leading order slope of the existence lines at $\mu = 0$ is fixed at 1, while the parameter a varies the opening of the stability sector (also in the elliptic case).

Remark 4.7 *To illustrate the absolute stability discussed in Remark 3.6, we computed* ω_{sn} *and* ω_{abs} *for (4.1). As to* ω_{sn}*, we refine equation (4.5) to*

$$-k^2 + \mu\partial_\mu f(1) + \frac{f''(1)}{2(g'(1))^2}(\omega^2 + a^2k^4 - 2a\omega k^2) + O(|\omega|^3 + k^6) = 0,$$

which, using $K = k^2$ *and* $c = (g'(1))^2/f''(1)$*, implies that to leading order*

$$K_\pm = \frac{c}{a^2} + \frac{\omega}{a} \pm \sqrt{\frac{c^2}{a^4} + \frac{2c}{a^3}(\omega - a\mu\partial_\mu f(1))}.$$

Saddle-nodes are located at $\mathrm{d}K/\mathrm{d}\omega = 0$*, which implies* $\omega = a\mu\partial_\mu f(1)$*, so (4.5) gives* $\omega_{\text{sn}} = a$*.*

As to ω_{abs}*, we solve* $\partial_\nu \det M(-\mathrm{i}\nu;\omega,k) = 4k + 4a\nu + h.o.t. = 0$ *to leading by* $\nu = -k/a$*. Substituting this and* $\gamma = -\mathrm{i}\nu$ *into the equation for* $\lambda_\pm$*, we readily see that the square root term is purely imaginary to leading order. Hence, double roots occur as a complex conjugate pair, and the real part vanishes if* $f'_* - 2\nu ak + \nu^2 = 0$*, which, to leading order, means*

$$\frac{f''(1)}{g'(1)}(ak^2 - \omega) + 2k^2 + \frac{k^2}{a^2} = 0.$$

We thus have $\omega_{\text{abs}} = a + (2 + 1/a^2)g'(1)/f''(1)$*, and* $\operatorname{sgn}(\omega_{\text{abs}} - \omega_{\text{sn}}) = \operatorname{sgn}(g'(1)/f''(1))$*.*

For $g'(1)f''(1) > 0$*, the stability sector lies in* $\omega > 0$*, so that branch points would be stable at saddle-nodes, if* $\omega_{\text{sn}} > \omega_{\text{abs}}$*, which is not possible in this case. On the other hand,* $g'(1)f''(1) < 0$ *implies* $\omega < 0$*, but then* $\omega_{\text{sn}} < \omega_{\text{abs}}$ *is excluded as well. Therefore, the nearly homogeneous wave trains of (4.1) that are close to a saddle-node are absolutely unstable, and thus not only unstable on the real line, but also on large bounded domains.*

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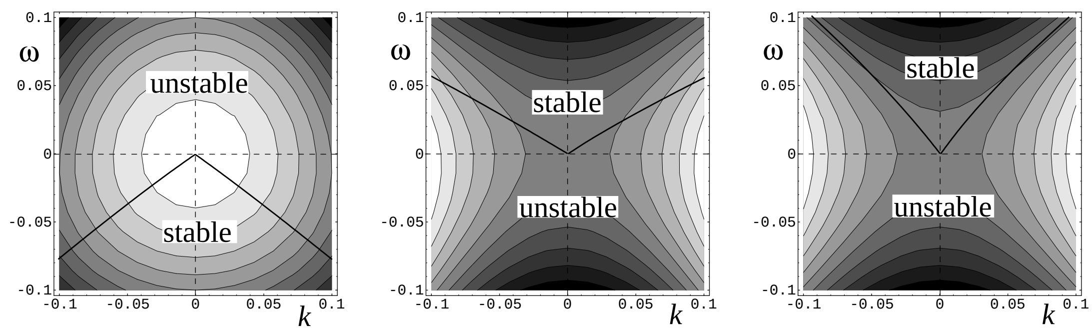

(a) (b) (c)

Figure 4.1: Stability sectors and existence of wave trains to the complex Ginzburg-Landau equation (4.1) and (4.9); some curves for fixed μ are plotted, darker regions have larger μ. (a) elliptic case and $a = -1$; (b) first hyperbolic case for $a = -1$; (c) second hyperbolic case for $a = -1/4$.

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Caption Figure 1.1:

Sketch of stability sectors (shaded) and curves of existence for wave trains for fixed μ. (a) the first hyperbolic cases where wave trains at $\mu = 0$ are unstable, (b) second hyperbolic case where wave trains at $\mu = 0$ are stable. (c) the elliptic case.

Caption Figure 1.2:

Sketches of the bifurcation and stability of wave trains for fixed k. Solid lines indicate stable wave trains, dashed lines unstable ones; the onset is always a sideband instability. (a) $k = 0$, the homogeneous oscillation. (b) first hyperbolic case for $k \neq 0$, (c) second hyperbolic case for $k \neq 0$. (d) elliptic case for $k \neq 0$.

Caption Figure 4.1:

Stability sectors and existence of wave trains to the complex Ginzburg-Landau equation (4.1) and (4.9); some curves for fixed μ are plotted, darker regions have larger μ. (a) elliptic case and $a = -1$; (b) first hyperbolic case for $a = -1$; (c) second hyperbolic case for $a = -1/4$.