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RELATIVE MORSE INDICES, FREDHOLM INDICES, AND GROUP VELOCITIES

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Abstract. We discuss Fredholm properties of the linearization of partial differential equations on cylindrical domains about travelling and modulated waves. We show that the Fredholm index of each such linearization is given by a relative Morse index which depends only on the asymptotic coefficients. Several strategies are outlined that help to compute relative Morse indices using crossing numbers of spatial eigenvalues, and the results are applied to prove the existence of small viscous shock waves in hyperbolic conservation laws upon adding small localized time-periodic source terms.

1. **Introduction.** The dynamics of many spatially extended physical systems is governed by the coherent structures, such as fronts, pulses, spiral waves, and boundary layers, that the system exhibits. We are interested in analysing the properties of the linearization of partial differential equation models (PDEs) about these coherent structures. Indeed, this is typically the first step in most bifurcation and stability analyses; see [5] for details.

In many situations, the resulting linear differential operators are posed on cylindrical domains with asymptotically constant or periodic coefficients. The simplest case is that of a travelling wave solution $u_*(x-ct)$ of a reaction-diffusion system

$$u_t = Du_{xx} + f(u), \quad u \in \mathbb{R}^n, \tag{1}$$

where $D \in \mathbb{R}^{n\times n}$ is a positive diagonal matrix and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function. The eigenvalue problem associated with the linearization of (1) about the wave $u_*(x-ct)$ in the comoving frame $\xi = x - ct$ is given by

$$\lambda u = Du_{\xi\xi} + cu_\xi + f_u(u_*(\xi))u =: \mathcal{L}_* u$$

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which can be cast as a first-order nonautonomous differential equation

$$U_\xi = A(\xi;\lambda)U, \quad U = (u, u_\xi)^T.$$

It can be shown that spectral properties of $\mathcal{T}(\lambda) := \frac{\mathrm{d}}{\mathrm{d}\xi} - A(\xi;\lambda)$ and $\mathcal{L}_*$ are strongly related: both can be viewed as closed unbounded operators on $L^2(\mathbb{R})$, and their Fredholm indices coincide for all $\lambda \in \mathbb{C}$ where at least one of the two is Fredholm; see Theorem A.1 or [12, p. 50]. In those regions where the Fredholm index is zero, one can also define point spectra, and it is readily seen that the geometric and algebraic multiplicities of eigenvalues coincide for both operators, with an appropriately generalized notion of multiplicity for the operator stencil $\mathcal{T}(\lambda)$.

If the underlying travelling wave is a front or a pulse so that $\lim_{\xi\to\pm\infty} u_*(\xi)$ exists, then the limiting matrices $A_\pm(\lambda) = \lim_{\xi\to\pm\infty} A(\xi;\lambda)$ also exist. In this situation, Palmer proved in [9] that $\mathcal{T}(\lambda)$ is Fredholm if and only if the matrices $A_\pm(\lambda)$ are hyperbolic, that is, provided that $\operatorname{spec} A_\pm(\lambda) \cap \mathrm{i}\mathbb{R} = \emptyset$. Furthermore, the Fredholm index of $\mathcal{T}(\lambda)$ is then given through

$$i(\mathcal{T}) = i_- - i_+ \tag{2}$$

where $i_\pm$ are the Morse indices of $A_\pm(\lambda)$: The Morse index of a hyperbolic matrix A is the dimension of its unstable subspace, which is the generalized eigenspace associated with all eigenvalues ν of A that have $\operatorname{Re}\nu > 0$. We use the term Morse index even though the underlying equation will, in general, not have a variational structure.

The index formula (2) shows that the Fredholm index depends only on the difference of the Morse indices, that is, only on the *relative Morse index* between A_- and A_+. The importance of (2) lies in the fact that the Fredholm index of the linearization encodes crucial information for perturbation analyses. Morse indices, on the other side, are often much easier to compute.

The fact that the Fredholm index depends only on the difference of two Morse indices makes it possible to extend (2) to situations where the individual Morse indices may not make sense: In [12], we showed how the relative Morse index can be defined in cases where $A_\pm$ are unbounded operators for which both stable and unstable eigenspaces are infinite-dimensional (i.e. have Morse index $i = \infty$). This case is of interest when formulating travelling-wave problems

$$u_{\xi\xi} + cu_\xi + \Delta_y u + f(u) = 0, \quad (\xi, y) \in \mathbb{R} \times \Omega$$

on cylinders as dynamical systems in ξ or when studying time-periodic travelling-wave solutions

$$u = u(x - ct, \omega t), \quad u(\xi, \tau) = u(\xi, \tau + 2\pi)$$

of the reaction-diffusion system (1).

In this paper, we collect and prove several results that facilitate the computation of relative Morse indices, and therefore of Fredholm indices, of the PDE linearization about travelling or modulated waves on cylindrical domains.

Outline: The first part of this paper is devoted to the linearization about travelling waves on cylindrical domains whose properties we introduce in §2. In §3-§4, we state several results that allow us to calculate Fredholm indices via relative Morse indices, through appropriate homotopies to simpler systems or by computing crossing numbers of eigenvalues and Floquet exponents of the asymptotic problems. We then comment in §5 and §6 on the applicability of these results to modulated waves and to systems with boundary conditions. In §7, we apply our results to prove that small localized time-periodic source terms create weak shock waves when added to viscous hyperbolic conservation laws. Section 8 contains a brief discussion of possible generalizations and restrictions of the approach outlined here. Lastly, for completeness, we prove in the appendix that first-order and second-order formulations have the same Fredholm properties and indices.

Notation: Null space and range of a linear operator $\mathcal{T}$ are denoted by $\mathrm{N}(\mathcal{T})$ and $\mathrm{Rg}(\mathcal{T})$, respectively. We denote by $i(\mathcal{T}) := \dim \mathrm{N}(\mathcal{T}) - \operatorname{codim} \mathrm{Rg}(\mathcal{T})$ the index of a Fredholm operator $\mathcal{T}$. Morse indices will be denoted by i_j where $j = +, -, 1, 2, \ldots$.

2. **Travelling waves on cylindrical domains.** For ease of exposition, we focus on the case of travelling waves on cylindrical domains with Dirichlet boundary conditions and consider therefore the elliptic operator

$$\begin{aligned} \mathcal{L}u &= D[u_{\xi\xi} + \Delta_y u] + cu_\xi + a(\xi, y)u && (3)\\ \mathcal{L}: &\quad H^2(\mathbb{R}\times\Omega) \cap H^1_0(\mathbb{R}\times\Omega) \longrightarrow L^2(\mathbb{R}\times\Omega), && (4) \end{aligned}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ with smooth boundary, and Δ_y denotes the Laplace operator in the variable $y \in \mathbb{R}^N$. Associated with the eigenvalue problem for this operator is the equation

$$\begin{aligned} u_\xi &= v\\ v_\xi &= -\Delta_y u - D^{-1}[cv + a(\xi, y)u - \lambda u], && (5) \end{aligned}$$

where $(u, v)(\xi, \cdot) \in X := H^1_0(\Omega) \times L^2(\Omega)$ when considered as functions of y for fixed ξ. We may write this equation as the abstract differential equation

$$U_\xi = A(\xi; \lambda)U \tag{6}$$

where

$$A(\xi;\lambda) = \begin{pmatrix} 0 & \mathrm{id} \\ -\Delta_y + D^{-1}[\lambda - a(\xi, y)] & -D^{-1}c \end{pmatrix} \tag{7}$$

has domain $\mathcal{D}(A) = X^1 := (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)$ for all (ξ, λ). We may also view equation (6) as the abstract first-order operator

$$\mathcal{T}(\lambda): \quad L^2(\mathbb{R}, X^1) \cap H^1(\mathbb{R}, X) \longrightarrow L^2(\mathbb{R}, X), \quad U \longmapsto \mathcal{T}(\lambda)U = U_\xi - A(\xi;\lambda)U.$$

We shall assume that the coefficients $a(\xi, y)$ converge to ξ-periodic functions $a_\pm(\xi, y)$ as $\xi \to \pm\infty$, uniformly in y, where $a_\pm(\xi, y) = a_\pm(\xi + \zeta_\pm, y)$ for some $\zeta_\pm > 0$ and all ξ and y. Asymptotically constant coefficients $a_\pm(\xi, y) = a_\pm(y)$ arise as a special case. We then define the asymptotic operators $\mathcal{L}_\pm$, $A_\pm(\lambda)$, and $\mathcal{T}_\pm(\lambda)$ by replacing the coefficient matrix a in their definition by the asymptotic coefficients $a_\pm$.

Sometimes, we shall also use the adjoint equation $U_\xi = -A(\xi;\lambda)^* U$, where A^* is the adjoint of A in X. The associated operator $\mathcal{T}^*$ is the L^2-adjoint of $\mathcal{T}$.

Hypothesis 2.1 (Cauchy-Uniqueness). *Assume that each solution of (6) and its adjoint equation on $\mathbb{R}^+$ or on $\mathbb{R}^-$ which vanishes at $\xi = 0$ vanishes everywhere.*

Theorem 2.2 ([11, 12], Theorem A.1). *Assume Hypothesis 2.1 holds, then the following statements are equivalent:*

(i) *The operator $\mathcal{T}(\lambda)$ is Fredholm.*
(ii) *The operator $\mathcal{L} - \lambda$ is Fredholm.*
(iii) *The operators $\mathcal{T}_\pm(\lambda)$ are both invertible.*
(iv) *The operators $\mathcal{L}_\pm - \lambda$ are both invertible.*
(v) *Constant coefficients: The operators $A_\pm(\lambda) - \mathrm{i}k$ are invertible on X for all $k \in \mathbb{R}$.*

(v)' *Periodic coefficients with period $\zeta_\pm$: The operators $\frac{\mathrm{d}}{\mathrm{d}\xi} - A_\pm(\xi;\lambda) - \mathrm{i}k$ are invertible on $L^2_{\mathrm{per}}((0,\zeta_\pm), X)$ for all $k \in \mathbb{R}$.*

We will refer to the characterizations (v) and (v)' in the preceding theorem as *hyperbolicity* of $A_\pm(\cdot;\lambda)$. We record that the results in [12] actually do not require that $A(\xi;\lambda)$ has constant or periodic limits as $\xi \to \pm\infty$: instead, Fredholm properties are characterized by the existence of exponential dichotomies for (6) on $\mathbb{R}^+$ and $\mathbb{R}^-$.

Fredholm properties and the Fredholm index are robust under relatively compact perturbations [6, Theorem IV.5.26], and the Fredholm index of $\mathcal{T}(\lambda)$ depends therefore only on the asymptotic operators:

Lemma 2.3. *The Fredholm indices of $\mathcal{T}(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\xi} - A(\xi;\lambda)$ and of $\tilde{\mathcal{T}}(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\xi} - \tilde{A}(\xi;\lambda)$ with*

$$\tilde{A}(\xi;\lambda) = \begin{cases} A_-(\xi;\lambda) & \xi \le 0 \\ A_+(\xi;\lambda) & \xi > 0 \end{cases}$$

coincide whenever one of them is defined.

Hence, it suffices to consider the operator $\tilde{\mathcal{T}}(\lambda)$, and we focus therefore from now on exclusively on this operator. The computation of its Fredholm index involves more detailed knowledge of the asymptotic operators $A_\pm$, as we shall explain in the next two sections.

3. **Relative Morse indices.** We shall assume that all asymptotic coefficient operators $A(\xi;\lambda)$ encountered in this section are hyperbolic in the sense outlined in the previous section after Theorem 2.2.

We begin by considering a pair of hyperbolic operators $A_1(\xi)$ and $A_2(\xi)$, both constant or periodic with possibly different periods ζ_1 and ζ_2, of the form (7). To these operators, we assign the operator $A(\xi)$ defined through

$$A(\xi) = \begin{cases} A_1(\xi) & \xi \le 0 \\ A_2(\xi) & \xi > 0 \end{cases} \tag{8}$$

and the associated operator

$$\mathcal{T} = \frac{\mathrm{d}}{\mathrm{d}\xi} - A(\xi). \tag{9}$$

From Theorem 2.2, we infer that $\mathcal{T}$ is Fredholm, and we define the index of the operator pair (A_1, A_2) as

$$i_{\mathrm{F}}(A_1, A_2) := i(\mathcal{T}). \tag{10}$$

This definition, although seemingly simple, makes it difficult to actually compute the index of $\mathcal{T}$. A natural refinement would be to consider the operators $\mathcal{T}_\pm$ given by

$$\begin{aligned} \mathcal{T}_\pm : \quad & L^2(\mathbb{R}^\pm, X^1) \cap H^1(\mathbb{R}^\pm, X) \subset L^2(\mathbb{R}^\pm, X) \longrightarrow L^2(\mathbb{R}^\pm, X) \\ & U \longmapsto \frac{\mathrm{d}}{\mathrm{d}\xi} U - A_\pm(\xi) U \end{aligned} \tag{11}$$

with $A_- = A_1$ and $A_+ = A_2$. Using hyperbolicity of A_1 and A_2, which we assumed to hold, it is not difficult to see that the operators $\mathcal{T}_\pm$ are both onto and therefore semi-Fredholm. Under the uniqueness hypothesis 2.1 on the Cauchy problem, the bounded trace maps

$$\mathrm{tr}_\pm : \quad \mathrm{N}(\mathcal{T}_\pm) \longrightarrow X, \quad U \longmapsto U(0)$$

are one-to-one, and we define

$$E^{\rm s}_+ := {\rm Rg}({\rm tr}_+), \quad E^{\rm u}_- := {\rm Rg}({\rm tr}_-).$$

Elements in the kernel of $\mathcal{T}$ have traces in the intersection $E^{\rm s}_+ \cap E^{\rm u}_-$, and we therefore consider the natural immersion

$$\iota_{12}: \quad E^{\rm s}_+ \times E^{\rm u}_- \longrightarrow X, \quad (U^{\rm s}_+, U^{\rm u}_-) \longmapsto U^{\rm s}_+ - U^{\rm u}_-. \tag{12}$$

Exploiting again the uniqueness of the Cauchy problem and using the adjoint equation, one finds [12, §5] that

$$i(\iota_{12}) = i(\mathcal{T}). \tag{13}$$

Under the assumptions made so far, one can actually say more about the operators $\mathcal{T}$ and $\mathcal{T}_\pm$. If $\mathcal{T}_\pm$ is Fredholm, or equivalently if the equation

$$\frac{\rm d}{{\rm d}\xi} U = A_\pm(\xi) U \tag{14}$$

does not have any purely imaginary eigenvalues or Floquet exponents, then (14) admits a hyperbolic splitting: Choose $\eta > 0$ so that all eigenvalues or Floquet exponents of (14) have distance strictly larger than η from the imaginary axis, then there exist a constant C and a ξ-dependent decomposition $X = E^{\rm u}_\pm(\xi) \oplus E^{\rm s}_\pm(\xi)$ into subspaces that are periodic in ξ with the same period as $A_\pm(\xi)$ so that the following holds. For each $U \in E^{\rm s}_\pm(\xi_*)$, there is a solution $U(\xi)$ of (14) which is defined for $\xi > \xi_*$ and for which

$$|U(\xi)|_X \le C{\rm e}^{-\eta|\xi-\xi_*|}|U(\xi_*)|_X, \quad \xi > \xi_*.$$

Similarly, for each $U \in E^{\rm u}_\pm(\xi_*)$, there exists a solution $U(\xi)$ of the above equation (14) which is defined for $\xi < \xi_*$ so that

$$|U(\xi)|_X \le C{\rm e}^{-\eta|\xi-\xi_*|}|U(\xi_*)|_X, \quad \xi < \xi_*.$$

We denote by $P^{\rm u,s}_\pm(\xi)$ the projections onto $E^{\rm u,s}_\pm(\xi)$ with kernel $E^{\rm s,u}_\pm(\xi)$; for constant coefficients, these projections are simply the spectral projections associated with the stable and unstable eigenvalues of $A_\pm$. One can then measure the relative dimensions of $E^{\rm u}_\pm(\xi)$ by projecting $E^{\rm u}_-(\xi)$ onto $E^{\rm u}_+(\xi)$ along $E^{\rm s}_+(\xi)$. Consider therefore the map

$$p_{12}(\xi): \quad E^{\rm u}_-(\xi) \longrightarrow E^{\rm u}_+(\xi), \quad U \longmapsto P^{\rm u}_+(\xi)U. \tag{15}$$

We claim that

$$i(\iota_{12}) = i(p_{12}(\xi)), \tag{16}$$

which implies, in particular, that the index on the right-hand side is independent of ξ. To prove (16), one shows by explicit comparison that the dimensions of the null spaces of $p_{12}(0)$ and i_{12} coincide, as do the codimensions of their ranges. Neither of these dimensions for $p_{12}(\xi)$ depends on ξ which can be shown by using the uniqueness assumption 2.1 for (6) and its adjoint.

We record that (16) together with (13) shows that the index i_{12} indeed measures the difference of the unstable dimensions of the two asymptotic equations, thus justifying the terminology of a *relative Morse index*. We summarize our three characterizations of the relative Morse index in the following proposition.

Proposition 3.1. *Assume that the operator $\mathcal{T}$ as defined in (8)-(9) is Fredholm. Then its Fredholm index is equal to the Fredholm index of ι_{12}, as defined in (12), and equal to the Fredholm index of $p_{12}(\xi)$, as defined in (15), for any ξ:*

$$i(\mathcal{T}) = i(\iota_{12}) = i(p_{12}(\xi)).$$

We emphasize that this proposition reduces the dimension of the underlying spaces by one: the operator $\mathcal{T}$ is posed on $L^2(\mathbb{R}, X)$, while both ι_{12} and p_{12} are defined on X. This typically restores compactness properties when the direction ξ is the only unbounded variable in the problem under consideration. In the next section, we shall give a number of useful consequences of this geometric characterization of $i(\mathcal{T})$ through $i(\iota_{12})$ and $i(p_{12})$.

4. **Computation of relative Morse indices.** We start with a simple transitivity lemma, which also justifies our index terminology. Consider the equations

$$\frac{\mathrm{d}}{\mathrm{d}\xi}U = A_j(\xi)U, \qquad j = 1, 2, 3$$

where $A_j(\xi)$ are arbitrary constant or periodic coefficient operators for $j = 1, 2, 3$, and define the relative Morse indices

$$i_{kl} := i(\iota_{kl})$$

for $1 \le k, l \le 3$, whenever possible.

Lemma 4.1 (Transitivity)**.** *We have $i_{13} = i_{12} + i_{23}$ where, in particular, the relative Morse index i_{13} is defined whenever the relative Morse indices i_{12} and i_{23} are defined.*

Proof. The result follows from the characterization of the indices i_{jk} via the projections p_{jk} and the additivity of Fredholm indices under composition of operators. $\square$

We remark that the preceding lemma is not so obvious from the characterization of i_{jk} as the Fredholm index of $\mathcal{T}_{jk}$.

In the remainder of this section, we outline several strategies for computing relative Morse indices. These strategies are facilitated by the robustness properties of Fredholm indices under small perturbations which allow us to calculate relative Morse indices via continuation and homotopies.

4.1. **Continuation.** Suppose therefore that we are interested in computing the relative Morse index i_{12} of the hyperbolic operators $A_1(\xi)$ and $A_2(\xi)$ which we assume to be periodic in ξ. Their relative Morse index is given by $i_{12} = i(\mathcal{T})$ with $\mathcal{T} = \mathrm{d}/\mathrm{d}\xi - A(\xi)$ as in (9) and $A(\xi)$ as in (8):

$$A(\xi) = \begin{cases} A_1(\xi) & \xi \le 0 \\ A_2(\xi) & \xi > 0. \end{cases}$$

Assume furthermore that we have found a homotopy $A(\xi;\mu)$ between $A_1(\xi)$ and $A_2(\xi)$ so that $A_1(\xi) = A(\xi;0)$ and $A_2(\xi) = A(\xi;1)$, the domain of $A(\xi;\mu)$ is independent of μ and ξ, and $A(\mu;\xi)$ depends smoothly on μ as an operator from X^1 into X for each ξ. Possibly after rescaling ξ in a μ-dependent fashion, we may also assume that $A(\xi;\mu)$ has period L in ξ, independently of μ. Our first result states that the relative Morse index is zero if $A(\xi;\mu)$ is hyperbolic for all $\mu \in [0, 1]$.

Lemma 4.2 (Continuation)**.** *Assume that either*

1. *$A(\xi;\mu) = A(\mu)$ is independent of ξ, and $[\nu - A(\mu)]^{-1}$ exists for all $\nu \in \mathrm{i}\mathbb{R}$ and $\mu \in [0, 1]$, or*
2. *$A(\xi;\mu)$ is L-periodic in ξ, and $[\frac{\mathrm{d}}{\mathrm{d}\xi} + \nu - A(\xi;\mu)]^{-1}$ exists on $L^2_{\mathrm{per}}((0, L), X)$ for all $\nu \in \mathrm{i}\mathbb{R}$ and $\mu \in [0, 1]$,*

then $i_{12} = 0$, where i_{12} is the relative Morse index of $A_1(\xi) = A(\xi;0)$ and $A_2(\xi) = A(\xi;1)$.

Proof. We consider the operator $\mathcal{T}(\mu)$ associated with the coefficients

$$\begin{cases} A_1(\xi) & \xi \leq 0 \\ A(\xi;\mu) & \xi > 0 \end{cases}$$

which depends continuously on μ in the operator norm and, by Theorem 2.2, is Fredholm for all values of μ. Thus, its Fredholm index is constant, and therefore equal to zero since it vanishes at $\mu = 0$. □

4.2. **Crossing numbers.** Having established the continuation lemma, it remains to find a way to track the change of Morse indices at points of a homotopy where the asymptotic system fails to be hyperbolic: this would allow us to compute relative Morse indices through arbitrary homotopies. For the sake of clarity, we focus on constant-coefficients operators: the results below hold also for periodic coefficients by simply replacing, in the following discussion, eigenvalues by Floquet exponents.

Assume therefore that we wish to compute the relative Morse index i_{12} of the hyperbolic operators A_1 and A_2. Suppose that $A(\mu)$ is a homotopy with constant coefficients for all $\mu \in [-1, 1]$ which is hyperbolic for $\mu \neq 0$ and satisfies $A_1 = A(-1)$ and $A_2 = A(1)$. We also assume that $A(0)$ has a finite-dimensional center eigenspace at $\mu = 0$ where hyperbolicity fails. In this case, there exists a spectral projection $P^{\mathrm{c}}(0)$ onto the finite-dimensional center space which persists as a spectral projection $P^{\mathrm{c}}(\mu)$ for all μ near zero. Lastly, we assume that the spectral projections $P^{\mathrm{c}}(\mu)$ can be continued continuously, as spectral projections of $A(\mu)$, to $\mu \in [-1, 0) \cup (0, 1]$. In this situation, the relative Morse index i_{12} should be equal to the difference of the Morse indices inside the center space $\operatorname{Rg} P^{\mathrm{c}}(\mu)$ at $\mu = -1$, corresponding to $\xi = -\infty$, and $\mu = 1$, corresponding to $\xi = \infty$, as in (2). This is indeed the case: For $\mu \neq 0$, let $P^{\mathrm{c}}_{\mathrm{u}}(\mu)$ be the projection on the unstable subspace inside $\operatorname{Rg} P^{\mathrm{c}}(\mu)$, then we have the following result:

Lemma 4.3 (Crossing number)**.** *In the setting outlined above, the relative Morse index i_{12} is given by*

$$i_{12} = \dim \operatorname{Rg} P^{\mathrm{c}}_{\mathrm{u}}(-1) - \dim \operatorname{Rg} P^{\mathrm{c}}_{\mathrm{u}}(1).$$

In other words, it is equal to the crossing number, that is, the difference of the number of eigenvalues (or Floquet exponents) of $A(\mu)$ crossing from right to left and left to right, respectively, through the imaginary axis as μ increases through zero.

An alternative explanation for the formula in the preceding lemma goes as follows: Inside the finite-dimensional eigenspace $\operatorname{Rg} P^{\mathrm{c}}(\mu)$, the dynamics are described by constant- or periodic-coefficient ordinary differential equations (ODEs), and stable and unstable subspaces are consequently characterized completely through algebraic multiplicities of eigenvalues or Floquet exponents ν with $\operatorname{Re} \nu < 0$ and $\operatorname{Re} \nu > 0$, respectively. Rouché's theorem shows that we can continue eigenvalues continuously in μ. The relative Morse index is therefore given by the number of eigenvalues, counted with algebraic multiplicity, that cross the imaginary axis from right to left minus the number of eigenvalues that cross from left to right when increasing μ through zero.

Before proving Lemma 4.3, we record the following standard bordering lemma.

Lemma 4.4. *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and that $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ is a Fredholm operator with index $i(\mathcal{A})$. The operator*

$$\mathcal{S} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} : \quad \mathcal{X} \times \mathbb{R}^p \longrightarrow \mathcal{Y} \times \mathbb{R}^q$$

is then Fredholm with index $i(\mathcal{S}) = i(\mathcal{A}) + p - q$ provided $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are bounded and linear.

Proof of Lemma 4.3. The transitivity and continuation properties stated in Lemmas 4.1 and 4.2 imply that i_{12} is equal to the relative Morse index of

$$\begin{cases} A(-\epsilon) & \xi \leq 0 \\ A(\epsilon) & \xi > 0 \end{cases}$$

for each small $\epsilon > 0$. To compute its relative Morse index, we use the characterization of i_{12} through p_{12}. Due to continuity of $A(\mu)$ and the properties of the spectral projection $P^{\mathrm{c}}(\mu)$ stated above, we have

$$E^{\mathrm{u}}_{-}(-\epsilon) = \tilde{E}^{\mathrm{u}}(-\epsilon) \oplus \operatorname{Rg} P^{\mathrm{c}}_{\mathrm{u}}(-\epsilon), \quad E^{\mathrm{u}}_{+}(\epsilon) = \tilde{E}^{\mathrm{u}}(\epsilon) \oplus \operatorname{Rg} P^{\mathrm{c}}_{\mathrm{u}}(\epsilon)$$

where the subspace $\tilde{E}^{\mathrm{u}}(\mu)$ can be chosen to depend continuously on μ for μ near zero. Thus, we need to compute the Fredholm index of

$$p_{12} : \tilde{E}^{\mathrm{u}}(-\epsilon) \oplus \operatorname{Rg} P^{\mathrm{c}}_{\mathrm{u}}(-\epsilon) \longrightarrow \tilde{E}^{\mathrm{u}}(\epsilon) \oplus \operatorname{Rg} P^{\mathrm{c}}_{\mathrm{u}}(\epsilon), \quad U \longmapsto P^{\mathrm{u}}_{+}(\epsilon) U.$$

Due to continuity of $\tilde{E}^{\mathrm{u}}(\mu)$, we have that $P^{\mathrm{u}}_{+}(\epsilon)|_{\tilde{E}^{\mathrm{u}}(-\epsilon)} : \tilde{E}^{\mathrm{u}}(-\epsilon) \to \tilde{E}^{\mathrm{u}}(\epsilon)$ is close to the identity in the operator norm for ϵ close to zero, and therefore has Fredholm index zero. The preceding bordering lemma 4.4 for Fredholm operators shows that

$$i(p_{12}) = p - q = \dim \operatorname{Rg} P^{\mathrm{c}}_{\mathrm{u}}(-\epsilon) - \dim \operatorname{Rg} P^{\mathrm{c}}_{\mathrm{u}}(\epsilon).$$

This is the desired result as these dimensions cannot change from $\mu = \pm\epsilon$ to $\mu = \pm 1$ due to hyperbolicity for $\mu \neq 0$ and the assumed existence of the spectral projection $P^{\mathrm{c}}(\mu)$ for all μ. □

Lemma 4.3 and transitivity imply the following corollary.

Corollary 4.5. *The change of the relative Morse index during homotopies can be computed by adding crossing numbers at all parameter values where hyperbolicity fails.*

4.3. **Exponentially weighted spaces.** We now give an application of Lemma 4.3 to operators posed on exponentially weighted spaces. We formulate our results again only for constant-coefficients operators but emphasize that the results below hold also for periodic coefficients upon replacing eigenvalues by Floquet exponents.

Thus, assume that A has constant coefficients and consider the associated operator $\mathcal{T} = \frac{\mathrm{d}}{\mathrm{d}\xi} - A$ on the space $L^2_\eta(\mathbb{R}, X)$ with norm

$$|U|_\eta := |U(\xi)\mathrm{e}^{\eta\xi}|_{L^2}.$$

Using the isomorphism

$$L^2_\eta(\mathbb{R}, X) \longrightarrow L^2(\mathbb{R}, X), \quad U(\xi) \longmapsto V(\xi) = U(\xi)\mathrm{e}^{\eta\xi},$$

the operator $\mathcal{T}$ for U on L^2_η is readily seen to be conjugated to $\mathcal{T}^\eta = \frac{\mathrm{d}}{\mathrm{d}\xi} - (A + \eta)$ for V on L^2. We record that $\mathcal{T}^\eta$ is Fredholm for η in open subsets of the real line, that is, in a countable union of intervals. When A possesses a finite-dimensional center eigenspace, as discussed above in the context of crossing numbers, we can consider

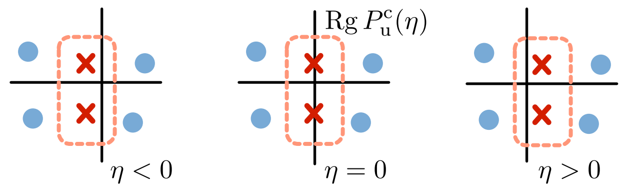

FIGURE 1. Plotted is the spectrum of $A + \eta$ which illustrates the dependence of the dimension of $\operatorname{Rg} P_u^c(\eta)$ on η: We have $\dim \operatorname{Rg} P_u^c(\eta) = 0$ for $\eta < 0$ and $\dim \operatorname{Rg} P_u^c(\eta) = 2$ for $\eta > 0$.

the family of operators $\mathcal{T}^\eta$ with η close to zero. Note that the eigenvalues in the center space depend trivially on η, namely through $\nu(\eta) = \nu(0) + \eta$, so that the crossing number is $-\dim E^c$; see also Figure 1. More generally, we can introduce a two-sided family of weights via

$$|U|_{\eta_-,\eta_+} := |U\chi_+|_{\eta_+} + |U\chi_-|_{\eta_-}$$

where

$$\chi_\pm(\xi) = \begin{cases} 1 & \pm\xi > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The operator $\mathcal{T}$ on $L^2_{\eta_-,\eta_+}$ is again conjugated to an operator $\mathcal{T}^{\eta_-,\eta_+}$ on L^2 whose coefficients are $A+\eta_+$ for $\xi > 0$ and $A+\eta_-$ for $\xi < 0$. An application of Lemma 4.3 gives the following result:

Corollary 4.6. *The operator $\mathcal{T}^{\eta_-,\eta_+}$ is Fredholm for all $\eta_\pm$ close to zero with $\eta_-\eta_+ \neq 0$, and its Fredholm index is given by*

$$i(\mathcal{T}^{\eta_-,\eta_+}) = \frac{1}{2}[\operatorname{sign}\eta_- - \operatorname{sign}\eta_+]\dim E^c.$$

Proof. The eigenvalues $\nu(\eta)$ of $A + \eta$ are given by $\nu(0) + \eta$. Thus, in the notation used in the paragraph preceding Lemma 4.3, we have

$$\dim \operatorname{Rg} P_u^c(\eta) = \begin{cases} 0 & \eta < 0 \\ \dim E^c & \eta > 0 \end{cases}$$

for all $\eta \neq 0$ close to zero. Using the formula given in Lemma 4.3, the result now follows. □

4.4. **Fredholm boundaries and group velocities.** In the case of homotopies in the complex λ-plane, hyperbolicity typically fails along curves. We call these curves *Fredholm boundaries* and refer to the dimension $\dim \operatorname{Rg} P^c(\lambda)$ as the multiplicity of the Fredholm boundary. The generic case of simple Fredholm boundaries, corresponding to crossing numbers equal to ± 1, has in many examples a physical interpretation. Assume that $A(\lambda)$ is analytic in λ and that $A(\lambda_*)$ has a single simple eigenvalue $\nu = \mathrm{i}k$ on the imaginary axis. This eigenvalue can be continued analytically as an eigenvalue $\nu(\lambda)$ for all near values of λ. We assume that

$$\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}(\lambda_*) \neq 0 \tag{17}$$

and define the *group velocity* to be

$$c_g = -\operatorname{Re}\frac{\mathrm{d}\lambda}{\mathrm{d}\nu} = -\operatorname{Re}\left(\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\right)^{-1} = -\frac{\mathrm{d}\operatorname{Re}\lambda}{\mathrm{d}\operatorname{Re}\nu} = -\frac{\mathrm{d}\operatorname{Im}\lambda}{\mathrm{d}\operatorname{Im}\nu}, \tag{18}$$

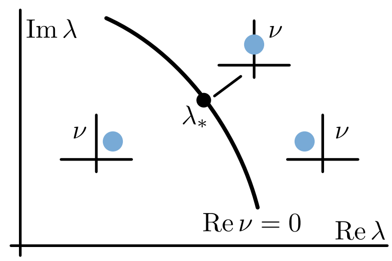

FIGURE 2. The crossing number of $A(\lambda_*+\mu)$ as μ increases through $\mu = 0$ is illustrated for positive group velocity $c_g > 0$. Using that $\operatorname{Re}\lambda = \operatorname{Re}\lambda_* - c_g \operatorname{Re}\nu$, we see that the eigenvalue ν crosses from right to left as μ increases, so that the dimension of the unstable subspace decreases by one, and the crossing number is equal to one.

evaluated at $\lambda = \lambda_*$ or $\nu = \mathrm{i}k$. We assume that $c_g \neq 0$ and refer to [2] for an interpretation of group velocities in terms of transport of perturbations along wave trains.

Lemma 4.7. *Under the above assumptions, that is, for a single simple eigenvalue $\nu = \mathrm{i}k$ on the imaginary axis that satisfies (17) and for a nonzero group velocity c_g, the crossing number of $A(\lambda_* + \mu)$ across the curve $\operatorname{Im}\nu(\lambda) = 0$ at $\mu = 0$ is given by the sign of the group velocity:*

$$i_{12} = \operatorname{sign} c_g.$$

We refer to Figure 2 for an illustration of this result.

Proof. Without loss of generality, we may assume that $\nu = 0$. Writing

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(0) = -c_g + \mathrm{i}b,$$

we find that

$$\lambda = \lambda_* - c_g \operatorname{Re}\nu - b \operatorname{Im}\nu + \mathrm{i}[-c_g \operatorname{Im}\nu + b \operatorname{Re}\nu] + \mathrm{O}(|\nu|^2),$$

and the Fredholm boundary $\{\lambda;\ \operatorname{Re}\nu = 0\}$ is given by

$$\lambda_{\mathrm{fb}} = \lambda_* - [b + \mathrm{i}c_g] \operatorname{Im}\nu + \mathrm{O}(|\nu|^2).$$

In particular, the Fredholm boundary is not horizontal in the λ-plane as $c_g \neq 0$, and computing the crossing number of $A(\lambda_* + \mu)$ as the real parameter μ varies near zero makes sense. For $\operatorname{Im}\nu = 0$, we have

$$\operatorname{Re}[\lambda - \lambda_*] = -c_g \operatorname{Re}\nu.$$

Thus, as μ increases through zero from left to right (meaning that we cross the Fredholm boundary from left to right), the eigenvalue ν crosses from left to right for $c_g < 0$, and from right to left when $c_g > 0$. Thus, the crossing number is equal to $\operatorname{sign} c_g$ as claimed. □

An analogous statement holds in the case of periodic coefficients with the change that the eigenvalue ν needs to be replaced by a Floquet exponent ν.

In summary, the Fredholm index of $\mathcal{T}(\lambda)$ jumps up or down by one when crossing a simple Fredholm boundary, and which jump occurs depends on the sign of the group velocity; see also [4, §2.3] for a similar discussion. The following corollary is central to the characterization of coherent structures in terms of group velocities, linear stability, and multiplicity.

Corollary 4.8 ([13]). *Given a simple Fredholm boundary with eigenvalue $\nu = \mathrm{i}k_+$ at $\lambda = 0$ for $A_+(\lambda)$, then the Fredholm boundary near $\lambda = 0$ is a curve tangent to the imaginary axis. The Fredholm index to the left of this curve is given by $i(\mathcal{T}(0-)) = i(\mathcal{T}(0+)) - \operatorname{sign} c_{\mathrm{g}}^+$ where $i(\mathcal{T}(0\pm))$ are the Fredholm indices for λ to the right and left of the imaginary axis. Moreover, the Fredholm boundary is shifted into the left half plane on weighted spaces when $\eta_+ c_{\mathrm{g}}^+ < 0$. Similarly, for a simple eigenvalue at $\xi = -\infty$ with finite group velocity, we have $i(\mathcal{T}(0-)) = i(\mathcal{T}(0+)) + \operatorname{sign} c_{\mathrm{g}}^-$, and the Fredholm boundary is shifted into the left half plane on weighted spaces when $\eta_- c_{\mathrm{g}}^- > 0$.*

5. Modulated waves.

The case when travelling waves are periodic in time in a comoving frame deserves special attention. Consider therefore a modulated wave $u(x,t) = u_*(x - c_* t, \omega_* t)$ where $u_*(\xi,\tau) = u_*(\xi, \tau + 2\pi)$ for some frequency $\omega_* > 0$. The linearization of a reaction-diffusion system about this wave is given by

$$w_t - D w_{\xi\xi} - c_* w_\xi - a(\xi, \omega_* t) w = 0, \quad a(\xi,\omega_* t) = f_u(u_*(\xi, \omega_* t)). \tag{19}$$

The Floquet ansatz $w(x,t) = u(\xi, \omega_* t)\mathrm{e}^{\lambda t}$ leads to

$$\omega_* u_\tau - D u_{\xi\xi} - c_* u_\xi - a(\xi,\tau) u + \lambda u = 0 \tag{20}$$

and to the first-order system

$$\begin{aligned} u_\xi &= v \\ v_\xi &= D^{-1}[\omega_* \partial_\tau u - c_* v - a(\xi,\tau) u + \lambda u] \end{aligned} \tag{21}$$

where $U = (u,v)^T \in Y = H^{1/2}(S^1) \times L^2(S^1)$ is 2π-periodic in τ for each fixed ξ. We may rewrite (21) as an abstract equation

$$U_\xi = A(\xi;\lambda) U, \tag{22}$$

with the associated operator $\mathcal{T}(\lambda) : L^2(\mathbb{R}, Y^1) \cap H^1(\mathbb{R}, Y) \mapsto L^2(\mathbb{R}, Y)$ where $Y^1 = H^1(S^1) \times H^{1/2}(S^1)$. We may also consider the operator $\mathcal{L} : L^2(\mathbb{R}, H^1(S^1)) \cap H^2(\mathbb{R}, L^2(S^1)) \to L^2(\mathbb{R}, L^2(S^1))$ defined by the left-hand side of the parabolic boundary-value problem (20), and the period map

$$\Phi : \quad L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}), \quad w(\xi,0) \longmapsto w(\xi, \frac{2\pi}{\omega})$$

where w solves (19). We also define $A_\pm$, $\mathcal{L}_\pm$, $\mathcal{T}_\pm$, and $\Phi_\pm$ via the limiting equation at $\xi = \pm\infty$ in a fashion completely analogous to §2.

Theorem 5.1 ([12], Theorem A.1). *The following statements are equivalent:*

(i) *The operator $\mathcal{T}(\lambda)$ is Fredholm;*
(ii) *The operator $\mathcal{L}(\lambda)$ is Fredholm;*
(iii) *The operator $\Phi - \mathrm{e}^{2\pi\lambda/\omega_*}$ is Fredholm;*
(iv) *The operators $\mathcal{T}_\pm(\lambda)$ are both invertible;*
(v) *The operators $\mathcal{L}_\pm - \lambda$ are both invertible;*
(vi) *The operators $\Phi_\pm - \mathrm{e}^{2\pi\lambda/\omega_*}$ are both invertible;*
(vii) *Constant coefficients: The operators $A_\pm(\lambda) - \mathrm{i}k$ are invertible for all $k \in \mathbb{R}$;*
(vii)' *Periodic coefficients: The operators $\frac{\mathrm{d}}{\mathrm{d}\xi} - A_\pm(\xi;\lambda) - \mathrm{i}k$ are invertible on $L^2_{\mathrm{per}}((0, L_\pm), Y)$ for all $k \in \mathbb{R}$.*

Furthermore, the Fredholm indices of $\mathcal{T}(\lambda)$, $\mathcal{L}-\lambda$, and $\Phi-\mathrm{e}^{2\pi\lambda/\omega_*}$ coincide when they exist and depend only on the operators $A_\pm(\lambda)$.

The uniqueness hypothesis 2.1 is automatically satisfied; see [12] and the references therein.

A very common example are modulated travelling waves that are asymptotic as $x\to\pm\infty$ to spatially periodic travelling waves $u_\pm(\kappa_\pm x-\omega_\pm t)$ with $u_\pm(\tau)=u_\pm(\tau+2\pi)$: We shall refer to spatially periodic travelling waves as wave trains. In the frame $\xi=x-c_*t$ of the modulated wave, the convergence of its profile $u_*(\xi,\omega_*t)$ to the asymptotic wave trains requires that $u_*(\xi,\omega_*t)\to u_\pm(\kappa_\pm(\xi+c_*t)-\omega_\pm t)$ as $\xi\to\pm\infty$ uniformly in t. We focus now on the resulting asymptotic problems which involve only the wave trains as their relative Morse index determines, on account of Theorem 5.1, the Fredholm index of the linearization about the modulated wave u_*.

Every wave train u_{wt} is temporally periodic in *any* comoving frame since $u_{\mathrm{wt}}(\kappa x-\omega t)=u_{\mathrm{wt}}(\kappa(\xi+ct)-\omega t)$ is periodic in t with frequency $\tilde\omega(c)=\omega-c\kappa$. If we choose $c=c_{\mathrm{p}}=\omega/\kappa$, then the wave train becomes stationary in this frame. For any given c, one can actually calculate the relative Morse index from a ξ-independent problem: We set $\xi=x-ct$ and $\tau=\tilde\omega(c)t=(\omega-c\kappa)t$ so that the linearization about the wave train u_{wt} becomes

$$U_\xi \quad = \quad \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}[\tilde\omega\partial_\tau+\lambda-f_u(u_{\mathrm{wt}}(\kappa\xi-\tau))] & -D^{-1}c \end{pmatrix} U = A(\xi;\lambda)U. \quad (23)$$

We compare (23) with the corotating version

$$V_\xi \quad = \quad \begin{pmatrix} -\kappa\partial_\sigma & \mathrm{id} \\ D^{-1}[\tilde\omega\partial_\sigma+\lambda-f_u(u_{\mathrm{wt}}(\sigma))] & -\kappa\partial_\sigma-D^{-1}c \end{pmatrix} V = \tilde A(\lambda)V, \quad (24)$$

which arises from (23) via the shear transformation

$$(\xi,\tau)\mapsto(\xi,\sigma)=(\xi,\kappa\xi-\tau). \quad (25)$$

Theorem 5.2. *Equation (23) is hyperbolic if and only if $[\tilde A(\lambda)-\mathrm{i}k]$ is invertible for all $k\in\mathbb{R}$, that is, if and only if the operator on the right-hand side of (24) is hyperbolic. If (24) is hyperbolic, there exists a spectral projection $\tilde P^{\mathrm{u}}$ on the generalized eigenspace associated with spectrum of $\tilde A(\lambda)$ in the right half plane $\mathrm{Re}\,\nu>0$, and relative Morse indices can be computed using the projection $\tilde P^{\mathrm{u}}$ in §3 and (16) instead of the projection $P^{\mathrm{u}}(\xi)$ associated with (23).*

Proof. Assume first that $\tilde A$ is hyperbolic and (23) is not. Since the Floquet spectrum of (23) is discrete, see [8, Lemma 6.1] or [12, §4], this equation has a nontrivial solution $U(\xi,\tau)=\mathrm{e}^{\mathrm{i}k\xi}U_0(\xi,\tau)$ with $U_0(\xi,\tau)$ periodic in ξ and τ. Using the shear transformation (25), we find a bounded nontrivial solution $V(\xi,\sigma)=\mathrm{e}^{\mathrm{i}k\xi}U_0(\xi,\kappa\xi-\sigma)$ of (24) which contradicts hyperbolicity of $\tilde A(\lambda)$. The reversed conclusion can be obtained in a similar fashion. Lastly, evaluating at $\xi=0$, say, one finds that the stable and unstable subspaces of (23) and (24) coincide, which proves the claim on the Morse indices. ∎

An interesting consequence of this description is a unique characterization of Floquet exponents of (23). For each eigenvalue ν of $\tilde A$, we find an associated solution $V(\xi,\sigma)=\mathrm{e}^{\nu\xi}V_0(\sigma)$ of (24). Applying the shear transformation (25), this yields a solution $U(\xi,\tau)=\mathrm{e}^{\nu\xi}V_0(\kappa\xi-\tau)$ of (23), which is associated with the Floquet exponent ν. One can rewrite this solution in the form

$$U(\xi,\tau) = \mathrm{e}^{\nu\xi}V_0(\kappa\xi-\tau) = \mathrm{e}^{(\nu+\mathrm{i}\ell)\xi}U_0(\xi,\tau) \quad \text{with } U_0(\xi,\tau) = \mathrm{e}^{-\mathrm{i}\ell\xi}V_0(\kappa\xi-\tau)$$

for any $\ell \in \mathbb{Z}$, which illustrates the well known fact that Floquet exponents are not unique: If ν is a spatial Floquet exponent, so is $\nu+\mathrm{i}\ell$ for each $\ell \in \mathbb{Z}$. We emphasize, however, that the representation with $\ell = 0$ is special in that it involves a function that depends only on the argument $\kappa\xi - \tau$ of the wave train.

One can also compare relative Morse indices when varying the speed of the frame c. Purely imaginary Floquet exponents $\nu \in \mathrm{i}\mathbb{R}$ occur only when (23) or (24) have nontrivial bounded solutions, that is, when λ belongs to the Floquet spectrum. We discuss now the most relevant case of a stable wave train when $\Phi - \rho$ is invertible for all $|\rho| \geq 1$ with $\rho \neq 1$, and the Fredholm boundary at $\lambda = 0$ is simple. Firstly, from §4, we know that the relative Morse index of $\Phi - \rho$, with $\rho = 1 - \delta$ for δ small, is given by the sign of the group velocity computed in the frame that moves with speed c: The group velocity of wave trains transforms nicely under frame changes in that $\tilde{c}_{\mathrm{g}}(c) = c_{\mathrm{g}}(c = 0) - c$ is the group velocity in the frame moving with speed c [13, §3.2]. Thus, the relative Morse index changes precisely when $c = c_{\mathrm{g}}(0)$, that is, if we sit in the frame that moves with the group velocity computed in the original coordinate x. The second distinguished frame occurs when $c = c_{\mathrm{p}} = \omega/\kappa$, since we then have $\tilde{\omega} = 0$, and the regularity properties of equations (23) and (24) degenerate. In fact, (23) becomes the ODE

$$U_\xi \quad = \quad \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}[\lambda - f_u(u(\kappa\xi))] & -D^{-1}c \end{pmatrix} U = A(\xi;\lambda)U \tag{26}$$

which has only finitely many Floquet multipliers. Actually, the wave train is time periodic with *any* period in this special comoving frame, so that one can also consider the relative Morse indices of

$$U_\xi \quad = \quad \begin{pmatrix} 0 & \mathrm{id} \\ D^{-1}[\hat{\omega}\partial_\tau + \lambda - f_u(u(\kappa\xi))] & -D^{-1}c \end{pmatrix} U = A(\xi;\lambda)U \tag{27}$$

for *any* frequency $\hat{\omega}$. Under the above assumptions of stability, this relative Morse index would then not depend on $\hat{\omega}$. More generally, (27) decouples into an infinite product of ODEs for the Fourier modes $U_\ell \mathrm{e}^{\mathrm{i}\ell\tau}$ with $\ell \in \mathbb{Z}$. For $\hat{\omega}$ large, these ODES are hyperbolic, as can be readily concluded by considering the principal part

$$U_\xi \quad = \quad \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{i}\ell\hat{\omega}D^{-1} & 0 \end{pmatrix} U. \tag{28}$$

The relative Morse index can therefore be inferred from the ODE (26).

6. **Boundary conditions.** In this section, we consider problems posed on half-cylinders $\mathbb{R}^+ \times \Omega$ in the setup considered in §2. In this case, boundary conditions need to be imposed on the end section $\{0\} \times \Omega$ of the cylinder, and we introduce these conditions for the abstract first-order system. Let $E_{\mathrm{bc}} \subset X^{1/2}$ be some closed subspace. We are interested in solutions of $U_\xi = A(\xi)U$ with $U(0) \in E_{\mathrm{bc}}$. Equivalently, we can study the kernel of the operator

$$\mathcal{T}_{\mathrm{bc}}: \quad \mathcal{D}(\mathcal{T}_{\mathrm{bc}}) \subset L^2(\mathbb{R}^+, X) \longrightarrow L^2(\mathbb{R}^+, X), \quad U \longmapsto \frac{\mathrm{d}}{\mathrm{d}\xi}U - A(\xi)U, \tag{29}$$

where

$$\mathcal{D}(\mathcal{T}_{\mathrm{bc}}) := \{U \in L^2(\mathbb{R}^+, X^1) \cap H^1(\mathbb{R}^+, X);\ U(0) \in E_{\mathrm{bc}}\},$$

which, by the trace theorem and interpolation theory, is a closed subspace of $L^2(\mathbb{R}^+, X^1) \cap H^1(\mathbb{R}^+, X)$.

We may then define the relative Morse index of the boundary condition E_{bc} and $A(\xi)$ as the Fredholm index of $\mathcal{T}_{\mathrm{bc}}$:

$$i_F(\mathrm{bc}, A) := i(\mathcal{T}_{\mathrm{bc}}).$$

Using the projection P_+^{u} associated with the asymptotic equation at $\xi = \infty$, we find that

$$i_F(\mathrm{bc}, A) = i(p_{\mathrm{bc}}) \quad \text{where} \quad p_{\mathrm{bc}} : \; E_{\mathrm{bc}} \longrightarrow \mathrm{Rg}\, P_+^{\mathrm{u}}, \quad U \longmapsto P_+^{\mathrm{u}} U. \tag{30}$$

The proof of this assertion is again a consequence of the spatial dynamics formulation in [12]. From the relation (30), one immediately concludes transitivity

$$i_F(\mathrm{bc}, A_2) = i_F(\mathrm{bc}, A_1) + i_{12}. \tag{31}$$

We also record that the Fredholm index is stable under perturbations of the boundary conditions. Here, continuity of subspaces is measured in the usual graph norm by writing a subspace E_2 as a graph over E_1 with values in the normal direction in $X^{1/2}$.

For problems on $\mathbb{R}^-$, one would reflect ξ or, equivalently, construct a "negative relative Morse index" via the stable projection of A and the boundary subspace. By adding the negative Morse index of a boundary-value problem on $\mathbb{R}^- \times \Omega$ and the Morse index on $\mathbb{R}^+ \times \Omega$, one can then compute Morse indices on bounded cylinders $J \times \Omega$ for finite intervals J. We note, however, that transitivity results between boundary conditions need not hold in general.

We conclude this section by discussing the important special cases of Dirichlet ($u = 0$) and Neumann ($v = 0$) boundary conditions.

Lemma 6.1. *The operator $\mathcal{T}_{\mathrm{bc}}$ with Dirichlet or Neumann boundary conditions is Fredholm if and only if A is hyperbolic. The relative Morse index is zero for the two special cases associated with $\mathcal{L}_1 u = D(u_{\xi\xi} + u_{yy}) - u$ and $\mathcal{L}_2 u = \omega u_\tau - D u_{\xi\xi} + u$.*

Proof. Invertibility is easily shown in the two examples using eigenfunction expansions in the τ- or y-direction. The transitivity formula (31) then gives the result for general hyperbolic operators A. □

7. **Example: Localized source terms in hyperbolic conservation laws.** We now give an application of our results that illustrates the relevance of Fredholm indices and their computation through relative Morse indices. Consider the viscous conservation law

$$u_t = B u_{xx} + f(u)_x + \varepsilon g(t, x, u, u_x), \quad u \in \mathbb{R}^n \tag{32}$$

for a positive definite, symmetric viscosity matrix B, a smooth hyperbolic flux f with

$$\operatorname{spec} f_u(0) = \{-c_1 > -c_2 > \ldots > -c_n\}, \quad \operatorname{spec}[B^{-1} f_u(0)] \cap \mathrm{i}\mathbb{R} = \emptyset, \tag{33}$$

and a smooth, spatially localized, temporally periodic source term g so that for appropriate positive constants $T, C, \delta > 0$

$$g(t, x, u, v) = g(t + T, x, u, v), \quad |g(t, x, u, v)| \leq C \mathrm{e}^{-\delta |x|} \tag{34}$$

for all (t, x) and all (u, v) near zero.

We are interested in finding time-periodic solutions $u(x, t) = u(x, t + T)$ of (32) and their asymptotic rest states for $\varepsilon \approx 0$. If g does not depend on t and is in

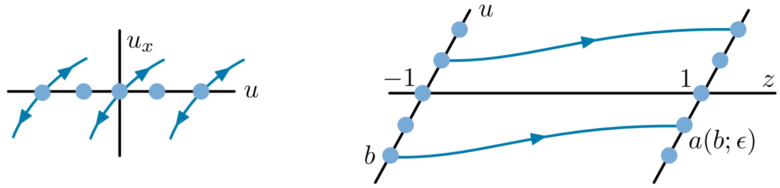

FIGURE 3. The autonomous flows of (35) for $\epsilon = 0$ [left] and on the two-dimensional center manifold of (36) for $\varepsilon \neq 0$ [right] are shown for $n = 1$ with $c_1 > 0$, illustrating the existence of shock profiles for $\varepsilon \neq 0$ when $g = g(x, u, u_x)$.

conservation form, $g = h(x, u)_x$, and the characteristic speeds $c_j \neq 0$ are nonzero for all j, one can easily obtain a perturbation result for solutions close to $u = 0$ by inspecting the integrated steady-state equation

$$Bu_x + f(u) + \varepsilon h(x, u) = 0$$

and using the hyperbolicity of the origin.

We consider first the more general case $g = g(x, u, u_x)$ and seek small bounded solutions of the second-order differential equation

$$Bu_{xx} + f(u)_x + \varepsilon g(x, u, u_x) = 0. \tag{35}$$

As illustrated in Figure 3, this can be achieved using dynamical-systems methods applied to the autonomous first-order system

$$\begin{pmatrix} u_x \\ v_x \\ z_x \end{pmatrix} = \begin{pmatrix} v \\ -B^{-1}[f_u(u)v + \varepsilon g(x(z), u, v)] \\ \frac{\delta}{4}(1 - z^2) \end{pmatrix}, \quad x = \frac{2}{\delta} \ln \frac{1+z}{1-z} \tag{36}$$

for $(u, v, z) \in \mathbb{R}^2 \times [-1, 1]$ with δ as in (34).

Alternatively, we may pursue an approach based only on functional analysis and Lyapunov–Schmidt reduction, thus exploiting the Fredholm constructions outlined above. We explain this latter approach in the case of time-independent g and for nonzero characteristic speeds $c_j \neq 0$. Afterwards, we outline the modifications that are necessary to deal with the general case of time-periodic source terms and vanishing characteristic speeds.

Key to the functional-analytic approach is the linearization

$$\mathcal{L}u = Bu_{xx} + f_u(0)u_x \tag{37}$$

at the origin, which can be viewed as a closed, densely defined operator on $L^2(\mathbb{R}, \mathbb{R}^n)$. We may also consider $\mathcal{L}$ as a closed, densely defined operator on $\tilde{L}^2_\eta(\mathbb{R}, \mathbb{R}^n)$ with norm

$$|u|_{\tilde{L}^2_\eta} = |u(x)\mathrm{e}^{\eta|x|}|_{L^2}. \tag{38}$$

For $\eta > 0$, elements in $\tilde{L}^2_\eta$ decay exponentially as $|x| \to \infty$.

Lemma 7.1. *Assume that $c_j \neq 0$ for all j, then there is an $\eta_* > 0$ with the following property. For each fixed η with $0 < \eta < \eta_*$, the operator $\mathcal{L}$ defined on $\tilde{L}^2_\eta(\mathbb{R}, \mathbb{R}^n)$ is Fredholm of index $-n$ and has trivial null space.*

Proof. The eigenvalues ν of the second-order system $\mathcal{L}u = 0$ are solutions of

$$\det[\nu^2 B + f_u(0)\nu] = \nu^n \det[\nu B + f_u(0)] = 0, \tag{39}$$

so that $\nu = 0$ is an eigenvalue with multiplicity n, and all other eigenvalues have nonzero real part due to (33). In particular, we have $\dim E^{\mathrm{c}} = n$. We apply Theorem 5.1 and Corollary 4.6 to the weighted norm (38), which corresponds to $\eta_- = -\eta < 0$ and $\eta_+ = \eta > 0$ in the notation of §4, and therefore find that the Fredholm index of $\mathcal{L}$ on $\tilde{L}^2_\eta$ is equal to $-n$ as claimed. Since (37) is a constant-coefficient operator, one can readily check that the kernel of $\mathcal{L}$ in the exponentially weighted space is trivial. $\square$

Lemma 7.1 implies that the kernel of the L^2-adjoint $\mathcal{L}^*$ considered on $\tilde{L}^2_{-\eta}$ is n-dimensional and spanned by the constants e_j for $j = 1, \dots, n$, where e_j are the canonical basis vectors in $\mathbb{R}^n$.

To find shock-like transition layers, caused by the inhomogeneity g for small ε, we make the ansatz

$$u(x) = \sum_{j=1}^{n} a_j e_j \chi(x) + \sum_{j=1}^{n} b_j e_j \chi(-x) + w(x), \tag{40}$$

where $a_j, b_j \in \mathbb{R}$, $w \in \tilde{L}^2_\eta$, and $\chi(x) = (1 + \tanh(x))/2$. Substituting this ansatz into (32), we obtain an equation of the form

$$F(a, b, w; \varepsilon) = 0, \quad F(\cdot; \varepsilon) : \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{D}(\mathcal{L}) \subset \mathbb{R}^{2n} \times \tilde{L}^2_\eta \longrightarrow \tilde{L}^2_\eta \tag{41}$$

for $a = (a_j), b = (b_j)$ and w. For $0 < \eta \ll 1$, the map F is smooth, as elements in $\tilde{L}^2_\eta$ and the source term g decay exponentially as $|x| \to \infty$, and its linearization at $(a, b, w) = 0$ is given by

$$F_w(0; 0) = \mathcal{L},\ F_{a_j}(0; 0) = B e_j \chi'' + f_u(0) e_j \chi',\ F_{b_j}(0; 0) = B e_j \chi'' - f_u(0) e_j \chi'$$

where $F_a(0; 0)$ and $F_b(0; 0)$ lie in $\tilde{L}^2_\eta$.

Lemma 7.2. *Under the hypotheses of Lemma 7.1, the operator*

$$F_{a,w}(0; 0) :\ \mathbb{R}^n \times \tilde{L}^2_\eta \longrightarrow \tilde{L}^2_\eta, \quad (a, w) \longmapsto F_a(0; 0)a + F_w(0; 0)w$$

is invertible.

Proof. Lemma 7.1 shows that $\mathcal{L}$ has trivial kernel and Fredholm index $-n$. First, we observe that the n partial derivatives with respect to a_j are linearly independent: To see this, evaluate $F_{a_j}(0; 0)$ at $x = 0$ where $\chi''(0) = 0$ and $\chi'(0) = \frac{1}{2}$, and exploit that $f_u(0)$ is invertible. Next, we compute the scalar products of $F_{a_j}(0; 0)$ with the elements e_i of the kernel of the adjoint $\mathcal{L}^*$:

$$\int_{\mathbb{R}} \langle F_{a_j}(0; 0)(x), e_i \rangle \,\mathrm{d}x = \int_{\mathbb{R}} \langle B e_j \chi'' + f_u(0) e_j \chi', e_i \rangle \,\mathrm{d}x = \langle f_u(0) e_j, e_i \rangle = [f_u(0)]_{ij} \tag{42}$$

which, for fixed j, is nonzero for some i by invertibility of $f_u(0)$. Hence, the partial derivatives $F_{a_j}(0; 0)$ are not in the range of $\mathcal{L}$. Taking these facts together proves the lemma. $\square$

We can therefore solve (41) with the implicit function theorem and obtain unique solutions $(a, w)(b; \varepsilon)$. The physically interesting quantity is the jump $u(\infty) - u(-\infty) = a(b; \varepsilon) - b$. A straightforward expansion gives

$$u(\infty) - u(-\infty) = a(b; \varepsilon) - b = \varepsilon \int_{\mathbb{R}} f_u(0)^{-1} g(x, 0, 0) \,\mathrm{d}x + \mathrm{O}(\varepsilon^2)$$

which, to leading order, is independent of b. By continuity, the characteristics of the PDE associated with the new shock profile are close to those of $u = 0$ at the origin, and we conclude that the number of positive characteristic speeds at ∞ and $-\infty$ are equal: thus, the inhomogeneity allows for transmission, and the resulting shock profiles are undercompressive shocks of index 1.

Using the results in §5 on Fredholm indices and relative Morse indices for modulated waves, the preceding analysis generalizes immediately to the case where g depends periodically on t with period $T = 2\pi/\omega$: The only extra hypothesis needed is that the linearization $\mathcal{L}$ does not have essential spectrum at $\lambda = \mathrm{i}\omega\ell$ for $\ell \in \mathbb{Z}$, which is guaranteed to hold, for instance, when $B = \mathrm{id}$.

The functional-analytic approach allows us also to study the case where precisely one characteristic speed c_j vanishes. In this situation, we may, without loss of generality, assume that $f_u(0)e_1 = 0$. We see that the Morse index of $\mathcal{L}$ in $\tilde{L}^2_\eta$ is $-(n+1)$, since $\nu = 0$ has multiplicity $n+1$ as a solution of (39). The kernel of the adjoint operator $\mathcal{L}^*$ on $\tilde{L}^2_{-\eta}$ is spanned by the constant functions e_j and the linear function xe_1, which lies in $\tilde{L}^2_{-\eta}$ for $\eta > 0$. Using the ansatz (40), we arrive again at the function F given in (41):

Lemma 7.3. *Assume that $f_u(0)$ has distinct real eigenvalues with a simple eigenvalue at $\nu = 0$ with eigenvector e_1. We also assume that* $\mathrm{spec}[B^{-1}f_u(0)]$ *does not contain nonzero purely imaginary eigenvalues. Then the linearization of F with respect to (a, b_1, w) is invertible at $(0; 0)$.*

Proof. One readily verifies that the partial derivatives with respect to $\{a_j\}_{j=1,\dots,n}$ and b_1 are linearly independent. Furthermore, for each fixed $j = 2, \dots, n$, we have

$$\int_{\mathbb{R}} \langle Be_j\chi'' + f_u(0)e_j\chi', e_i\rangle \,\mathrm{d}x = \langle f_u(0)e_j, e_i\rangle = [f_u(0)]_{ij}$$

which is nonzero for some i since $f_u(0)u = 0$ only when u is a multiple of e_1. Lastly,

$$\int_{\mathbb{R}} \langle F_{a_1}(0;0)(x), xe_1\rangle \,\mathrm{d}x = \int_{\mathbb{R}} \langle Be_1\chi'' + f_u(0)e_1\chi', xe_1\rangle \,\mathrm{d}x = -\langle Be_1, e_1\rangle < 0$$

and similarly

$$\int_{\mathbb{R}} \langle F_{b_1}(0;0)(x), xe_1\rangle \,\mathrm{d}x = \int_{\mathbb{R}} \langle Be_1\chi''(-x) - f_u(0)e_1\chi'(-x), xe_1\rangle \,\mathrm{d}x = \langle Be_1, e_1\rangle > 0$$

so that $F_{a_1}(0;0)$ and $F_{b_1}(0;0)$ do not lie in the range of $F_w(0;0)$. In summary, the $n+1$ linearly independent functions $F_{a_j}(0;0)$ and $F_{b_1}(0,0)$ lie outside the range of $\mathcal{L}$ which is one-to-one with Fredholm index $-(n+1)$. Thus, $F_{a,b_1,w}(0;0)$ is invertible as claimed. ◻

We can therefore solve (41) with the implicit function theorem and obtain unique solutions (a, b_1, w) as functions of $(b_2, \dots, b_n; \varepsilon)$. The interesting aspect in this situation is that the solution selects both a_1 and b_1 via

$$a_1 = M\varepsilon + \mathrm{O}(\varepsilon^2), \quad b_1 = -M\varepsilon + \mathrm{O}(\varepsilon^2), \quad M := \int_{\mathbb{R}} \frac{x\langle e_1, g(x,0,0)\rangle}{\langle Be_1, e_1\rangle} \,\mathrm{d}x.$$

Provided $M \neq 0$, the difference between the number of positive characteristic speeds at ∞ and $-\infty$ is therefore two, and the viscous profile is a Lax shock or an undercompressive shock of index 2, depending on the sign of ε (and of M).

In the simple case of a scalar conservation law, the preceding analysis parallels [13, §6.5] and [7, §3.2]. We emphasize, however, that the present analysis, by focusing on solutions which are exponentially localized at $x = \pm\infty$ with a uniform rate η, misses contact and transmission defects. In the language of shocks, contact defects correspond to shocks with glancing modes, while transmission defects correspond to the undercompressive shock waves of index 1 that we found in the case of inhomogeneities for non-vanishing characteristic speeds. We also note that there are many more Lax shocks in the vicinity of $u = 0$ for each sign of ε. The Lax shocks found in our analysis are special in that they are strongly decaying towards the asymptotic constant states.

8. **Discussion.** The results that we presented in this paper are valid for semilinear elliptic problems in cylindrical domains and for modulated waves in reaction-diffusion systems. Similar results hold for the linearization at radially symmetric stationary or time-periodic patterns [16] and at spiral waves [15], and also for perturbations of eigenvalues embedded in the essential spectrum of elliptic operators [3]. We expect that analogous results hold in far more general situations: In particular, the coefficients in front of the higher-order derivatives may depend on (ξ, y, λ) but we will not pursue this question here.

Lastly, we mention that the strategy we pursued in §7 for time-periodic perturbations of viscous conservation laws may also be successful when studying Hopf bifurcations from viscous shock profiles. This may give existence proofs that are simpler than those given in [14, 17, 18] though we do not know whether the stability proof given in [14] can be simplified using this approach.

Appendix A. **Fredholm properties of first- and second-order operators.** We consider the operator

$$\mathcal{L} = D\partial_{xx} + c(x)\partial_x + a(x)$$

as a densely defined operator on $X = L^2(\mathbb{R}, \mathbb{C}^n)$ with domain $X^1 = H^2(\mathbb{R}, \mathbb{C}^n)$ or, alternatively, on $X = C^0_{\text{unif}}(\mathbb{R}, \mathbb{C}^n)$ with domain $X^1 = C^2_{\text{unif}}(\mathbb{R}, \mathbb{C}^n)$. Here, D is a diagonal matrix with strictly positive entries, and $a, c \in C^0(\mathbb{R}, \mathbb{C}^{n\times n})$ are continuous matrix-valued functions. We also consider the associated first-order operator

$$\mathcal{T} = \frac{\mathrm{d}}{\mathrm{d}x} - \begin{pmatrix} 0 & 1 \\ -D^{-1}a(x) & -D^{-1}c(x) \end{pmatrix}$$

as a densely defined operator on $Y = L^2(\mathbb{R}, \mathbb{C}^{2n})$ with domain $Y^1 = H^1(\mathbb{R}, \mathbb{C}^{2n})$ or, alternatively, on $Y = C^0_{\text{unif}}(\mathbb{R}, \mathbb{C}^{2n})$ with domain $Y^1 = C^1_{\text{unif}}(\mathbb{R}, \mathbb{C}^{2n})$. Both $\mathcal{L}$ and $\mathcal{T}$ are closed operators.

Theorem A.1. *The following statements hold:*

(i) *The operator $\mathcal{T}$ is Fredholm on L^2 if and only if it is Fredholm on C^0_{unif}. The Fredholm indices coincide on these spaces.*
(ii) *The operator $\mathcal{L}$ is Fredholm on X if and only if the operator $\mathcal{T}$ is Fredholm on Y (with $X = L^2$ and $Y = L^2$ or with $X = C^0_{\text{unif}}$ and $Y = C^0_{\text{unif}}$).*
(iii) *If $\mathcal{L}$ and $\mathcal{T}$ are Fredholm on X and Y, respectively, then their Fredholm indices coincide.*

Using the material in [12], the proof of Theorem A.1 given below applies also to the situations described in Theorems 2.2 and 5.1 to show the equivalence of first-order and second-order formulations, and we will omit the details.

Proof. Palmer proved in [9, 10] that $\mathcal{T}$ is Fredholm on C^0_{unif} if and only if the ordinary differential equation $\mathcal{T}U = 0$ has exponential dichotomies on $\mathbb{R}^+$ and on $\mathbb{R}^-$. Ben-Artzi and Gohberg [1] proved the same result for L^2 spaces. Alternatively, the equivalence of exponential dichotomies on $\mathbb{R}^\pm$ and Fredholm properties on L^2 was proved in [12] for a far more general class of operators that may depend on additional independent variables provided c is a constant; if there are no additional variables present as in our setting, the proof in [12] works for the operators considered here[1]. This establishes (i).

We define the operator

$$\mathcal{T}_{\mathrm{ref}} = \frac{\mathrm{d}}{\mathrm{d}x} - \begin{pmatrix} 0 & 1 \\ D^{-1} & 0 \end{pmatrix}$$

from Y into itself with domain Y^1 and record that $\mathcal{T}_{\mathrm{ref}}$ is invertible. We may then define the continuous operator

$$\mathcal{P}: \quad Y \longrightarrow X, \quad G \longmapsto -D^{-1}[(1+a(x))P_1 + c(x)P_2]\mathcal{T}_{\mathrm{ref}}^{-1}G \tag{43}$$

where $P_j : Y \to X,\ G = (g_1, g_2)^T \mapsto g_j$ projects onto the jth component.

Lemma A.2. *We have*

$$G \in \mathrm{Rg}(\mathcal{T}) \iff \begin{pmatrix} 0 \\ \mathcal{P}G \end{pmatrix} \in \mathrm{Rg}(\mathcal{T}), \quad \begin{pmatrix} 0 \\ h \end{pmatrix} \in \mathrm{Rg}(\mathcal{T}) \iff h \in \mathrm{Rg}(\mathcal{L}). \tag{44}$$

Specifically, $\mathcal{T}U = G$ *for some* $U \in Y^1$ *if and only if* $\mathcal{T}V = (0, \mathcal{P}G)^T$*, where* $U = V + \mathcal{T}_{\mathrm{ref}}^{-1}G$*.*

Proof. Assume that $\mathcal{T}U = G$ for some $U \in Y^1$. Define $V := U - \mathcal{T}_{\mathrm{ref}}^{-1}G \in Y^1$, then

$$\begin{aligned} \mathcal{T}V &= \mathcal{T}U - \mathcal{T}\mathcal{T}_{\mathrm{ref}}^{-1}G = \underbrace{G}_{=\mathcal{T}U} + \underbrace{[\mathcal{T}_{\mathrm{ref}} - \mathcal{T}]\mathcal{T}_{\mathrm{ref}}^{-1}G - G}_{=-\mathcal{T}\mathcal{T}_{\mathrm{ref}}^{-1}G} = [\mathcal{T}_{\mathrm{ref}} - \mathcal{T}]\mathcal{T}_{\mathrm{ref}}^{-1}G \\ &= \begin{pmatrix} 0 & 0 \\ -D^{-1}[1+a(x)] & -D^{-1}c(x) \end{pmatrix} \mathcal{T}_{\mathrm{ref}}^{-1}G \stackrel{(43)}{=} \begin{pmatrix} 0 \\ \mathcal{P}G \end{pmatrix} \in Y. \end{aligned}$$

The converse is proved by reversing the order of the above argument. The second equivalence in (44) is obvious. $\square$

Regarding statement (ii), it is not difficult to see that $\mathcal{L}$ is Fredholm whenever $\mathcal{T}$ is Fredholm. We therefore prove only the converse. Thus, assume that $\mathcal{L}$ is Fredholm.

We first claim that $\mathrm{Rg}(\mathcal{T})$ is closed in Y. To show this, take a sequence $G_n \in \mathrm{Rg}(\mathcal{T})$ with $G_n \to G$ in Y as $n \to \infty$. We need to prove that $G \in \mathrm{Rg}(\mathcal{T})$. Since $G_n \in \mathrm{Rg}(\mathcal{T})$, we know from Lemma A.2 that $\mathcal{P}G_n \in \mathrm{Rg}(\mathcal{L})$ for all n. Continuity of $\mathcal{P}$ and closedness of $\mathrm{Rg}(\mathcal{L})$ implies that the limit $\mathcal{P}G \in \mathrm{Rg}(\mathcal{L})$, and Lemma A.2 shows that $G \in \mathrm{Rg}(\mathcal{T})$. Thus, $\mathrm{Rg}(\mathcal{T})$ is closed in Y as claimed.

To complete the proof of of (ii) and of statement (iii), we first record that

$$\dim \mathrm{N}(\mathcal{L}) = \dim \mathrm{N}(\mathcal{T}).$$

We have shown above that if $\mathrm{Rg}(\mathcal{L})$ or $\mathrm{Rg}(\mathcal{T})$ is closed, then both spaces are closed. Assuming therefore that these spaces are closed, we need to show that their codimensions are equal, which will complete the proof of (ii) and (iii). Lemma A.2 implies that

$$G \notin \mathrm{Rg}(\mathcal{T}) \iff \begin{pmatrix} 0 \\ \mathcal{P}G \end{pmatrix} \notin \mathrm{Rg}(\mathcal{T}), \quad \begin{pmatrix} 0 \\ h \end{pmatrix} \notin \mathrm{Rg}(\mathcal{T}) \iff h \notin \mathrm{Rg}(\mathcal{L}). \tag{45}$$

[1]The assumption in [12] that c is constant is needed only to guarantee compactness with respect to the additional independent variables.

In particular, if $G_j \notin \mathrm{Rg}(\mathcal{T})$ are linearly independent in Y for $j = 1, \ldots, N$, then the elements $\mathcal{P}G_j$ are linearly independent in X; otherwise, the first equivalence in (45) would not hold.

Assume first that $\mathcal{L}$ is Fredholm, then $\operatorname{codim} \mathrm{Rg}(\mathcal{L}) =: N$ is finite, and we can choose a basis $\{h_j\}_{j=1,\ldots,N}$ of a complement of $\mathrm{Rg}(\mathcal{L})$. Thus, $H_j := (0, h_j)^T$ are linearly independent in Y and $H_j \notin \mathrm{Rg}(\mathcal{T})$ by (45), which shows that $\operatorname{codim} \mathrm{Rg}(\mathcal{T}) \geq \operatorname{codim} \mathrm{Rg}(\mathcal{L})$. If $\operatorname{codim} \mathrm{Rg}(\mathcal{T}) > \operatorname{codim} \mathrm{Rg}(\mathcal{L})$, then there exists an element $H_{N+1} \notin \mathrm{Rg}(\mathcal{T}) \oplus \operatorname{span}\{H_j\}_{j=1,\ldots,N}$. In particular, $H_j \notin \mathrm{Rg}(\mathcal{T})$ are linearly independent for $j = 1, \ldots, N+1$, and the discussion following (45) shows that $\hat{h}_j := \mathcal{P}H_j \notin \mathrm{Rg}(\mathcal{L})$ are also linearly independent for $j = 1, \ldots, N+1$ which contradicts our starting assumption. Thus, $\operatorname{codim} \mathrm{Rg}(\mathcal{T}) = \operatorname{codim} \mathrm{Rg}(\mathcal{L})$.

An analogous, and in fact simpler, argument leads to the same conclusion if we start from the assumption that $\mathcal{T}$ is Fredholm. This completes the proof of Theorem A.1. □

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