Spectra of coherent structures — why do we care?

I **Example: oscillations & small inhomogeneities**

II **Example: instabilities of spiral waves**

I Example: oscillations & small inhomogeneities

II Example: instabilities of spiral waves

Oscillatory media

Oscillatory reaction

$$\boldsymbol{U_t = F(U) \in \mathbb{R}^N, \qquad U_*(-\omega_* t) = U_*(-\omega_* t + 2\pi)}$$

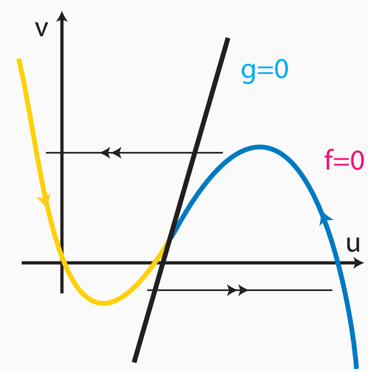
Diffusion

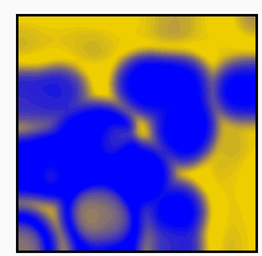
$$\boldsymbol{U_t = D\Delta U + F(U), \qquad x \in \Omega \subseteq \mathbb{R}^n,\ D = \mathrm{diag}\, d_j > 0,}$$ **Neumann b.c.**

Example: FitzHugh-Nagumo

$$\boldsymbol{u_t = \frac{1}{\mu}[u(1-u)(u-a) - v]}$$

$$\boldsymbol{v_t = u - \gamma v + b}$$





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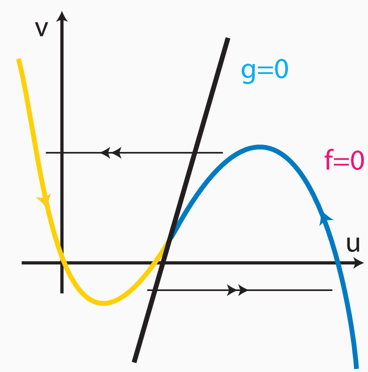
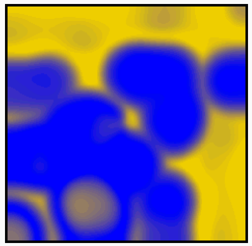


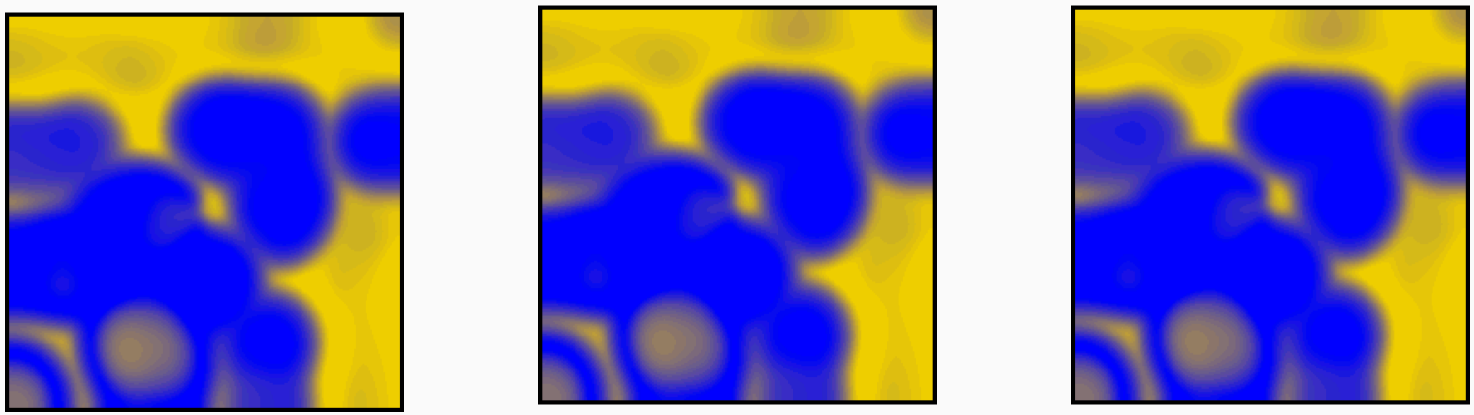


Periodic orbits are robust

size of perturbation $\ll$ spectral Floquet gap

Robustness — inhomogeneities

$\varepsilon = -0.05$ $\qquad$ $\varepsilon = 0$ $\qquad$ $\varepsilon = 0.05$



$$u_t = \Delta u + \frac{1}{\mu} u(1-u)(u-a)$$

$$v_t = \Delta v + u - v + b + \frac{\varepsilon}{1 + |x/3|^2}$$

with $a = 0.34, b = -0.045, \mu = 0.08$ on $\Omega = \{|x_j| \leq 90\}$

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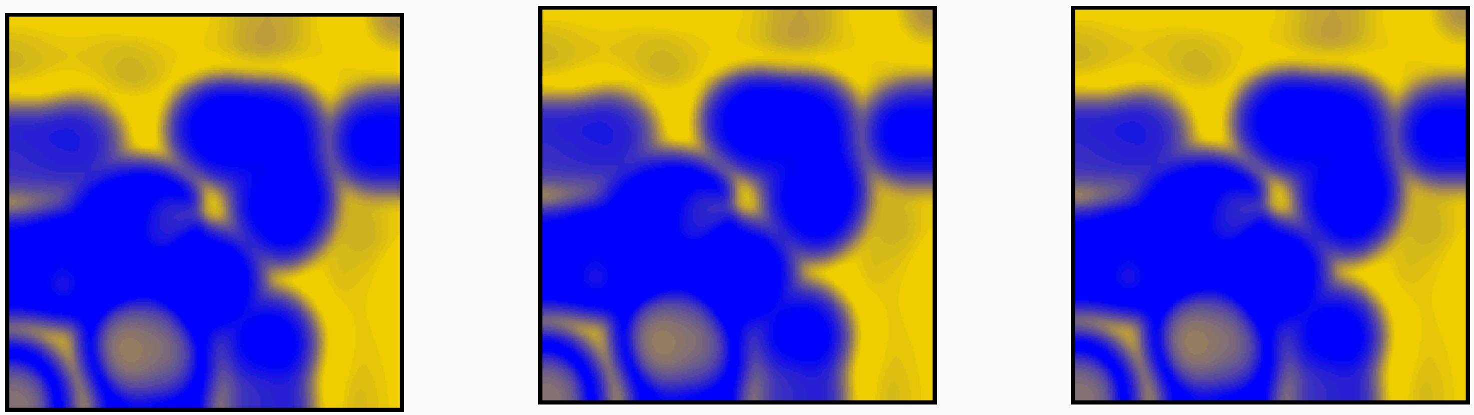

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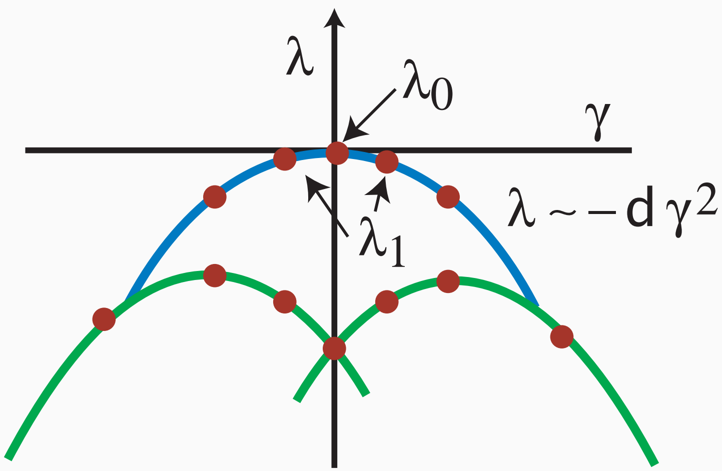
Coherent structures — what makes it difficult?

! Robustness analysis of the linearized the period map Φ !

Small spectral gaps in $|x| \leq L$

Floquet spectrum of Φ clusters near $\lambda = 0$:

$$\lambda_0 = 0 > \lambda_1 \sim -\frac{d}{4L^2} \geq \dots$$



In our example $L = 90$, and

$$\lambda_1 \sim 3 \times 10^{-5}$$

Fredholm boundaries in $\mathbb{R}^n$

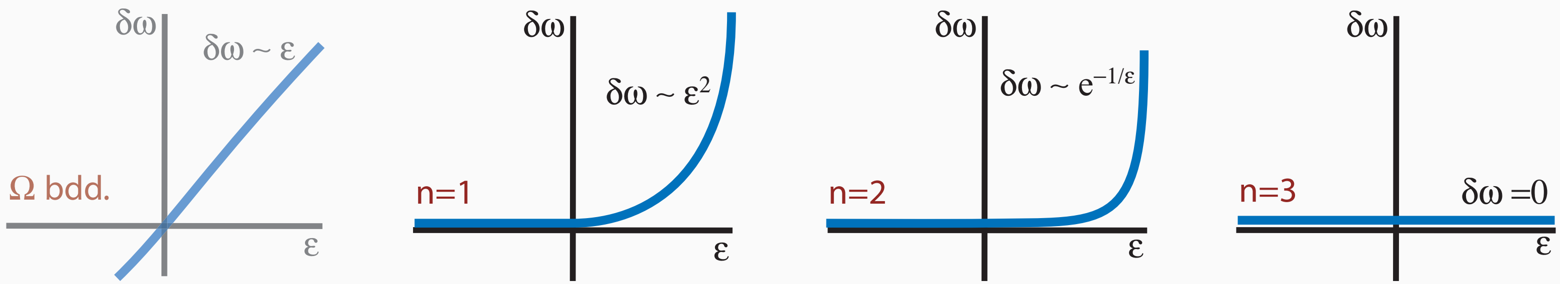
- **Φ is not Fredholm when posed on $L^2(\mathbb{R}^n)$**

Inhomogeneities: main results

$$U_t = D\Delta U + F(U) + \varepsilon G(|x|),\ x \in \mathbb{R}^n,\ |G(r)| \leq C(1+r)^{-2-\delta}$$

Theorem **[Kollár&Scheel]**

Assume minimal critical spectrum, normal dispersion, then



where

- $\delta\omega = \omega - \omega_*$
- **sources correspond to** $\delta\omega > 0$

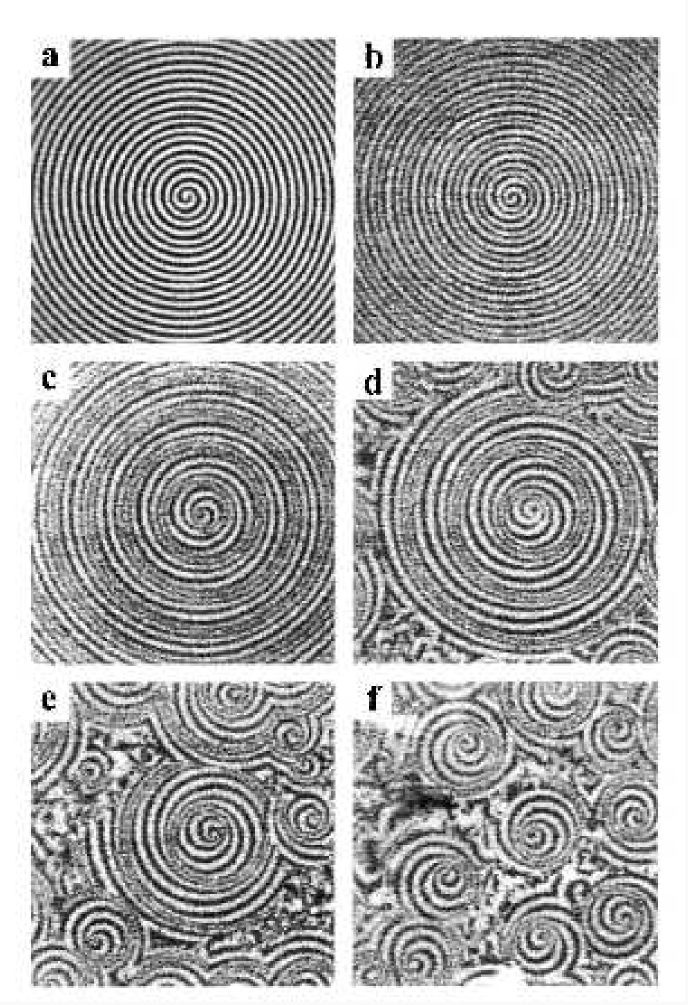
For anomalous dispersion, replace $\delta\omega \mapsto -\delta\omega$

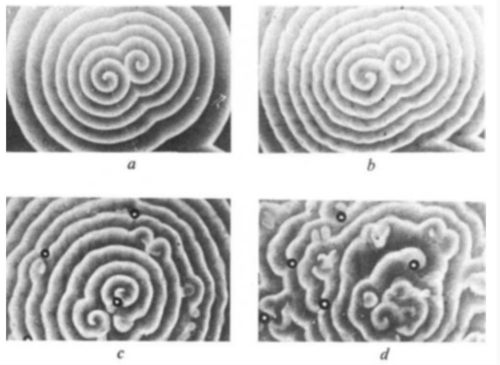
I Example: oscillations & small inhomogeneities

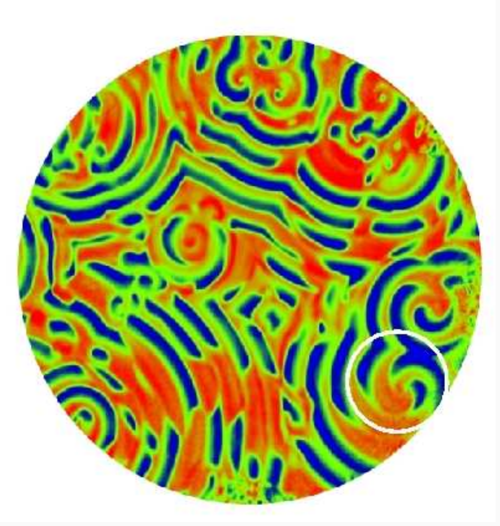
II **Example: instabilities of spiral waves**

Simple and complicated patterns

Patterns in the Belousov-Zhabotinsky reaction







[Park,Lee], [Zhou, Ouyang], [Agladze, Krinsky, Pertsov]

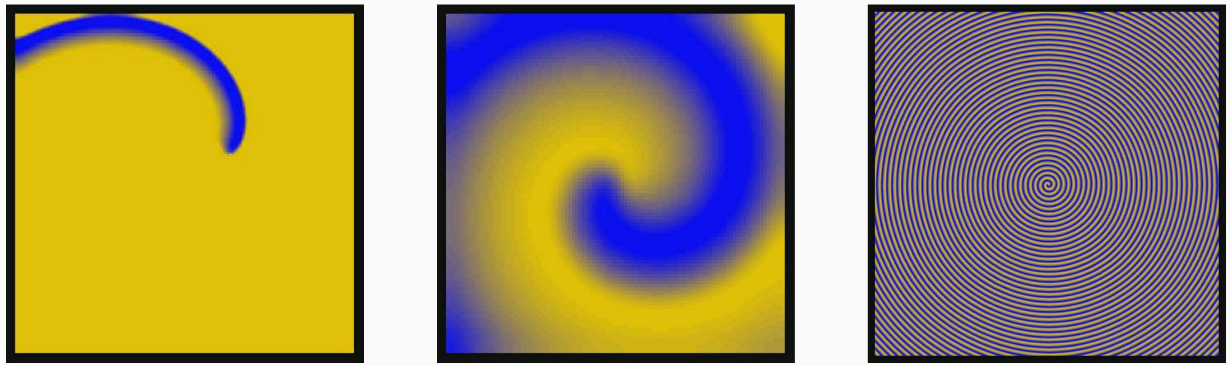
From simple to complicated patterns — spiral waves

$$\frac{\mathrm{d}}{\mathrm{d}t}u = D\Delta u + f(u), \quad u \in X = C^2(\mathbb{R}^n, \mathbb{R}^N)$$

Spirals

$\longleftrightarrow$

periodic orbits



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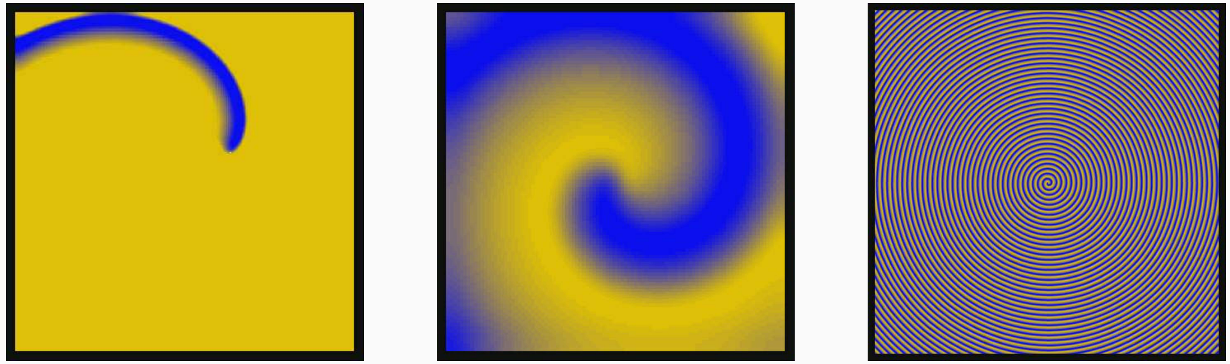
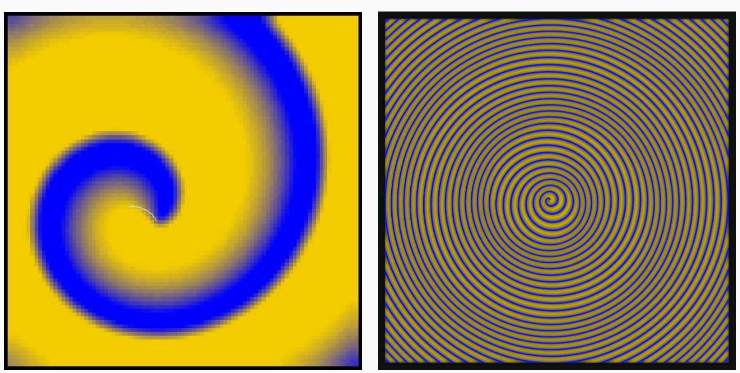


Two generic routes to chaos (ex. FHN, Roessler):

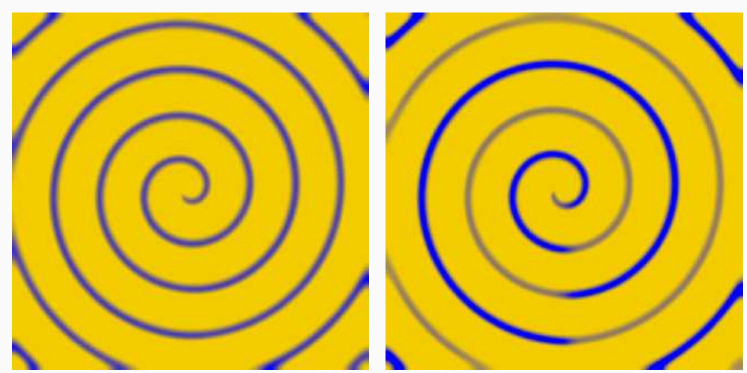
Hopf bifurcation

two frequencies



Period-doubling

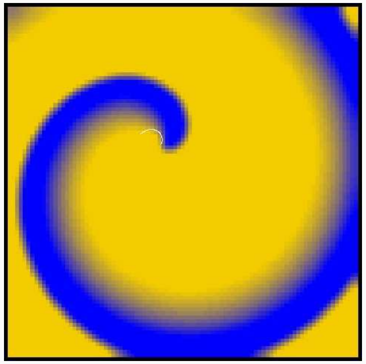
half frequency



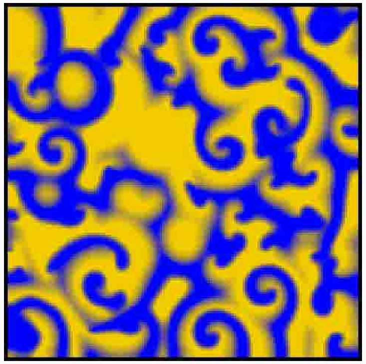
Bifurcations and spiral waves — oddities

More Hopf instabilities...

Drift



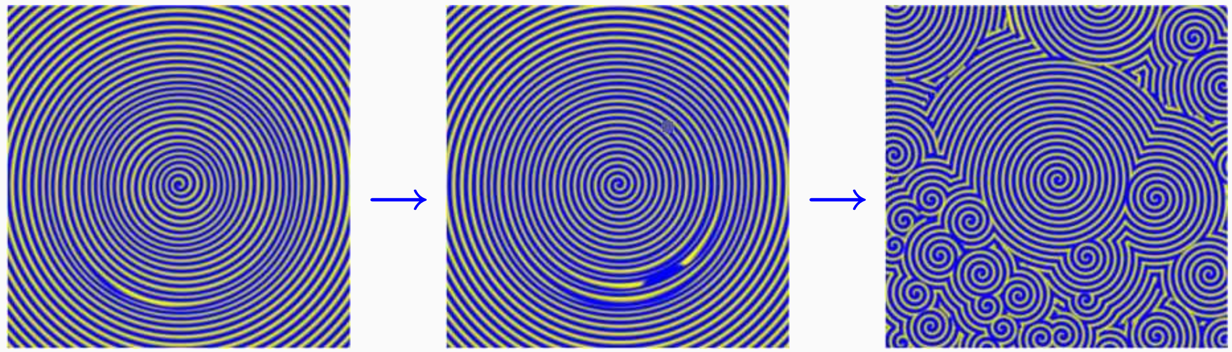
Breakup



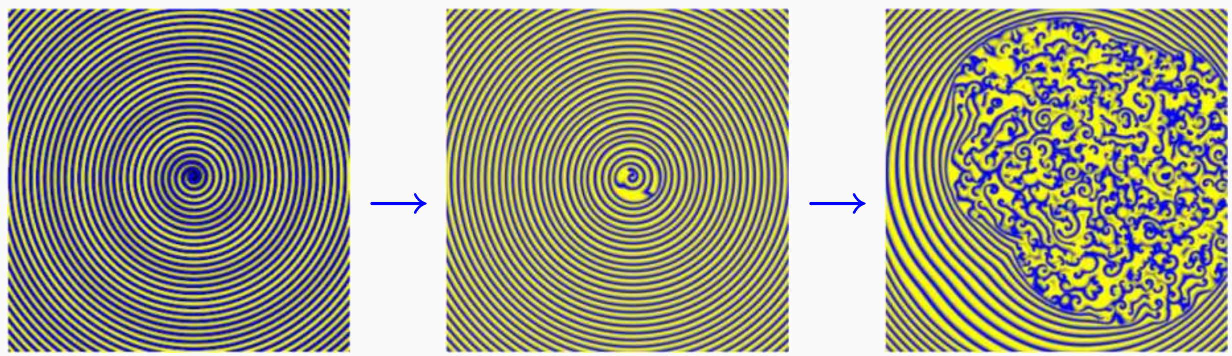
Break-up and routes to turbulence

Transitions to turbulence can be different. . .

Hopf-breakup I



Hopf-breakup II



Spectra of spirals [Barkley&Wheeler]