Patterns far from equilibrium

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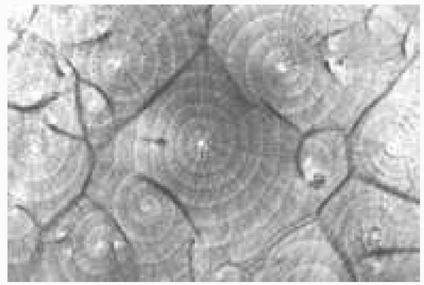

CAM Colloquium, University of St Thomas

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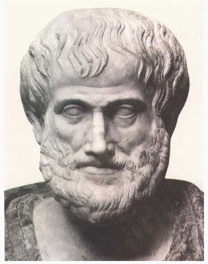
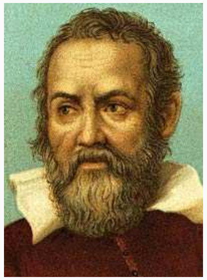
Equilibrium as the most natural state

Aristotle (384-322 BC):



For heavenly objects, natural motion is motion in a circle with the same speed. For base objects, natural motion is rest.

Galilei (1564-1642):



The natural state of motion is uniform motion.

Equilibrium statistics

Clausius (1822-1888):

The entropy of an isolated system not in equilibrium will tend to increase over time, approaching a maximum value at equilibrium.

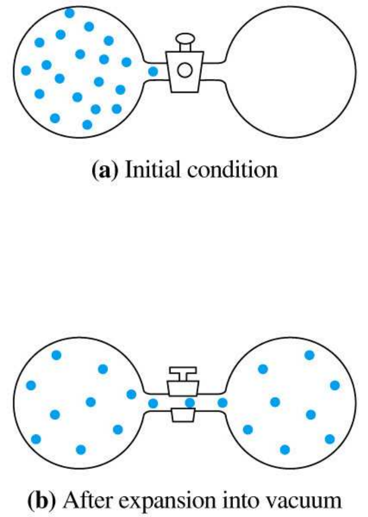
Belousov (1893-1970):

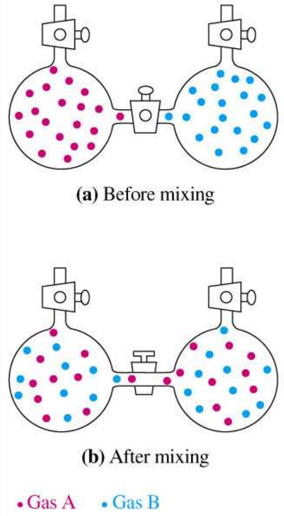
observed an *oscillating* chemical reaction.

Turing (1912-1954):

suggested that the simple interplay of diffusion and reaction is responsible for complicated biological patterns.

The most likely state





but. . .

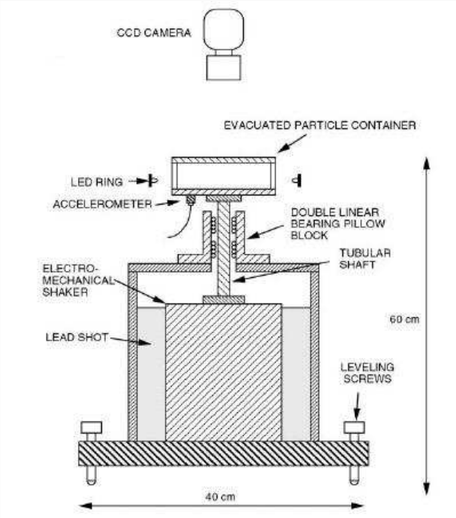
Initial Mixture of Uncooked Rice and Split Peas

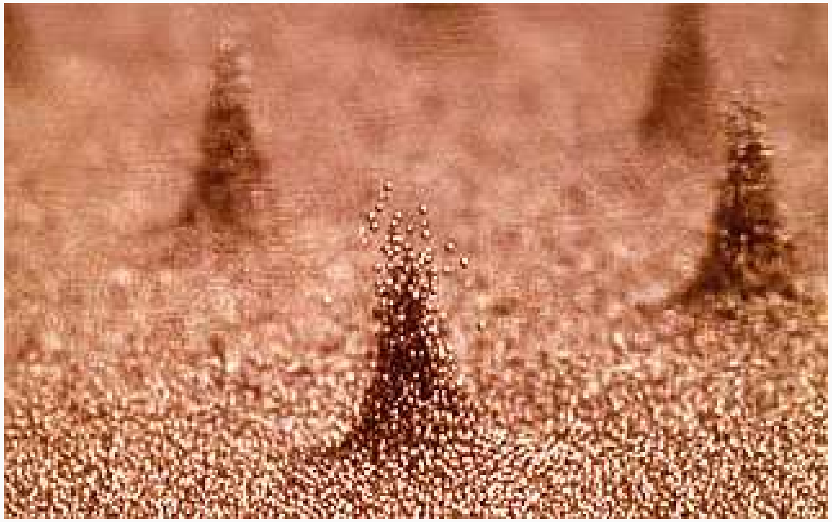
After Rotation About Horizontal Axis at 15 rpm for 2 hours

[James Kakalios], `groups.physics.umn.edu/sand/axial.shtml`

More unlikely things

somewhere in the desert

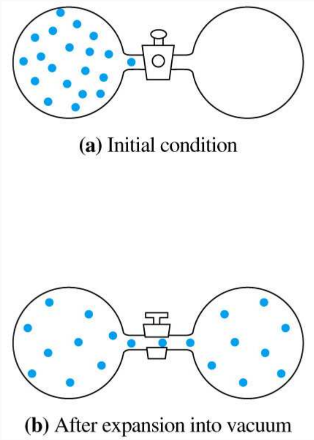




Faraday experiment ***chaos.ph.utexas.edu/research/granular.html***

— Watch the vibrated cornstarch movie —

Diffusion



Two containers, particles hop randomly between left and right u_1 and u_2 densities

$$\frac{\mathrm{d}}{\mathrm{d}t}u_1 = d(u_2 - u_1)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}u_2 = d(u_1 - u_2)$$

Of course we could look at more containers

$$\frac{\mathrm{d}}{\mathrm{d}t}u_i = d((u_{i+1} - u_i) - (u_i - u_{i-1})), \qquad i = 1, \ldots, m$$

and even a continuum of containers

$$\partial_t u(t,x) = d\partial_x^2 u(t,x), \qquad x \in \mathbb{R}$$

or arrays of containers

$$\partial_t u(t,x) = d\Delta u(t,x), \qquad x \in \mathbb{R}^n, \qquad \Delta = \partial_{x_1}^2 + \ldots + \partial_{x_n}^2$$

Diffusion and convergence to equilibrium

$$\frac{\mathrm{d}}{\mathrm{d}t}u_1 = d_u(u_2 - u_1) \qquad \qquad \frac{\mathrm{d}}{\mathrm{d}t}(u_1 + u_2) = 0$$
$$\Longrightarrow$$
$$\frac{\mathrm{d}}{\mathrm{d}t}u_2 = d_u(u_1 - u_2) \qquad \qquad \frac{\mathrm{d}}{\mathrm{d}t}(u_1 - u_2) = -2d_u(u_1 - u_2)$$

. . . so $u_1 - u_2 \sim \mathrm{e}^{-2d_u t} \longrightarrow 0$

More compact Matrix notation $U = (u_1, u_2)^T$,

$$\frac{\mathrm{d}}{\mathrm{d}t}U = D_u U, \quad D_u = \begin{pmatrix} -d_u & d_u \\ d_u & -d_u \end{pmatrix}$$

Eigenvalues λ of D_u are $\lambda = 0, -2d_u$, all negative

No Patterns!

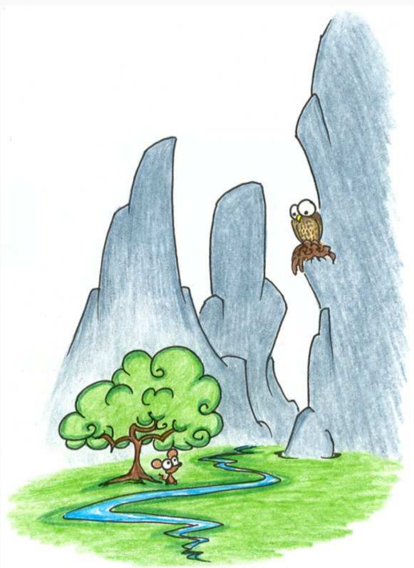
Reaction

Activator-inhibitor systems:

$$\left\{ \begin{array}{l} \frac{\mathrm{d}}{\mathrm{d}t}U = U - V \\ \frac{\mathrm{d}}{\mathrm{d}t}V = 8U - 5V \end{array} \right\} \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} U \\ V \end{pmatrix} = R\begin{pmatrix} U \\ V \end{pmatrix}, \quad R = \begin{pmatrix} 1 & -1 \\ 8 & -5 \end{pmatrix}$$

The eigenvalues of R are -1, -3, negative, so

$$(U, V) \sim a_{u/v}\mathrm{e}^{-t} + b_{u/v}\mathrm{e}^{-3t} \to 0 \quad \text{for } t \to \infty$$



Think mice (U) multiplying, owls (V) eating mice

. . . and they all die in the end

Reaction & Diffusion

Owls and mice, in Wisconsin and in Minnesota:

$$U = (U_W, U_M),\ V = (V_W, V_M)$$

Both react (feed) and diffuse (migrate)

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} U \\ V \end{pmatrix} = \mathcal{D}\begin{pmatrix} U \\ V \end{pmatrix} + \mathcal{R}\begin{pmatrix} U \\ V \end{pmatrix}, \quad \text{where}$$

$$\mathcal{D} = \begin{pmatrix} -d_u & d_u & 0 & 0 \\ d_u & -d_u & 0 & 0 \\ 0 & 0 & -d_v & d_v \\ 0 & 0 & d_v & -d_v \end{pmatrix}, \qquad \mathcal{R} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 8 & 0 & -5 & 0 \\ 0 & 8 & -0 & -5 \end{pmatrix}$$

We need the eigenvalues of $\mathcal{D} + \mathcal{R}$. We'd hope

$$\text{eigenvalues}\,(\mathcal{R}) + \text{eigenvalues}\,(\mathcal{D}) \stackrel{?}{=} \text{eigenvalues}\,(\mathcal{R} + \mathcal{D})$$

Turing patterns

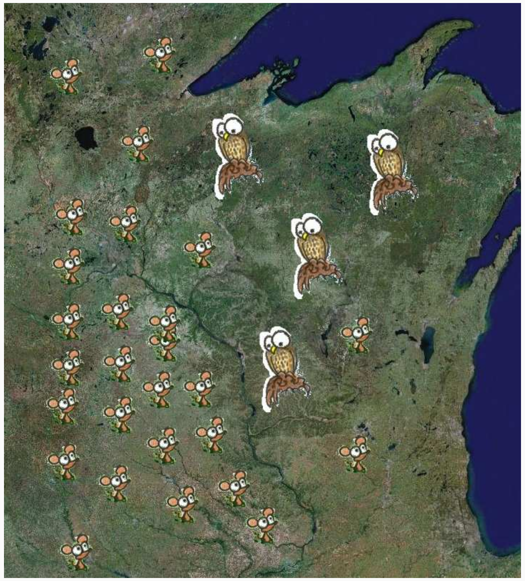
$$\textbf{eigenvalues}\,(\mathcal{R}) + \textbf{eigenvalues}\,(\mathcal{D}) \neq \textbf{eigenvalues}\,(\mathcal{R}+\mathcal{D})$$

. . . in most cases

In fact, Alan Turing observed that

$$\textbf{eigenvalue}\,(\mathcal{R}+\mathcal{D}) > 0 \text{ if } d_v \gg d_u$$

The sum of stable mechanisms creates instability and patterns!



Since owls cross the Mississippi more easily than mice, we actually do expect $d_v \gg d_u$ and many mice in MN

. . . or in WI

Oscillations

Back to reaction only:

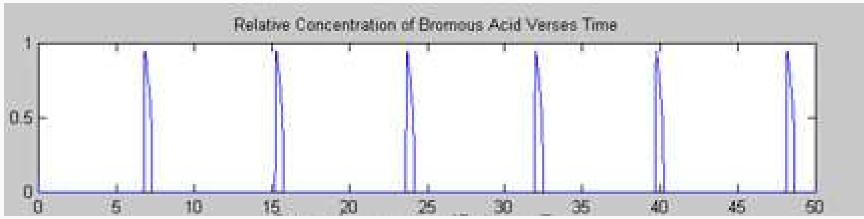
$$\frac{\mathrm{d}}{\mathrm{d}t}U = U - V$$
$$\frac{\mathrm{d}}{\mathrm{d}t}V = 8U + \mu V$$

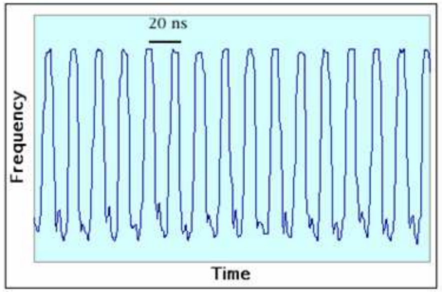
Computing eigenvalues shows

$$\mu < -1: \quad (U, V) \to 0$$
$$\mu = -1: \quad (U, V) \sim \sin(\omega t)$$
$$\mu > -1: \quad (U, V) \to \infty$$

Activator-Inhibitor systems can create oscillations!

chemical reactions (BZ, CIMA), gas discharges, semi conductors





Watch the BZ reaction oscillate —

Nonlinear activator-inhibitor

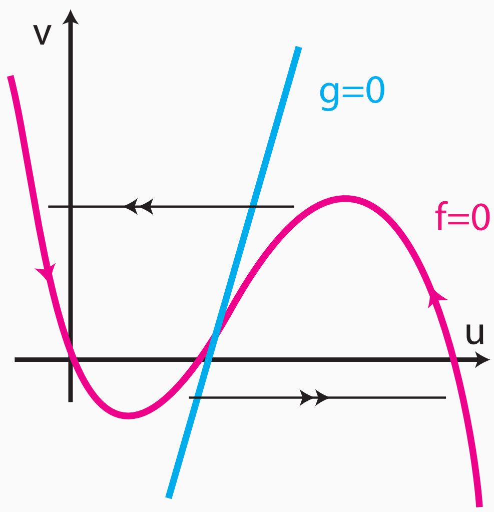
Rates do not depend linearly on concentrations

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}U &= f(U,V), \qquad \partial_U f > 0,\ \partial_V f < 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}V &= g(U,V), \qquad \partial_U g > 0,\ \partial_V g < 0
\end{aligned}$$

Ex: FitzHugh-Nagumo, $\mu \ll 1$

$$f = \frac{1}{\mu}[U(1-U)(U-a) - V]$$

$$g = U - \gamma V - \beta$$



— Nonlinear Oscillations —

Coupled oscillators

Two diffusively coupled oscillators $u_j = (U_j, V_j)$

$$\frac{\mathrm{d}}{\mathrm{d}t}u_1 = d(u_2 - u_1) + F(u_1)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}u_2 = d(u_1 - u_2) + F(u_2)$$

typically synchronize: $(u_1, u_2) \longrightarrow (u_*(\omega t), u_*(\omega t))$

Proof: **Weak coupling or being close to synchrony allows one to linearize**

$$\frac{\mathrm{d}}{\mathrm{d}t}u_1 = d(u_2 - u_1) + F'(u_*(\omega t))u_1$$
$$\frac{\mathrm{d}}{\mathrm{d}t}u_2 = d(u_1 - u_2) + F'(u_*(\omega t))u_2$$

and solutions are (Floquet theory)

$$u_1 + u_2 \sim u_{\mathrm{F}}(\omega t)\mathrm{e}^{\lambda t}, \quad u_1 - u_2 \sim u_{\mathrm{F}}(\omega t)\mathrm{e}^{(\lambda - 2d)t}$$

Now $\lambda \leq 0$ **since the single oscillator is stable, so** $u_1 - u_2 \longrightarrow 0$.

Synchronization and averaging

Varying parameters typically changes the frequency: we assume

$$u_*(\omega t) \quad \textbf{solves} \quad \frac{\mathrm{d}}{\mathrm{d}t}u = F(u;\omega)$$

This is a diffusively coupled family of m oscillators

$$\frac{\mathrm{d}}{\mathrm{d}t}u_j = d(u_{j+1} + u_{j-1} - 2u_j) + F(u_j;\omega_j), \qquad -m \leq j \leq m$$

Take all ω equal $\omega_j \equiv \omega_*$ for $j \neq 0$, then detune ω_0: $\omega_0 - \omega_* \sim 0$

Fact: Again all oscillators synchronize, but at which frequency?

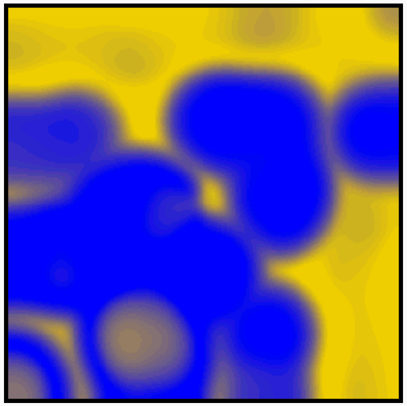
$m = 2$: $\quad \omega_{\mathrm{syn}} \sim \alpha\omega_0 + (1-\alpha)\omega_1$

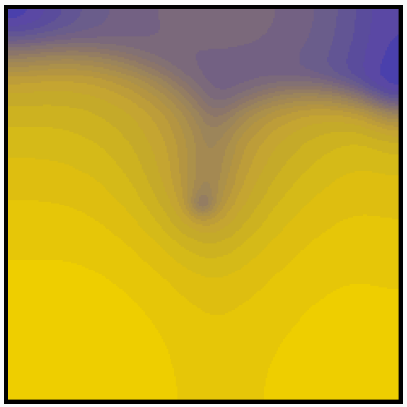
$m = 10^6$: $\quad \omega_{\mathrm{syn}} - \omega_* = \delta\omega \sim 10^{-6}$ **?**

$m = \infty$: $\quad \omega_{\mathrm{syn}} - \omega_* = \delta\omega = 0$ **?**

The many ways to reach consensus...

One's a little slow





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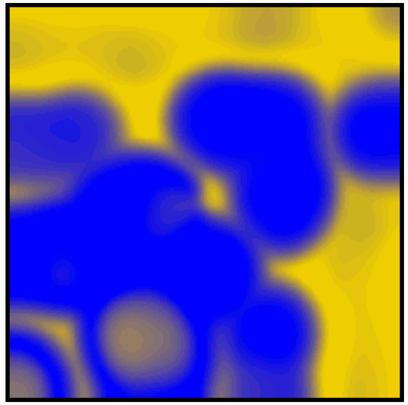
All equal



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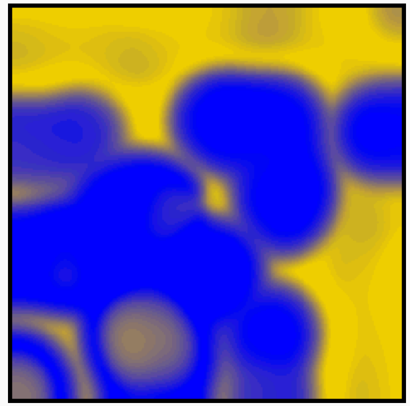
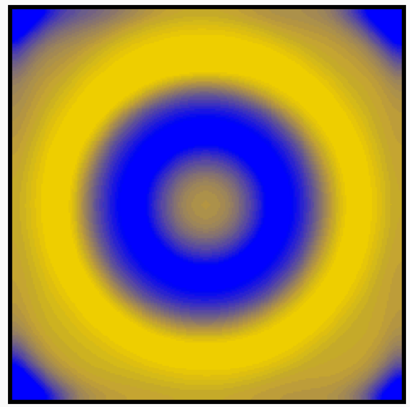
One's a little fast





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$$\partial_t u = \Delta u + \frac{1}{\mu} u(1-u)(u-a), \quad \partial_t v = \Delta v + u - v + b + \frac{\varepsilon}{1+|x/3|^2}$$

with $a = 0.34, b = -0.045, \mu = 0.08$ on $\Omega = \{|x_j| \leq 90\}$.

A Theorem

$$u_t = D\Delta u + F(u) + \varepsilon G(|x|),\ x \in \mathbb{R}^n,\ |G(r)| \le C(1+r)^{-2-\delta}$$

Define $M = \int_{\tau,x} u^{\mathrm{ad}}(\tau) \cdot G(|x|)$ **with** $u^{\mathrm{ad}}(\tau) \in \mathrm{Ker}\,\mathcal{L}_*^{\mathrm{ad}}$

Theorem **[Kóllar&Scheel]**

$n \le 2$: $M\varepsilon > 0$: **sources, with**

$$\delta\omega \sim (M\varepsilon)^{2/(2-n)},\ n < 2$$

$$\delta\omega \sim \exp(-1/(M\varepsilon)),\ n = 2$$

$M\varepsilon < 0$: **contact** ($\delta\omega = 0$)

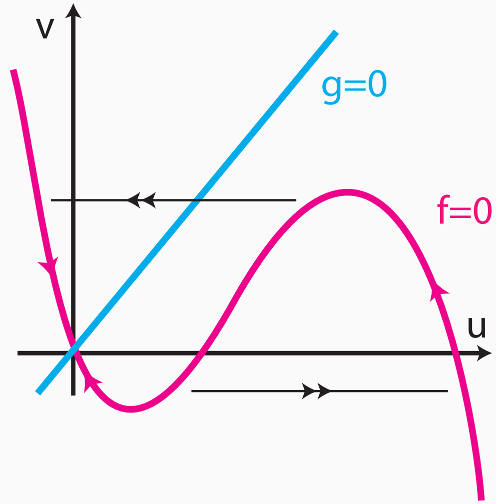
$n > 2$: $M\varepsilon > 0$: **contact** ($\delta\omega = 0$)

$M\varepsilon < 0$: **contact** ($\delta\omega = 0$)

$\longrightarrow$

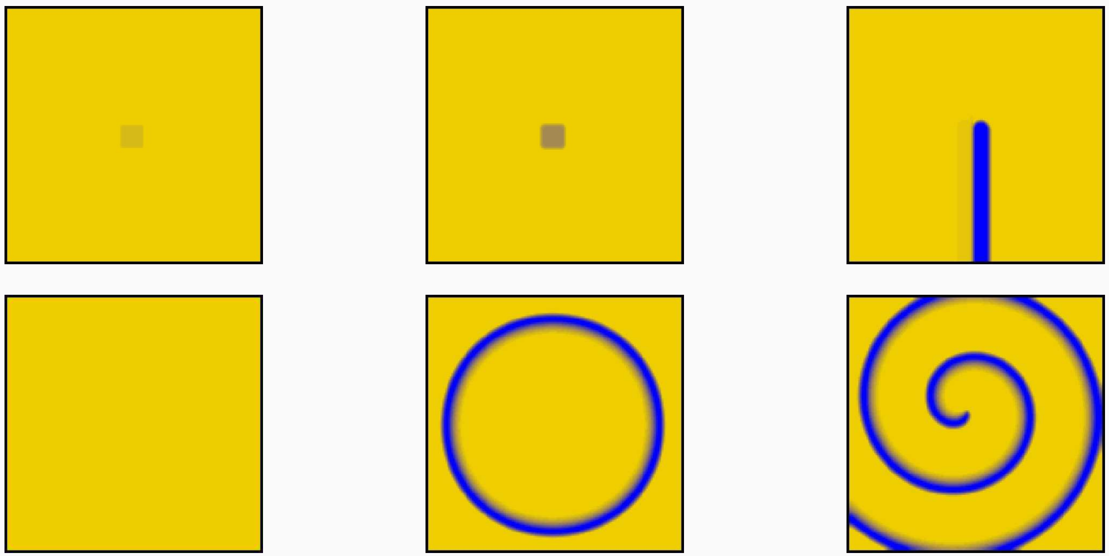
Excitable media

More activator-inhibitor dynamics



$$U_t = f(U, V)$$
$$V_t = g(U, V)$$

$$U_t = \Delta U + f(U, V)$$
$$V_t = g(U, V)$$

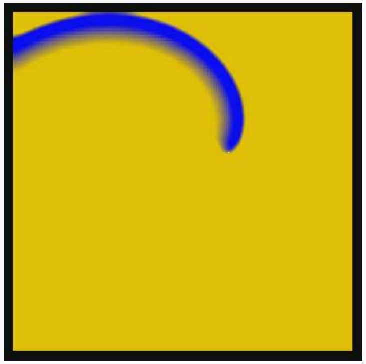
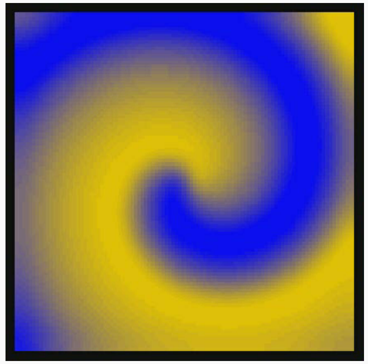
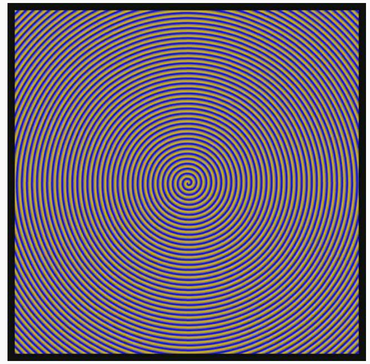


— movie — **— movie —** **— movie —**

From simple to complicated patterns — spiral waves

$$\frac{\mathrm{d}}{\mathrm{d}t}u = D\Delta u + f(u), \quad u \in X = C^2(\mathbb{R}^n, \mathbb{R}^N)$$

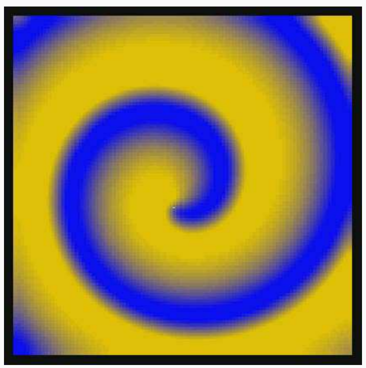
Spiral instabilities — click on pictures to play movies

Two routes to chaos (ex. FHN, Roessler):

Hopf bifurcation

two frequencies



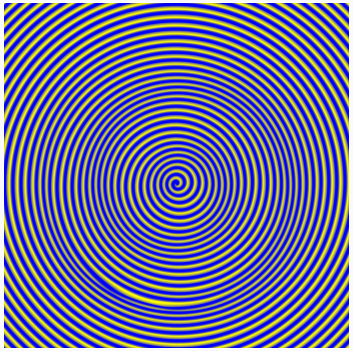
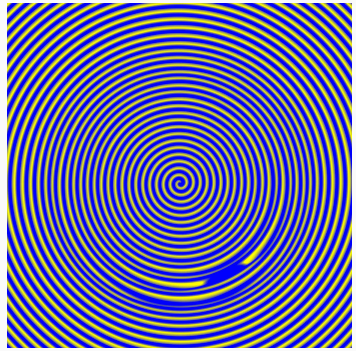

Period-doubling

half frequency

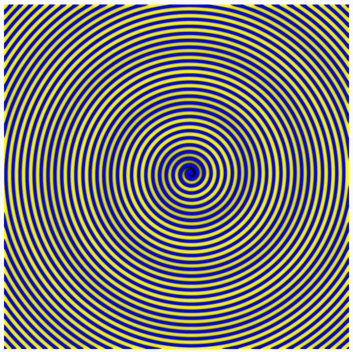
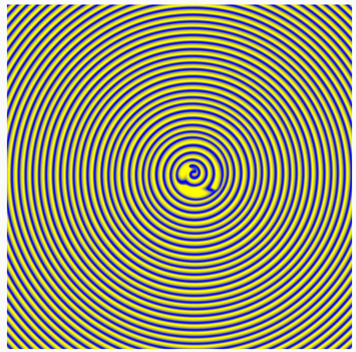
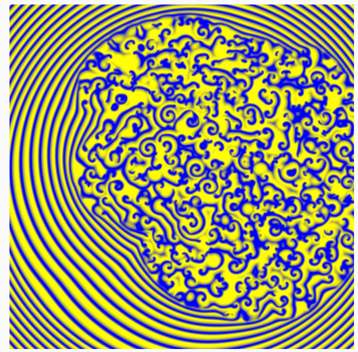

Break-up and routes to turbulence

Transitions to turbulence can be different. . .

Hopf-breakup I

 →  → 

Hopf-breakup II

 →  → 

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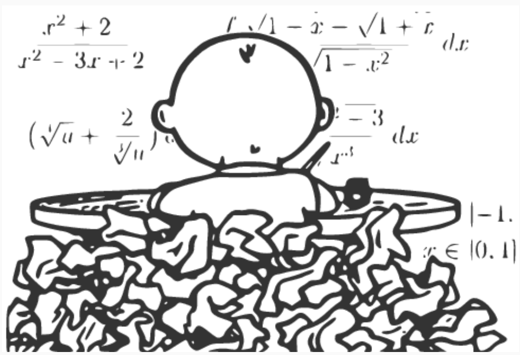
Summary

- **Our world is not in equilibrium**

. . . and this may well be a good thing

- **Things do not always add up**



. . . but you still need to know your math

Acknowledgments and references

- **Spiral dynamics joint work with**
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- **Math references**
  `www.math.umn.edu/~scheel`
- **Numerics based on EZSpiral**
  Dwight Barkley [Warwick]
- **Period-doubling of spirals based on the Roessler model**
  Ray Kapral [U Toronto]
- **Experiments on segregation**
  Steve Morris [U Toronto], Jim Kakalios [UMN]
- **Faraday experiments and corn starch**
  Harry Swinney's group [UT Austin]
- **Owls and mice**
  Noah [Kaukasus], who let them on the ark. . .