The impossible period-doubling of a spiral wave

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joint work with

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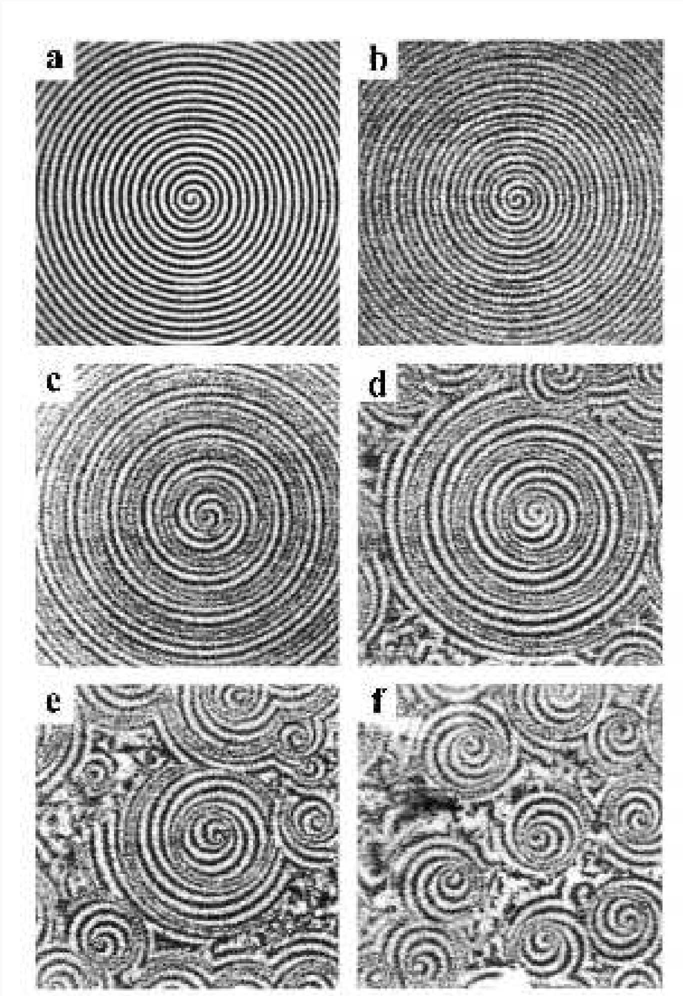
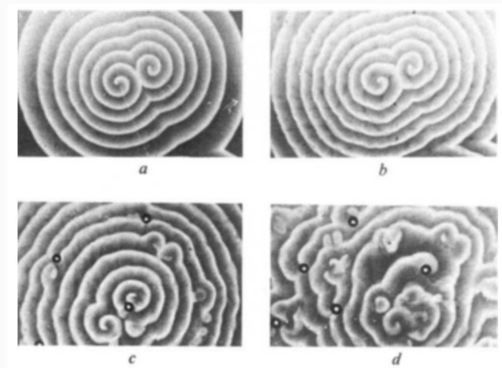
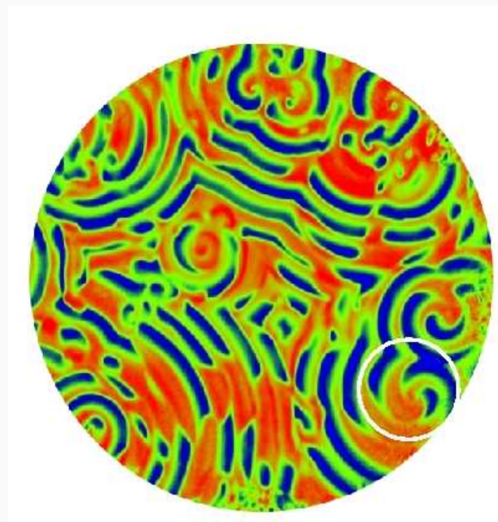
Simple and complicated patterns

Patterns in the Belousov-Zhabotinsky reaction







[Park,Lee], [Zhou, Ouyang], [Agladze, Krinsky, Pertsov]

From simple *to* complicated dynamics

Routes to chaos in dynamical systems

$$\frac{\mathrm{d}}{\mathrm{d}t}u = f(u;\mu), \quad u \in X = \mathbb{R}^N, \ \mu \in \mathbb{R}$$

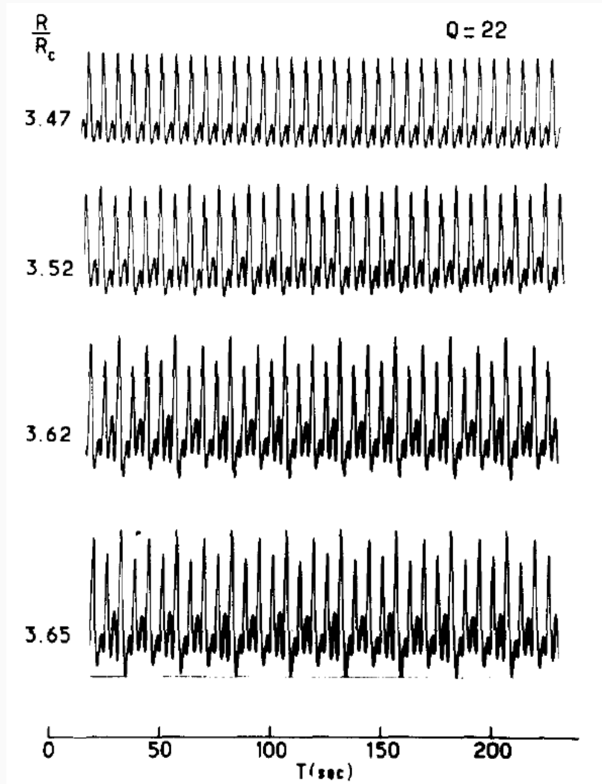
Hopf bifurcations

1 frequency	$\longrightarrow$	2 frequencies	$\longrightarrow$	3 frequencies
periodic orbit	$\longrightarrow$	two-torus	$\longrightarrow$	strange attractors

Occurrence of Strange Axiom A Attractors Near Quasi Periodic Flows on T^m, $m \geqq 3$

S. Newhouse[1], D. Ruelle[2*], and F. Takens[3*]

Period doubling cascades



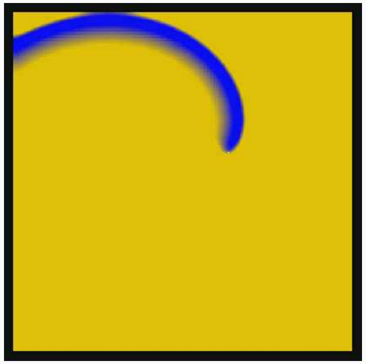
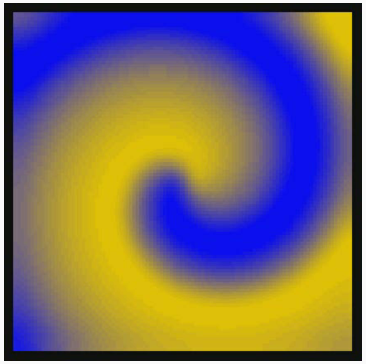
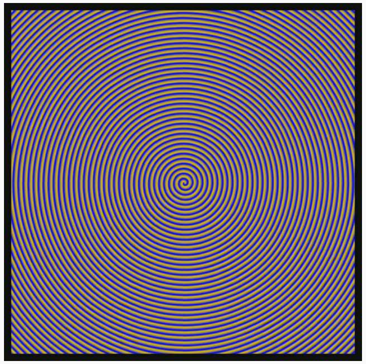
From simple to complicated patterns — spiral waves

$$\frac{\mathrm{d}}{\mathrm{d}t}u = D\Delta u + f(u), \quad u \in X = C^2(\mathbb{R}^n, \mathbb{R}^N)$$

Spirals

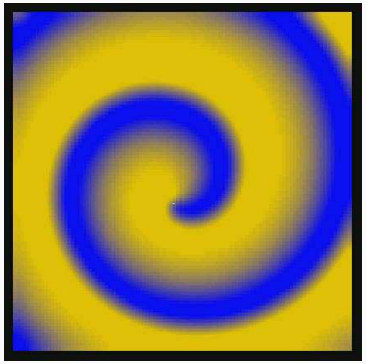
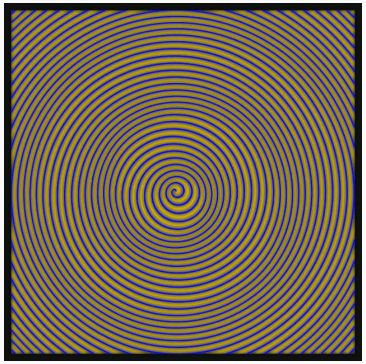
$\longleftrightarrow$

periodic orbits

Two generic instability mechanisms (ex. FHN, Roessler):

Hopf bifurcation
two frequencies

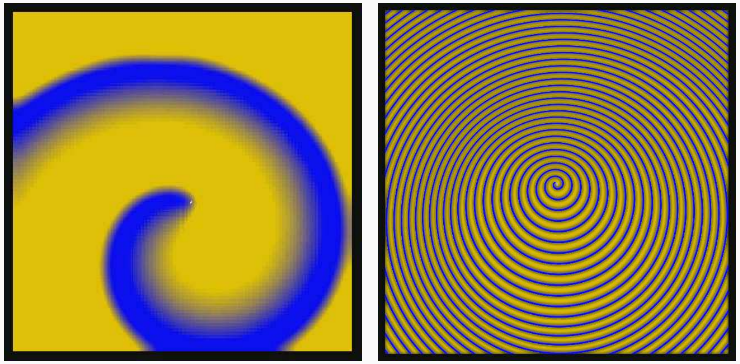
Period-doubling
half frequency


— click on images to play movies —

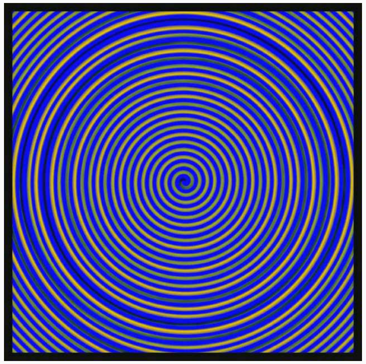
Bifurcations and spiral waves — oddities

More Hopf instabilities...

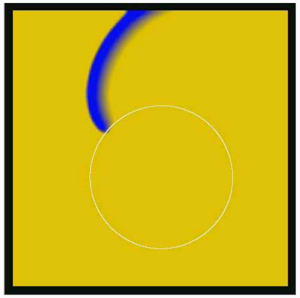
Drift



Breakup



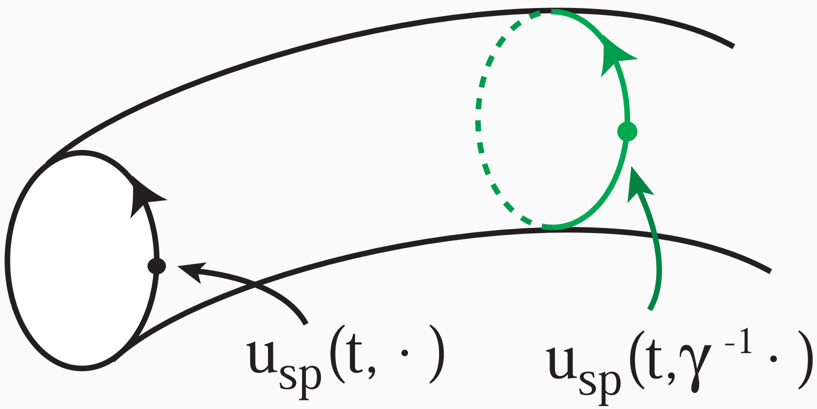
Euclidean symmetry



$\gamma \in SE(2)$, rotations and translations

$$\gamma = (\varphi, z) \in S^1 \times \mathbb{C} \qquad \gamma \cdot x = \mathrm{e}^{\mathrm{i}\varphi}(x + z) \in \mathbb{R}^2 \sim \mathbb{C}$$

$u(t,x)$solution $\iff$ $u(t,\gamma^{-1}x)$ solution

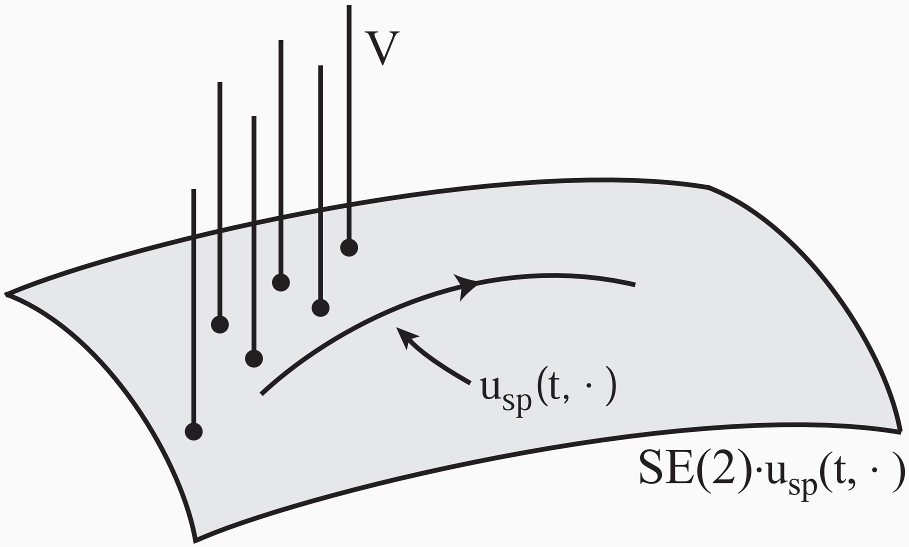


Spirals are relative equilibria

$$u_{\mathrm{sp}}(t,x) = u_{\mathrm{sp}}(0,\gamma^{-1}(t)x) \quad \gamma(t) = \exp(\omega_{\mathrm{sp}} t \partial_\varphi)$$

Bifurcations from Relative Equilibria

[Barkley], [Sandstede,AS,Wulff],[Fiedler, Sandstede, AS, Wulff], [Golubitsky, LeBlanc, Melbourne]



Reduction to principal fiber bundle

$$(\gamma, v) \in SE(2) \times V$$

$$\dot{\gamma} = \gamma \; a(v) \qquad \text{“}group\text{”}$$

$$\dot{v} = h(v) \qquad \text{“}shape\text{”}$$

Resonances and drift

$$\begin{aligned} \dot{\varphi} &= \omega_{\mathrm{sp}} && \textbf{rotation} \\ \dot{z} &= \mathrm{e}^{\mathrm{i}\varphi} v && \textbf{translation} \\ \dot{v} &= h(v) && \textbf{Hopf} \end{aligned}$$

Periodic Orbit $v(t) = \sum_k v_k \mathrm{e}^{-\mathrm{i}k\omega_{\mathrm{H}}t}$

Position $\dot{z} = \sum_k v_k \mathrm{e}^{\mathrm{i}(\omega_{\mathrm{sp}} - k\omega_{\mathrm{H}})t}$

Unbounded motion if $\omega_{\mathrm{H}} = \omega_{\mathrm{sp}}/k$ for some $k \in \mathbb{Z}$

The paradox

Spirals are relative *equilibria* $\Longrightarrow$ period-doubling is non-generic

...yet is *is* observed

More precisely...

Linearizing at a spiral wave $u_{\mathrm{sp}}(t,x)$ we find

Linearized period map: $\partial_u \Phi_{2\pi/\omega_{\mathrm{sp}}}$, where

$u(t) = \Phi_t(u(0))$ is the flow map

Linearization in corotating frame: $\mathcal{L} = D\Delta + \omega_{\mathrm{sp}}\partial_\varphi + f'(u_{\mathrm{sp}})$

Since spirals are equilibria,

$$\partial_u \Phi_{2\pi/\omega_{\mathrm{sp}}} = \mathrm{e}^{\mathcal{L}(2\pi/\omega_{\mathrm{sp}})}$$

The doubling eigenvalue -1 cannot be simple!

...but we would expect multiple eigenvalues to split generically.

An explanation with caveats

$\lambda = -1$ is double eigenvalue of $\partial_u \Phi$ since

$\alpha = \pm \mathrm{i}\omega_{\mathrm{sp}}/2$ are eigenvalues of $\mathcal{L}_*$

Problems:

- ***Genericity:*** **Why is the Hopf frequency in exact resonance?**
- ***Drift:*** **If there is an eigenvalue at $\lambda = \mathrm{i}\omega_{\mathrm{sp}}/2$, we expect drift!**

Instabilities — linearization

Reaction-diffusion system

$$\partial_t U = D \triangle U + F(U; \mu)$$

Spiral waves as rotating waves

$$U(t, x) = U_{\mathrm{sp}}(r, \varphi - \omega_{\mathrm{sp}} t)$$

Linearization in corotating frame

$$\mathcal{L}_{\mathrm{sp}} U = D \triangle U + F'(U_{\mathrm{sp}}; \mu) U + \omega_{\mathrm{sp}} \partial_\psi U$$

Stability: $\mathrm{Re}\,\mathrm{spec}\,\mathcal{L} \leq 0$

Eigenvalues enforced by symmetry

- $\lambda = 0$ — rotation
- $\lambda = \pm \mathrm{i}\omega$ — translation

Unbounded domains — the essential spectrum

Decompose the spectrum into continuous and discrete part:

$\mathrm{spec}\,\mathcal{L}_{\mathrm{sp}}$: $\mathcal{L}_{\mathrm{sp}} - \lambda$ not invertible

$\mathrm{spec}_{\mathrm{ess}}\,\mathcal{L}_{\mathrm{sp}}$: $\mathcal{L}_{\mathrm{sp}} - \lambda$ not Fredholm of index 0

$\mathrm{spec}_{\mathrm{pt}}\,\mathcal{L}_{\mathrm{sp}}$: $\mathcal{L}_{\mathrm{sp}} - \lambda$ Fredholm of index 0, not invertible

Localized changes of the spiral shape are compact perturbation of $\mathcal{L}_{\mathrm{sp}}$, and therefore leave $\mathrm{spec}_{\mathrm{ess}}\,\mathcal{L}_{\mathrm{sp}}$ unchanged

$$\Longrightarrow$$

Essential spectrum $\sim$ behavior in the far field

Point spectrum $\sim$ behavior in the core

Spectra of spiral waves

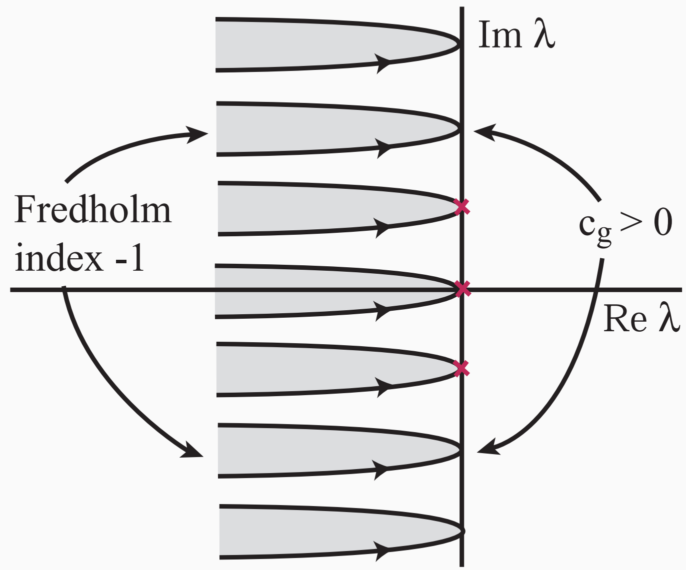
Spiral waves converge to wave trains

$$U_{\mathrm{sp}}(r,\varphi-\omega_{\mathrm{sp}}t)\sim U_{\mathrm{wt}}(kr+\varphi-\omega_{\mathrm{sp}}t)) \text{ for } r\to\infty,$$

they are *asymptotically Archimedean*

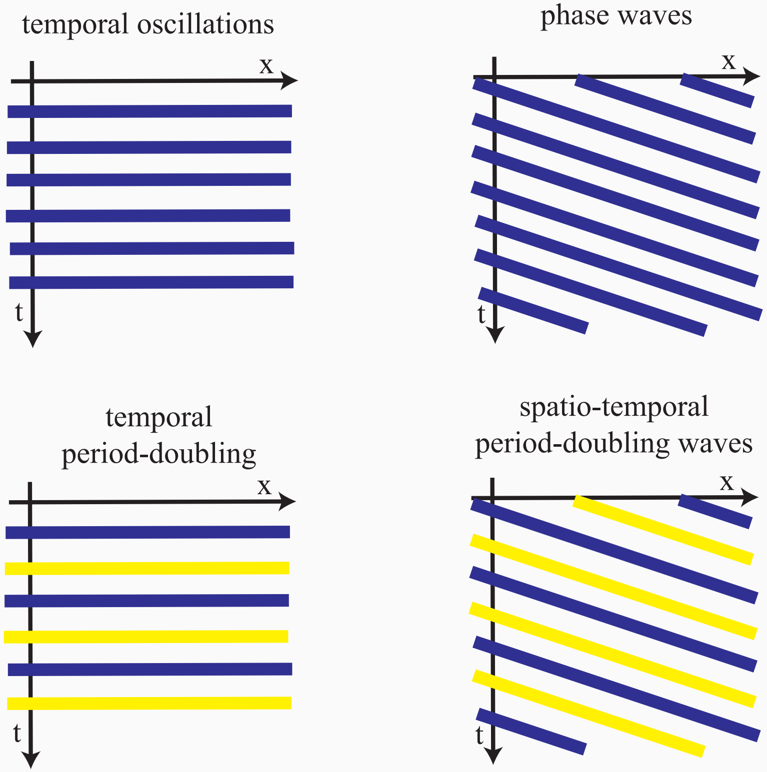
Theorem [Sandstede& AS]

The essential spectrum of $\mathcal{L}_{\mathrm{sp}}$ is given by the Floquet spectrum of the wave trains.

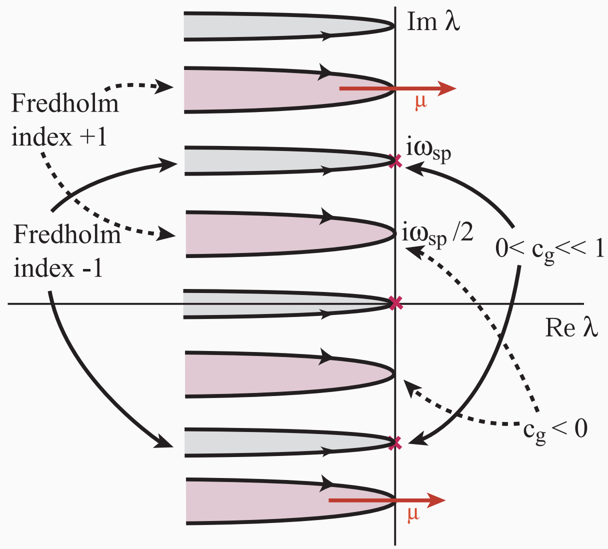


Spectra of wave trains

Instabilities of wavetrains close to homogeneous period-doubling



$\mathrm{spec}_{\mathrm{ess}}\mathcal{L}_{\mathrm{sp}}$ at period-doubling



Maximum at $\mathrm{i}\omega_{\mathrm{sp}}/2$ is robust:

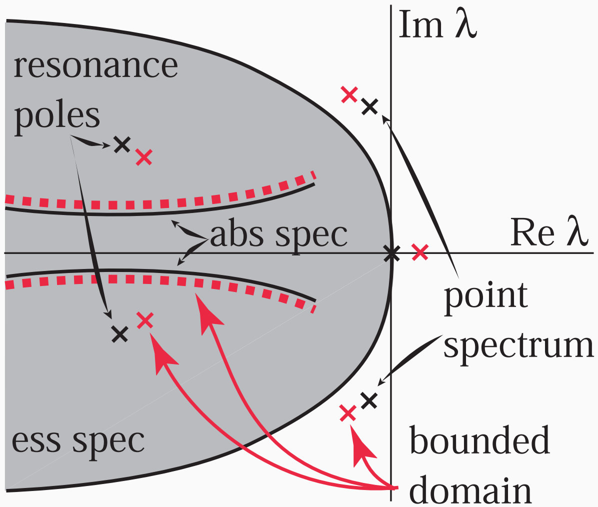
Floquet theory: period-doubling of wave trains is robust $\sim$ spatio-temporal symmetry breaking

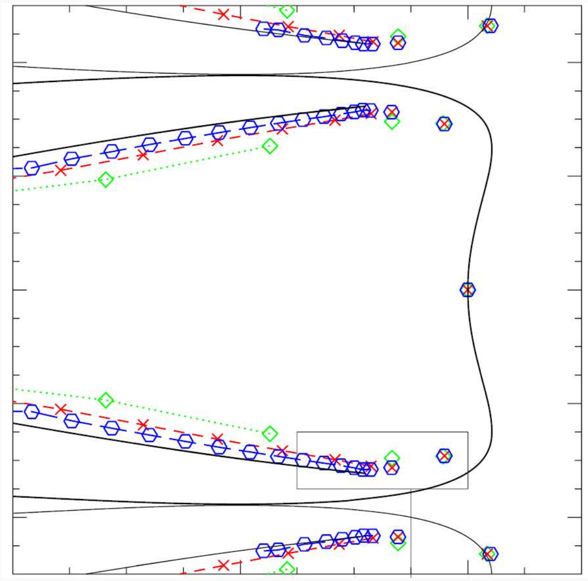
$$\left.\begin{array}{lcl} \lambda & \mapsto & \bar{\lambda} \\ \lambda & \mapsto & \lambda + \mathrm{i}\omega_{\mathrm{sp}} \end{array}\right\} \text{fix max}$$

Large domains [Sandstede,AS]

In a large disc $|x| \leq R$, with "compatible" boundary conditions

$$\mathbf{spec}_{|x|\leq R}\mathcal{L}_{\mathrm{sp}} \xrightarrow{R\to\infty} \mathbf{spec}_{\mathbf{abs}}\mathcal{L}_{\mathrm{sp}} \cup \mathbf{spec}_{\mathbf{expt}}\mathcal{L}_{\mathrm{sp}} \cup \mathbf{spec}_{\mathbf{bdy}}\mathcal{L}_{\mathrm{sp}}$$





[Barkley, Wheeler]

Absolute spectra are determined by wave trains only $\Longrightarrow$ **Robust "absolute" period-doubling in large domains**

A first summary

- **spirals resemble wave trains in the far field**
- **wave trains possess an additional translational symmetry**
- **period-doubling is symmetry-breaking of wave trains**
- **rigorous decomposition on the linearized level:**

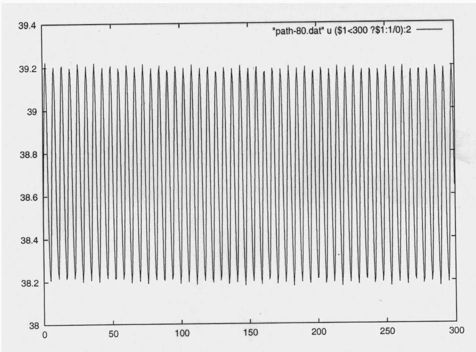
spiral core	$\longleftrightarrow$	**point spectrum**
far field	$\longleftrightarrow$	**essential and absolute spectra**

- **In unbounded and large bounded domains, period-doubling is typical when caused by essential or absolute spectrum.**
- ***Eigenfunctions predict a stationary line defect***
- ***In the Roessler system, the instability appears to be caused by boundary spectrum which happens to be resonant for a similar reason . . .***

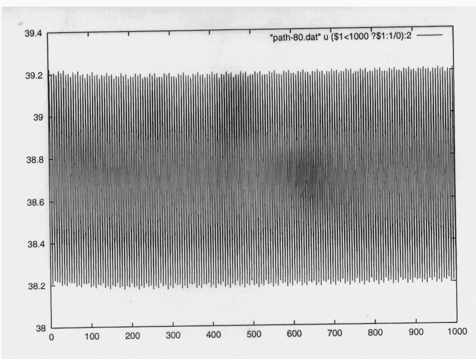
Drift?

In a fixed bounded domain, the instability caused by the first eigenvalue is a resonant Hopf bifurcation, so we predict drift:

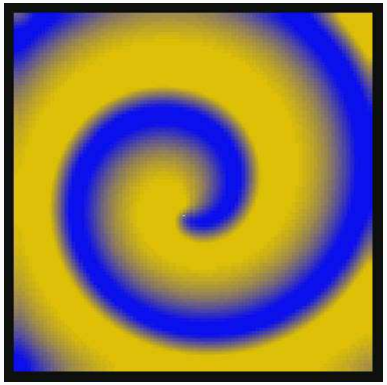
We plot the position of the spiral tip and wait. . .



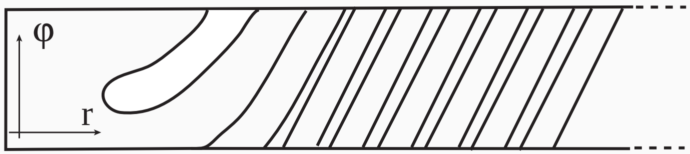
and wait some more. . .



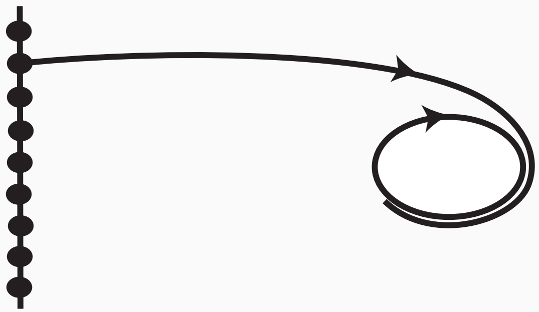
Proofs: spatial dynamics $\leftrightarrow$ functional analysis



$$\partial_t u = D\Delta u + f(u)$$

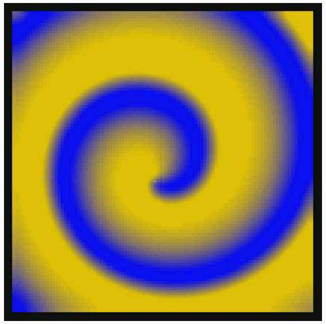
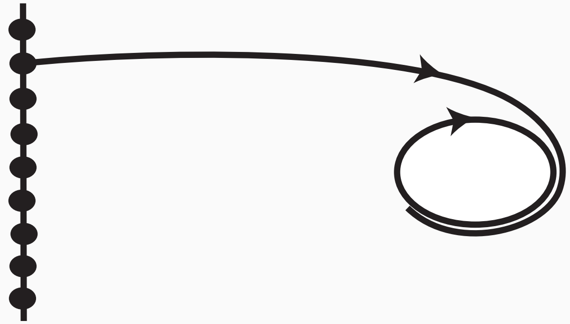


$$0 = \omega u_\varphi + D(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi}) + f(u)$$



$$\begin{aligned} u' &= v \\ v' &= -(\frac{1}{r}v + \frac{1}{r^2}\partial_{\varphi\varphi}u) \\ &\quad -D^{-1}(\omega\partial_\varphi u + f(u)) \\ r' &= 1 \end{aligned}$$

The spatial dynamics dictionary

	$\longleftrightarrow$	
$-\omega u_\varphi = D\Delta u + f(u)$	$\longleftrightarrow$	$U_r = F(\partial_\varphi, U, r)$
$\lambda v = D\Delta v + f'(u_{\mathrm{sp}})v$	$\longleftrightarrow$	$V_r = A(\partial_\varphi, u_{\mathrm{sp}}, r, \lambda)V$
spiral wave	$\longleftrightarrow$	heteroclinic orbit
linearization	$\longleftrightarrow$	linear bundle
Fredhom properties	$\longleftrightarrow$	hyperbolicity
eigenfunctions	$\longleftrightarrow$	heteroclinic orbits
point spectrum instability	$\longleftrightarrow$	non-transversality
essential spectrum instability	$\longleftrightarrow$	bif' of periodic orbits at ∞
$\ldots$	$\longleftrightarrow$	$\ldots$

Summary: Bifurcations in large domains

- **coherent structures: localized effects versus far field**
- **linear theory: point spectra versus essential and absolute spectra**
- **period-doubling is a robust wave train doubling in the far field**
- ***nonlinear theory more generally?***
- ***explain slow drift!***

Things are not always what they seem to be —

but aren't they pretty?

Acknowledgements and references

- **Pictures from BZ-reaction [Park,Lee], [Zhou, Ouyang], [Agladze, Krinsky, Pertsov]**
- **Period doubling model and experiments**
  - **Yoneyama, Fuji, Maeda, *J. Amer. Chem. Soc.* 1995**
  - **Goryachev, Chaté, Kapral, *Phys. Rev. Lett.* 1998**
  - **Park, Lee, *Phys. Rev. Lett.* 1999, 2002**
- **PD cascades from [A. Libchaber, S. Fauve and C. Laroche, 1983]**
- **Simulations based on EZSpiral [Barkley] and the Roessler model [Kapral]**
- ***Period-doubling of spiral waves and defects*, [Sandstede,AS] on `www.math.umn.edu/~scheel`**