Coherent Structures in Spatially Extended Systems

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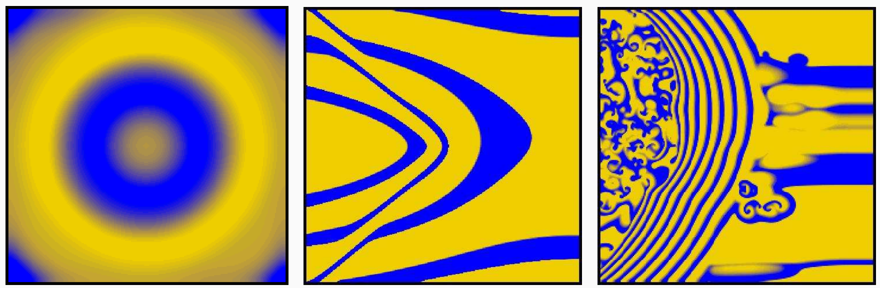

collaborators

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Research supported by NSF

Coherent structures — why do we care?

I Example: oscillations & small inhomogeneities

II Example: inhomogeneities & small oscillations

III Physics of coherent structures

IV Spatial dynamics of coherent structures

I **Example: oscillations & small inhomogeneities**

II **Example: inhomogeneities & small oscillations**

III **Physics of coherent structures**

IV **Spatial dynamics of coherent structures**

Oscillatory media

Oscillatory reaction

$$U_t = F(U) \in \mathbb{R}^N, \qquad U_*(-\omega_* t) = U_*(-\omega_* t + 2\pi)$$

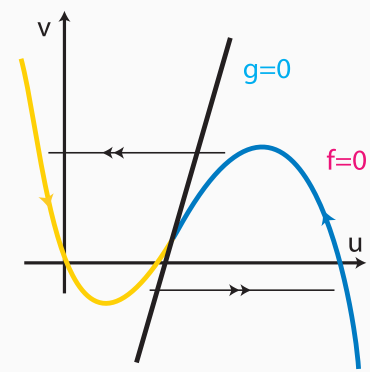
Diffusion

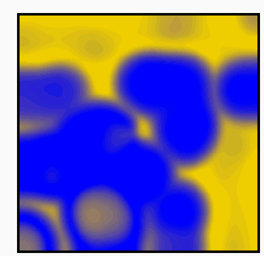
$$U_t = D\Delta U + F(U), \qquad x \in \Omega \subseteq \mathbb{R}^n,\ D = \operatorname{diag} d_j > 0,$$ **Neumann b.c.**

Example: FitzHugh-Nagumo

$$u_t = \Delta u + \frac{1}{\mu}[u(1-u)(u-a) - v]$$

$$v_t = \Delta v + u - \gamma v + b$$





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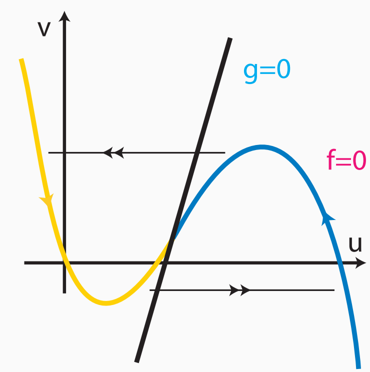
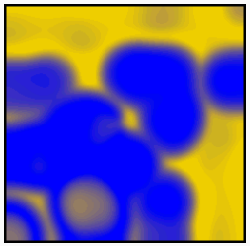
Example: FitzHugh-Nagumo

$$u_t = \Delta u + \frac{1}{\mu}[u(1-u)(u-a) - v]$$

$$v_t = \Delta v + u - \gamma v + b + \varepsilon$$



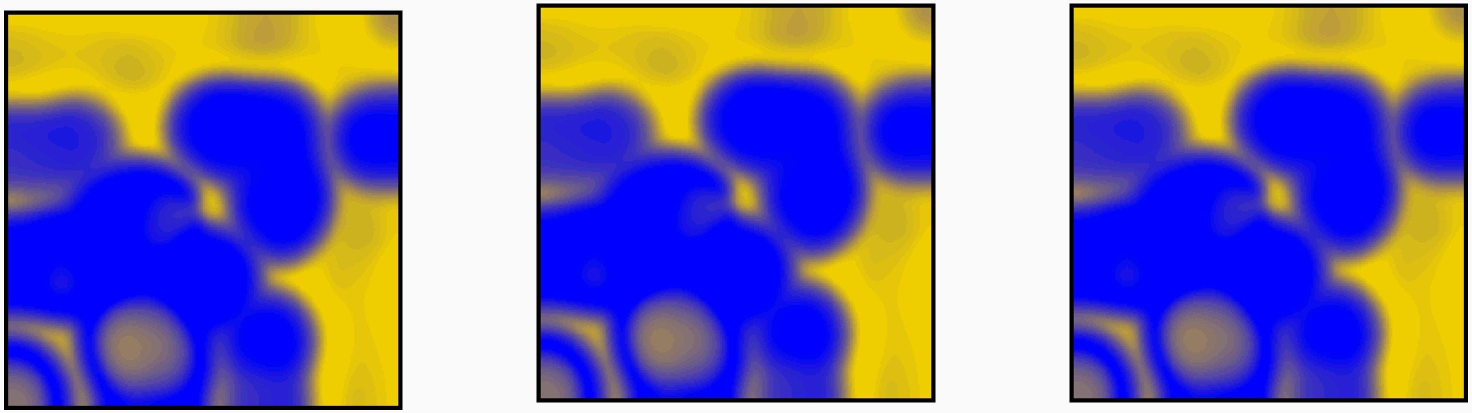


Periodic orbits are robust

size of perturbation $\ll$ spectral Floquet gap

Robustness — inhomogeneities

$\varepsilon = -0.05$ $\quad$ $\varepsilon = 0$ $\quad$ $\varepsilon = 0.05$



$$u_t = \Delta u + \frac{1}{\mu} u(1-u)(u-a)$$

$$v_t = \Delta v + u - v + b + \frac{\varepsilon}{1 + |x/3|^2}$$

with $a = 0.34, b = -0.045, \mu = 0.08$ on $\Omega = \{|x_j| \leq 90\}$

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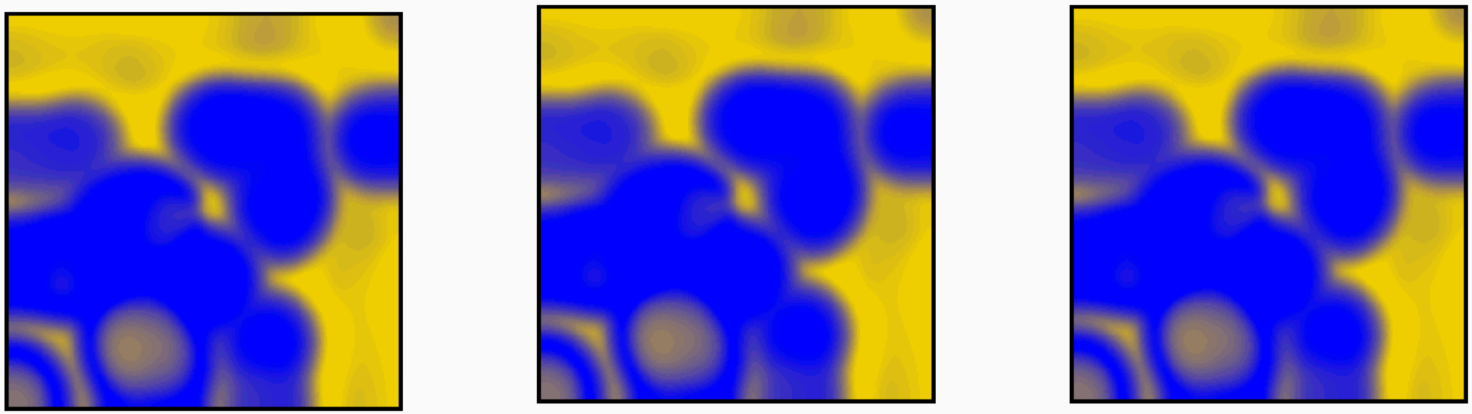

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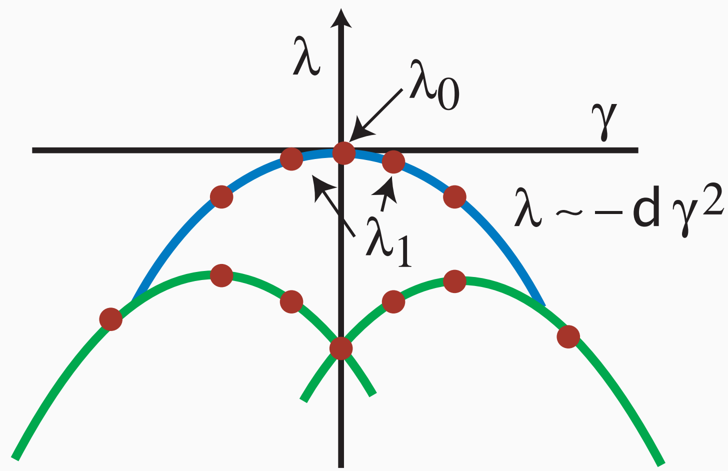
Coherent structures — what makes it difficult?

! Robustness analysis of the linearized the period map Φ !

Small spectral gaps in $|x| \leq L$

Floquet spectrum of Φ clusters near $\lambda = 0$:

$$\lambda_0 = 0 > \lambda_1 \sim -\frac{d}{4L^2} \geq \dots$$



In our example $L = 90$, and

$$\lambda_1 \sim 3 \times 10^{-5}$$

Fredholm boundaries in $\mathbb{R}^n$

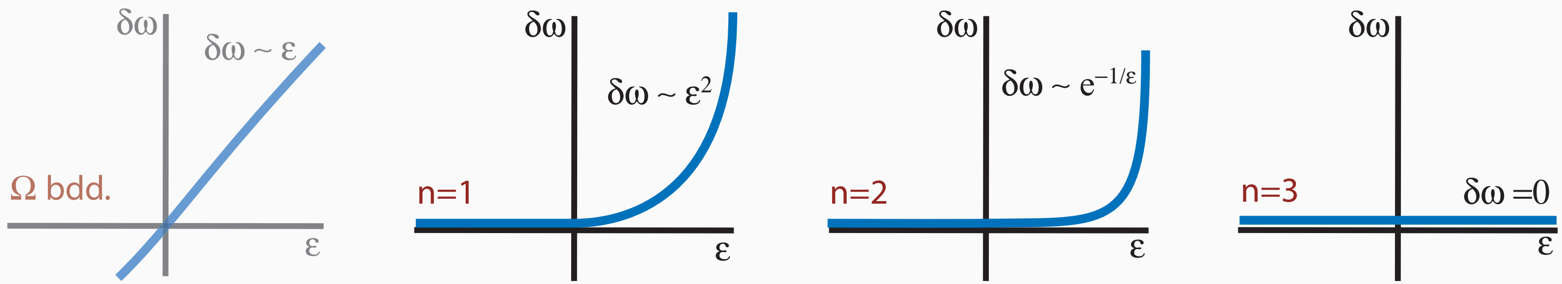
- **Φ is not Fredholm when posed on $L^2(\mathbb{R}^n)$**
- **Φ is Fredholm with index -1 when posed on $L^2_\eta(\mathbb{R}^n)$,**
  $0 < \eta \ll 1$,
  where $L^2_\eta = \{U;\ \mathrm{e}^{\eta|x|}U(x) \in L^2\}$, U radially symmetric

Inhomogeneities: main results

$$U_t = D\Delta U + F(U) + \varepsilon G(|x|),\ x \in \mathbb{R}^n,\ |G(r)| \leq C(1+r)^{-2-\delta}$$

Theorem **[Kollár&Scheel]**

Assume minimal critical spectrum, normal dispersion, then



where

- $\delta\omega = \omega - \omega_*$
- **sources correspond to** $\delta\omega > 0$

For anomalous dispersion, replace $\delta\omega \mapsto -\delta\omega$

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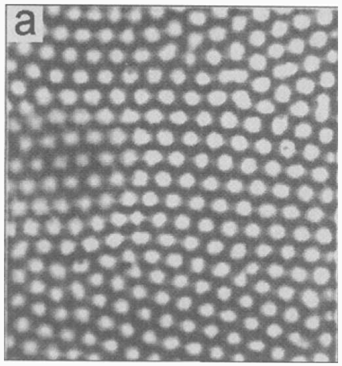
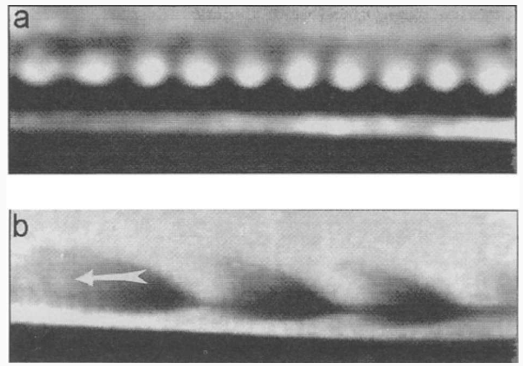
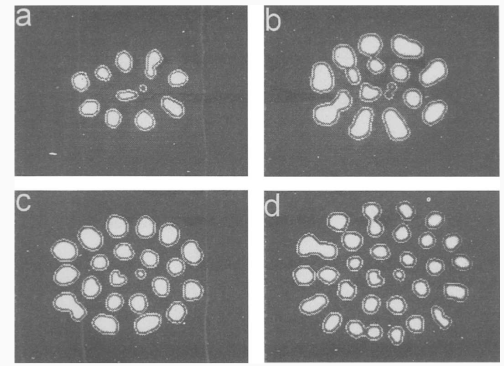
Chemical flip-flops

Turing patterns, turing spots, waves, splitting CIMA reaction

[Castets,Dulos, Boissonade,De Kepper]

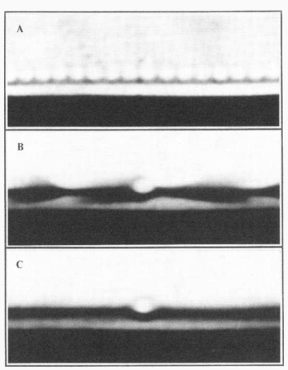


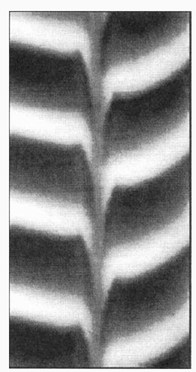


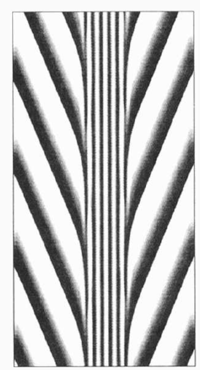


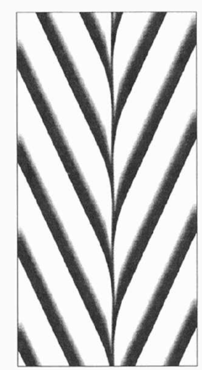
Chemical flip-flop

[Perraud, De Wit, Dulos, De Kepper, Dewel, Borckmans]







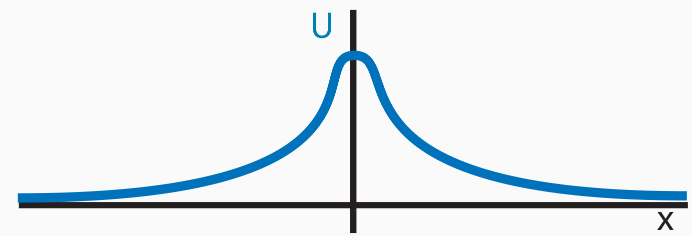


Turing spots and Hopf bifurcation

Reaction-diffusion system

$$U_t = DU_{xx} + F(U;\mu), \qquad x \in \mathbb{R},\ U \in \mathbb{R}^N$$

Turing spot (standing pulse)



$$Q(x) = Q(-x) \to 0 \text{ for} |x| \to \infty$$

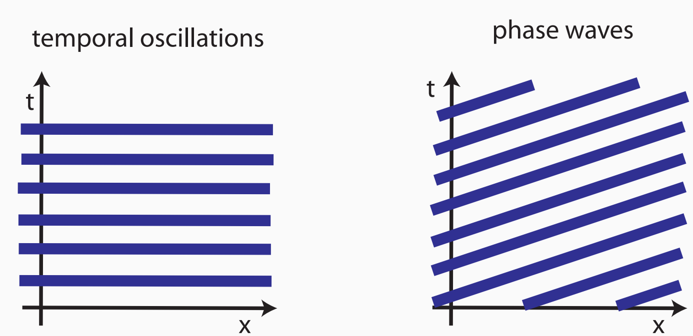
"self-organized inhomogeneity"

Hopf bifurcation

$$U_t = F(U;\mu)$$

undergoes supercritical Hopf at

$$U = 0, \mu = 0$$



One-dimensional target patterns and spiral waves

Theorem **[Sandstede&Scheel]**

Assume
- **standing pulse**
- **Hopf, outside Benjamin-Feir**
- **minimal critical spectrum**

Then unique 1d-target patterns and 1d-spiral waves bifurcate:

- **There do not exist patterns with homogeneous oscillations in the far field**
- **The 1d-target or the 1d-spiral is stable, the other one is unstable**

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Physics of oscillations — modulations

Modulations of the phase:

Ansatz

$$U(t,x) = U_*(\Phi(T,X) - \omega_* t) + \ldots,$$

with

$$\Phi = \Phi(X,T), \quad X = \varepsilon x, T = \varepsilon^2 t$$

gives viscous eikonal/Burgers equation for Φ, $u = \Phi_X$

$$\Phi_T = d\Phi_{XX} - \frac{1}{2}\Omega''\Phi_X^2 \qquad\qquad u_T = du_{XX} - \Omega'' u u_X$$

- d **effective diffusion**
- Ω **nonlinear dispersion**
- **Derivation:** [Howard,Kopell]
- **Validity:** [Doelman, Sandstede, S., Schneider], **Memoirs AMS**

Physics of oscillations — nonlinear dispersion

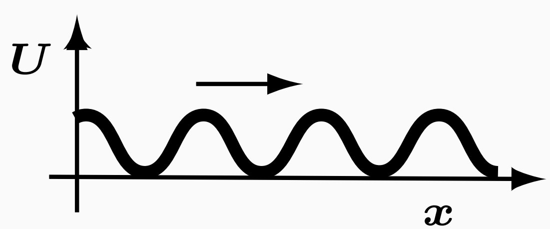
Wave trains $U = U_{\mathrm{wt}}(kx - \omega t; k)$**, 2π-periodic**

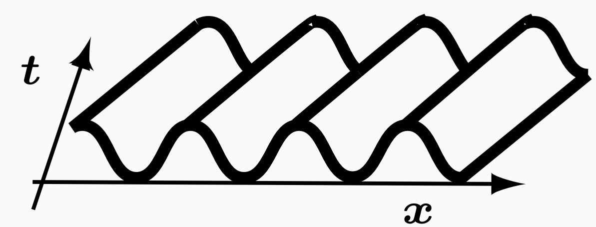
$$-\omega U'_{\mathrm{wt}} = k^2 D U''_{\mathrm{wt}} + F(U_{\mathrm{wt}}), \qquad x \in S^1$$

Dispersion relation $\omega = \Omega(k)$ $\quad \longrightarrow \Omega''$ **in eikonal eq.**

Phase speed $c_{\mathrm{ph}} = \Omega(k)/k$

Group velocity $c_{\mathrm{g}} = \Omega'(k)$





Physics of oscillations — linear dispersion

Linearize at wave train $\quad U_t = DU_{xx} + F'(U_{\mathrm{wt}}(kx - \omega t))U$

Floquet-Bloch $\quad U(t,x) = \mathrm{e}^{\lambda t + \nu x} U_{\mathrm{per}}(kx - \omega t)$

$$U_{\mathrm{per}}(\xi) = U_{\mathrm{per}}(\xi + 2\pi)$$

Boundary-value problem for $V = U_{\mathrm{per}}$

$$\lambda V - \omega \frac{\mathrm{d}}{\mathrm{d}\xi} V = D\left(\frac{\mathrm{d}}{\mathrm{d}\xi} + \nu\right)^2 V + F'(U_{\mathrm{wt}}(kx - \omega t))V$$

Typically, solve for $\lambda = \lambda(\nu) \Rightarrow$ **linear dispersion:**

$$\lambda(\mathrm{i}\gamma) = -c_{\mathrm{g}}\mathrm{i}\gamma - d\gamma^2 + \mathrm{O}(\gamma^3)$$

$\longrightarrow$ **viscosity in eikonal eq.**

group velocity $\sim$ slope of eikonal characteristics

Classifying coherent structures

Sink

k_+, k_- **free**

$$c_{\mathbf{g}}^- > c_{\mathbf{cs}} > c_{\mathbf{g}}^+$$

Contact

$k_+ = k_-$ **free**

$$c_{\mathbf{g}}^- = c_{\mathbf{cs}} = c_{\mathbf{g}}^+$$

Transmission

k_+ **free,** k_- **selected**

$$c_{\mathbf{g}}^-, c_{\mathbf{g}}^+ < c_{\mathbf{cs}}$$

Source

k_+, k_- **selected**

$$c_{\mathbf{g}}^- < c_{\mathbf{cs}} < c_{\mathbf{g}}^+$$

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Characterizing coherent structures

Characterize coherent structures as

- **Periodic in comoving frame:**

$$U = U(x - c_{\rm cs} t, t) = U(x - c_{\rm cs} t, t + T)$$

- **Asymptotic to wave trains:**

$$U(x - c_{\rm cs} t, t) \longrightarrow U_{\rm wt}(k_\pm x - \omega_\pm t + \theta_\pm(x); k_\pm)$$

for $x \to \pm\infty$**,** $\omega_\pm = \Omega(k_\pm)$**, asymptotic phases** $\theta'_\pm(x) \to 0$

Conditions at ∞ and periodicity imply Rankine-Hugoniot!

$$c_{\rm cs} = \frac{\Omega(k^+) - \Omega(k^-)}{k^+ - k^-}$$

Coherent structures satisfy modulated traveling-wave equation:

$$DU_{\xi\xi} + c_{\rm cs} U_\xi + F(U) - U_t = 0, \qquad U(\xi, 0) = U(\xi, T)$$

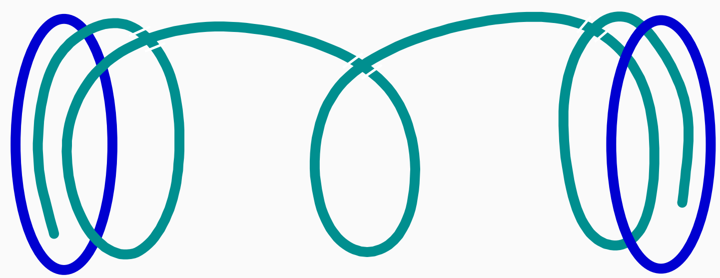
Spatial dynamics

Modulated wave equation [Iooss,Mielke]

$$\begin{pmatrix} u \\ v \end{pmatrix}_\xi = \begin{pmatrix} v \\ D^{-1}[\omega u_\tau - cv - F(u)] \end{pmatrix}$$

where $(u, v)(\xi, \cdot)$ **is time-periodic with period** 2π

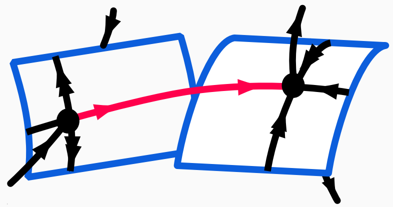
- **Parameters** $(\omega, c) \mapsto (k_+, k_-)$ **through Rankine-Hugoniot**
- S^1**-Symmetry** $(u, v)(\tau) \mapsto (u, v)(\tau + \sigma)$



Coherent structures are heteroclinic orbits that connect wave trains

Coherent structures as heteroclinic orbits

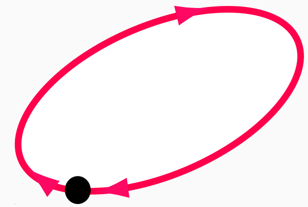
Sink



transverse heteroclinic

codimension 0

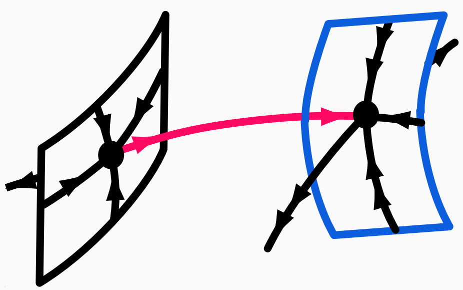
Contact



saddle-node homoclinic

codimension 1

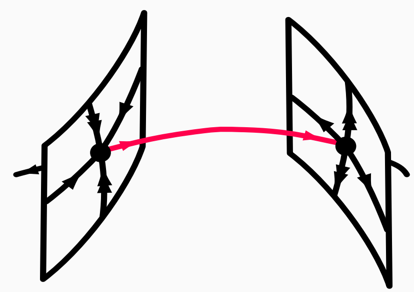
Transmission



homo-/heteroclinic

codimension 1

Source



heteroclinic

codimension 2

Existence and robustness of coherent structures

Theorem [Sandstede,S.]

The four types of elementary coherent structures occur in open, nonempty classes of reaction-diffusion systems

$$\boldsymbol{F} \in \boldsymbol{\mathcal{F}} \subset \boldsymbol{C}^{\mathbf{2}}(\mathbb{R}^{\boldsymbol{N}}, \mathbb{R}^{\boldsymbol{N}}), \quad \boldsymbol{D} \in \boldsymbol{\mathcal{D}} \subset \mathbb{R}^{\boldsymbol{N}}_{+},$$

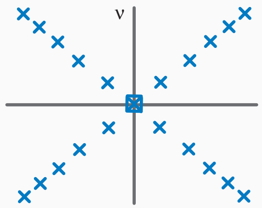
with spectra and multiplicities as described before.

$\Longrightarrow$ **Enables a pathfollowing approach to coherent structures**

Spatial dynamics — Fredholm properties

Modulated wave equation is ill-posed, e.g.

$$u_x = v,\ v_x = \partial_t u, \qquad u \sim \mathrm{e}^{\nu x + \mathrm{i}\ell t},\ \nu = \pm\sqrt{\mathrm{i}\ell}$$



Stable and unstable manifolds exist, both infinite-dimensional!

Define:

- **Intersection map**

$$\iota : TW^{\mathrm{u}}_{-} \times TW^{\mathrm{s}}_{+} \to X, \quad (w^{\mathrm{u}}, w^{\mathrm{s}}) \mapsto w^{\mathrm{u}} + w^{\mathrm{s}}$$

- **Linearized period-map**

$$u(t = T, \xi) = \Phi[u(t = 0, \xi)]$$

Proposition

Fredholm properties of ι and $\Phi - \mathrm{id}$ coincide,

Fredholm index $= i_{\mathrm{F}}(\iota) = i_{\mathrm{F}}(\Phi - \mathrm{id}) = \dim W^{\mathrm{u}}_{-}(\lambda) - \dim W^{\mathrm{u}}_{+}(\lambda)$

Spatial dynamics meets physics

Fredholm indices $\sim$ multiplicities of solutions: compute them!

- **homotope period map $\Psi_\lambda = \Phi - \mathrm{e}^{\lambda T}$ and associate spatial dynamics, $TW_\pm^{\mathrm{u/s}}(\lambda)$, and ι_λ:**

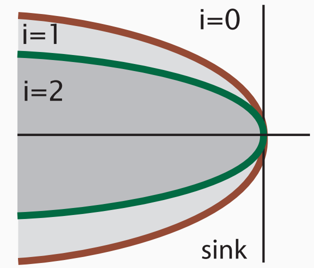
$$\begin{pmatrix} u \\ v \end{pmatrix}_\xi = \begin{pmatrix} v \\ D^{-1}[\omega u_\tau - cv - F'(u_*)u + \lambda u] \end{pmatrix}$$

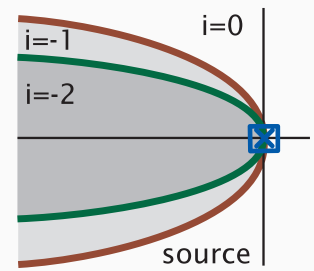
- $i_{\mathrm{F}}(\iota_\lambda) = i_{\mathrm{F}}(\Psi_\lambda) = \dim W_-^{\mathrm{u}}(\lambda) - \dim W_+^{\mathrm{u}}(\lambda)$
- **$\dim W_\pm^{\mathrm{u}}$ changes when there are solutions**

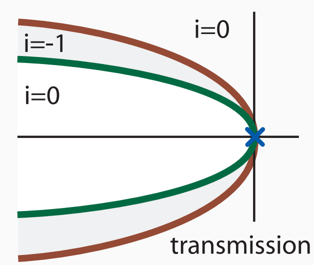
$$\mathrm{e}^{\nu x} U_{\mathrm{per}}(ki\xi - \omega t),\ \nu = \mathrm{i}\gamma,\ \text{at } \xi = \pm\infty$$

- **change of eigenvalue ν when varying γ is**

$$\partial_\lambda \nu = (\partial_\nu \lambda)^{-1} = (c_{\mathrm{g}})^{-1}$$







Fredholm indices are determined by group velocities!

Following coherent structures

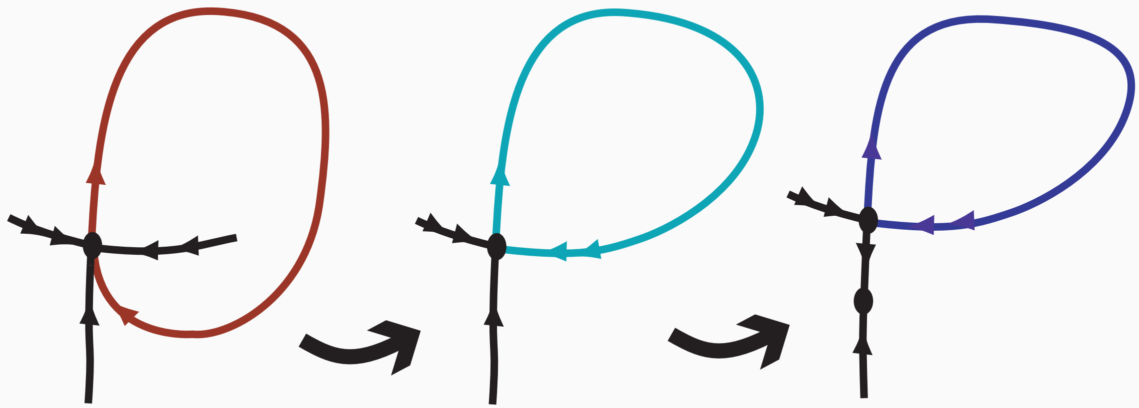
Following heteroclinic and homoclinic orbits	$\Longleftrightarrow$	**Following coherent structures in parameter space**

or

Homoclinic and heteroclinic bifurcations	$\Longrightarrow$	**Phase transitions in non-equilibrium systems**

Two examples...

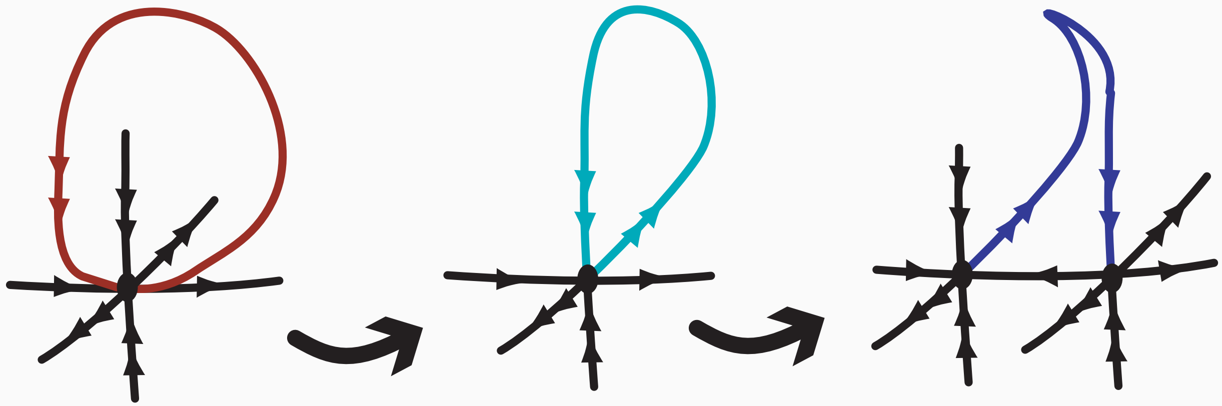
Bifurcations I: contact $\leftrightarrow$ transmission



Homoclinic saddle-node-flip [Chow,Lin]

- **Codimension zero**
- **Unfolded by wavenumber $k_- = k_+$ and speed c**
- **Interpretation: Locking and unlocking from group velocity**

Bifurcations II: contact $\leftrightarrow$ source

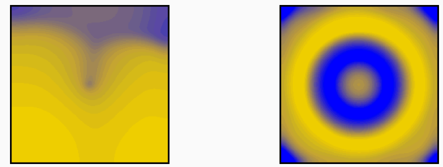


Homoclinic saddle-node-double-flip

- **Codimension one**
- **Unfolded by k_-, k_+, and parameter ε**
- **Example: oscillations with small inhomogeneities**

Summary

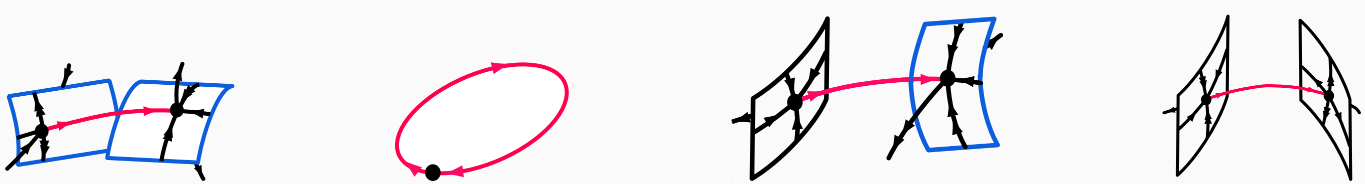
I **Example: oscillations & small inhomogeneities**



II **Example: inhomogeneities & small oscillations**

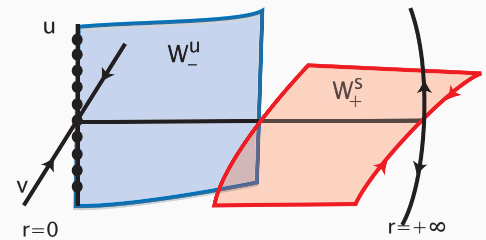
III **Physics of coherent structures**

IV **Spatial dynamics of coherent structures**

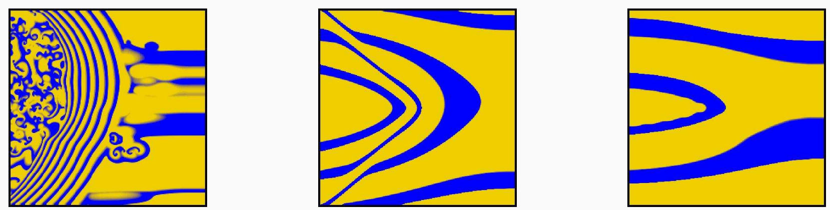


Outlook

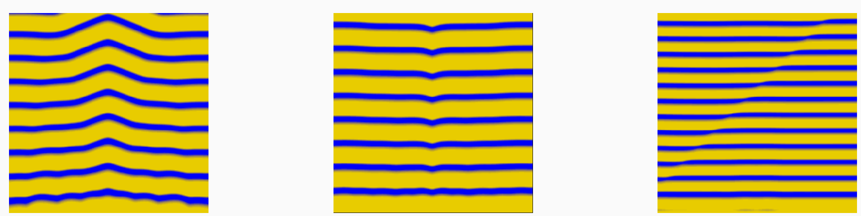
- **higher dimensions: radial dynamics**



- **higher dimensions: beyond radial symmetry**



- **higher dimensions: line defects [Haragus&Scheel]**



- **boundaries and interaction**

- **stability: point spectra, essential and absolute spectra, extended point spectra, Evans functions, nonlinear stability**