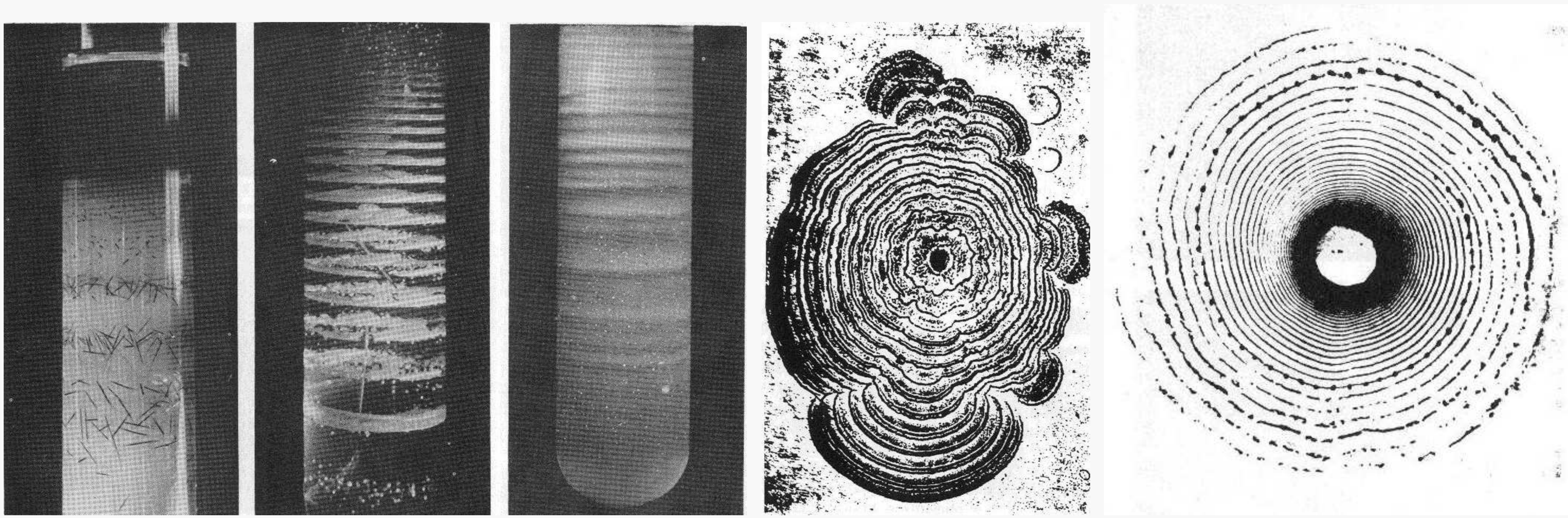
Pattern selection in the wake of fronts

Arnd Scheel, University of Minnesota



[Rothmund,1907],[Knöll,1939]

IMA, 2012

Research supported by NSF

Outline

- **Motivation & Models**
- **Invasion fronts**
- **Speed and pattern selection**

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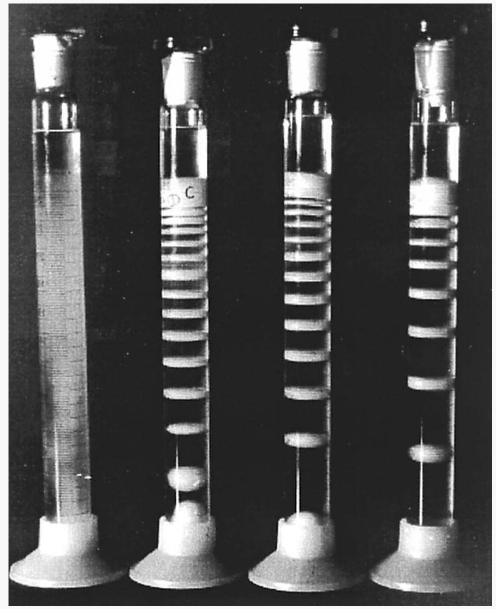
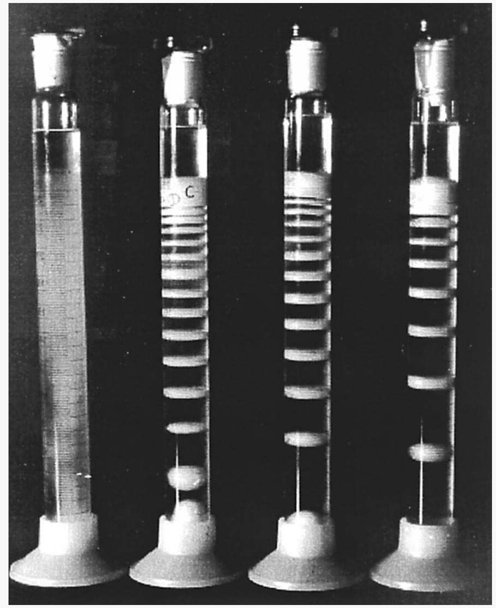
Liesegang patterns in nature and experiment



$$2NaOH + MgCl_2 \longrightarrow Mg(OH)_2$$

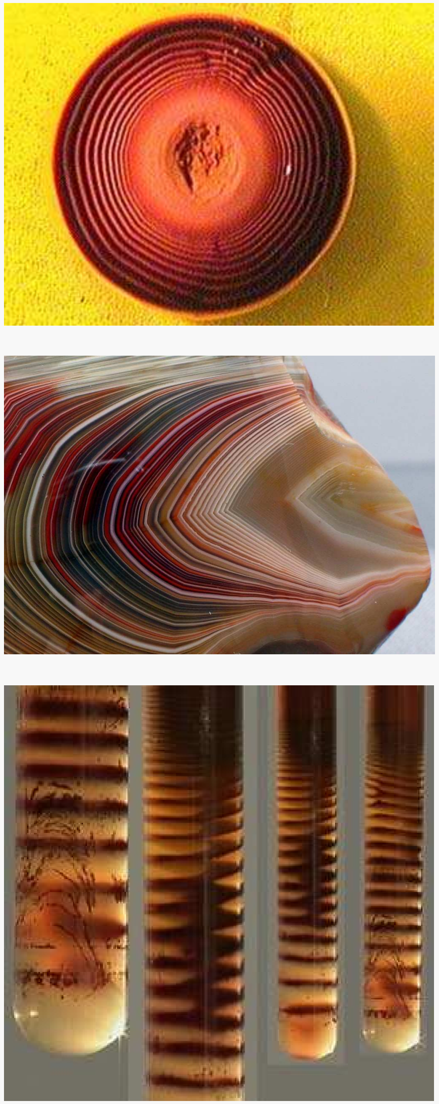
Liesegang patterns in nature and experiment



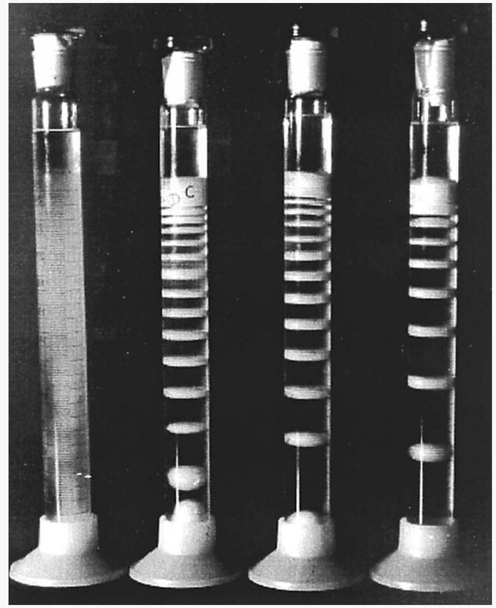
Gel	B	A
Agar	Zn^{2+}	OH^-
Agar	Fe^{2+}	NH_3
Agar	I^-	Pb^{2+}
Agar	F^-	Pb^{2+}
Agar	Mn^{2+}	S^{2-}
Agar	Cu^{2+}	S^{2-}
Agar	Cd^{2+}	S^{2-}
Agarose	Al^{3+}	OH^-
Gelatin	Ba^{2+}	SO_4^{2-}
Gelatin	Co^{2+}	NH_3
Gelatin	Ni^{2+}	NH_3
Gelatin	$Cr_2O_7^{2-}$	Ag^+
Gelatin	$Cr_2O_7^{2-}$	Pb^{2+}
Gelatin	OH^-	Mg^{2+}
Gelatin	Co^{2+}	OH^-
Gelatin	Ni^{2+}	NH_3
Gelatin	Cd^{2+}	NH_3
Gelatin	Mg^{2+}	NH_3
Silica	HPO_4^{2-}	Ca^{2+}
Poly(vinyl alcohol)	Cu^{2+}	OH^-
Poly(vinyl alcohol)	Co^{3+}	OH^-

[George&Varghese]

Liesegang patterns in nature and experiment



[Lagzi]



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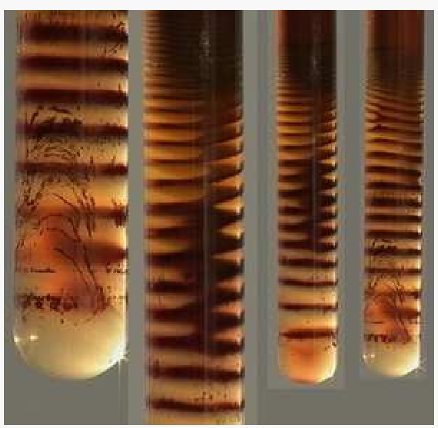
Liesegang patterns in nature and experiment



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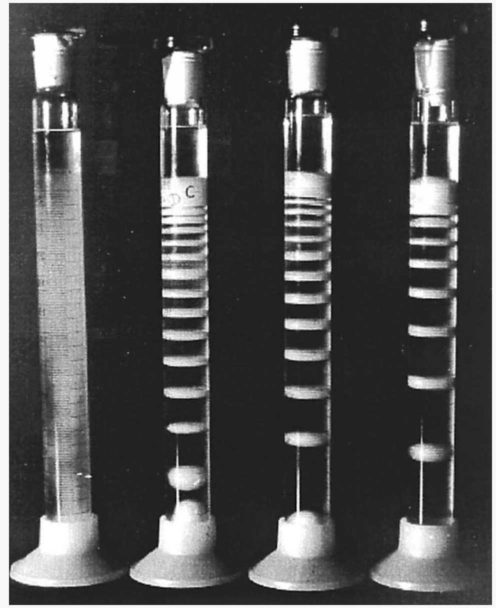

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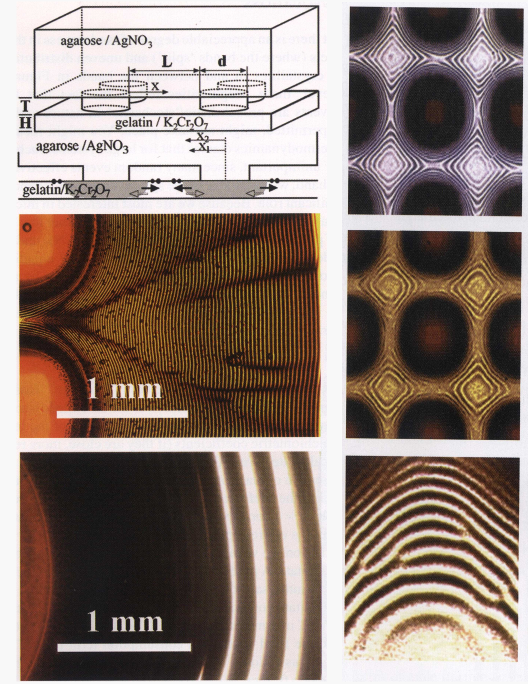
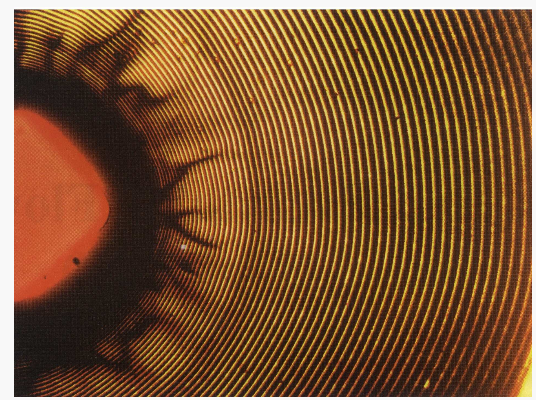




[Grzybowski]

Some History

Reaction-diffusion in gels

- **Raphael Liesegang (1896)**
- **Jablzinsky (1923)** $\Delta x_{j+1}/\Delta x_j \to \eta > 1$
- **Matalon-Packter (1955):** $\eta \sim g_1([B]) + g_2([B])/[A]$, $g_j \searrow$
- **Books:** **H.K. Henisch (1988), Crystals in gels and Liesegang rings; B.A. Grzybowski (2009), Chemistry in Motion**
- **Math:** **Keller & Rubinow (1981), Hilhorst, vd Hout, Mimura, Ohnishi (2007,2009)**

Naturwissenschaftliche Wochenschrift.

Redaktion: Dr. H. Potonié.

Verlag: Ferd. Dümmlers Verlagsbuchhandlung, Berlin SW. 12, Zimmerstr. 94.

XI. Band. | Sonntag, den 26. Juli 1896. | Nr. 30.

Abonnement: Man abonnirt bei allen Buchhandlungen und Postanstalten, wie bei der Expedition. Der Vierteljahrspreis ist ℳ 4.– Bringegeld bei der Post 15 ₰ extra. Postzeitungsliste Nr. 4827.

Inserate: Die viergespaltene Petitzeile 40 ₰. Grössere Aufträge entsprechenden Rabatt. Beilagen nach Uebereinkunft. Inseratenannahme bei allen Annoncenbureaux wie bei der Expedition.

Abdruck ist nur mit vollständiger Quellenangabe gestattet.

Ueber einige Eigenschaften von Gallerten.

Von. R. Ed. Liesegang.

Fast alle Arbeiten, welche eine Aufklärung über die geheimnissvollen Vorgänge in den lebenden Wesen Auskunft geben sollten, waren Studien an den lebenden Organismen selbst. Nur wenige Forscher der Neuzeit versuchten wie die Alchemisten diese Erscheinungen des Lebenden an nichtlebender Materie nachzuahmen. Es sind eigentlich nur Bütschli und Quincke hervorzuheben. Ersterer vermochte mit structurloser Materie Formen zu erzeugen, welche den Formen der Lebewesen sehr ähnlich sind. Letzterer beobachtete die Bewegung eines Quecksilber- und Oeltropfens und stellte ihre Aehnlichkeit mit der Bewegung niederer Organismen fest.

Derartige Untersuchungen halte ich für ebenso wichtig, wie die eigentliche Arbeit des Physiologen. Ueberhaupt haben wir meistens erst durch eine zufällige Nachahmung das Wesen der Organe verstehen gelernt: z. B. durch die optischen Vorrichtungen und durch die Photographie die Function des Auges.

So wird es wohl auch mit jenen Kräften gehen, welche bis jetzt nur in lebenden Organismen beobachtet worden sind und welche ich unter dem Begriff „Lebenskraft" zusammenfassen möchte. Eine neue Beobachtung am Nichlebenden kann zu einer Erklärung führen, auf welche Weise z. B. die Contraction des Muskels geschieht und was der Nervenstrom sei, welcher sie auslöst.

Dieses wegweisende Experiment wird sich um so eher einstellen, je mehr man unter Bedingungen arbeitet, die auch bei den Lebewesen vorhanden sind. Eine solche günstige Bedingung ist die, dass man weder mit Flüssigkeiten noch mit festen Körpern arbeitet, sondern mit Massen in einem Zustand, welcher zwischen beiden liegt: mit Gallerten.

Einige Eigenschaften derselben sollen im Folgenden beschrieben werden.

Allerdings bleiben die Resultate dieser Vorarbeit noch weit von dem vorgesteckten Ziel entfernt. Aber die Biologie wird dabei vielleicht einige Anhaltspunkte finden, deren weitere Verfolgung das eine oder andere kleine Problem lösen kann. So betrachte ich besonders die Structurbildung in einer structurlosen Materie und namentlich die merkwürdigen rhythmischen Erscheinungen als wichtig für die Entwickelungsmechanik der Organismen.

I.

Ich stellte eine Lösung von 50 Gramm reiner Gelatine in 1 Liter warmem Wasser her und übergoss hiermit Glasplatten, so dass auf einer Platte vom Format 13 × 18 cm etwa 50 ccm blieben. Diese Gallerte liess ich erstarren (nicht eintrocknen!) und benutzte sie bald danach zu den Versuchen.

1. Ein Tropfen reinen Wassers, welchen man darauf fallen lässt, zieht bald ein. Die Gelatine quillt darunter noch ein wenig stärker auf. Es zeigt sich kein bleibender Eindruck.

2. Gleich grosse Tropfen verschiedener wässeriger Salzlösungen ziehen ganz verschieden rasch in die Gelatine hinein. Einige ebenso rasch wie das Wasser, andere dagegen äusserst langsam. Ein Tropfen Kochsalzlösung verschwindet fast ebenso rasch wie Wasser.

3. Ein Tropfen Eisenchlorid oder Uranylchlorid bleibt dagegen mehrere Tage auf der Gallerte stehen. Das Wasser einer 10procentigen Eisenchloridlösung zieht überhaupt nicht in die Gelatine hinein. Es verschwindet nur durch Verdunstung in die Atmosphäre.

4. Ist die Salzlösung gefärbt, so kann man die Diffusion des Salzes in der Gelatine leicht mit dem Auge verfolgen. Es zeigt sich dabei, dass das Eisenchlorid ziemlich rasch weiter dringt. Da der Tropfen stehen bleibt,

Reaction-Diffusion Models

Outer and inner electrolyte A, B; product C solute, E precipitate.

$$A + B \to C, \qquad C \rightleftharpoons E$$

Models on $x \in \mathbb{R}_+$

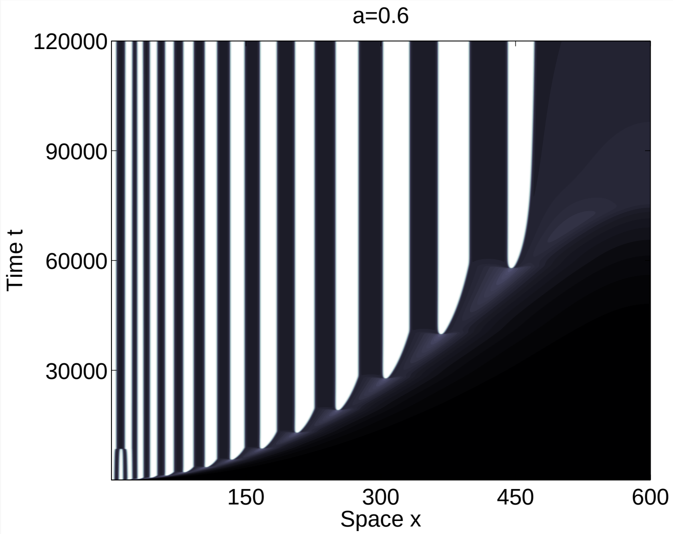
$$a_t = d_a \Delta a - ab$$

$$b_t = d_b \Delta b - ab$$

$$c_t = d_c \Delta c - f(c,e) + ab$$

$$e_t = d_e \Delta e + f(c,e)$$

$$f(c,e) = e(1-e)(e-a) + \gamma c$$



w/ R Goh, S Mesuro, REU

Initial and boundary conditions

$$t = 0 : b \equiv b_0 > 0, \; a, c, e \equiv 0 \qquad b.c. : a|_{x=0} = a_0 \,\&\, \text{Neumann}$$

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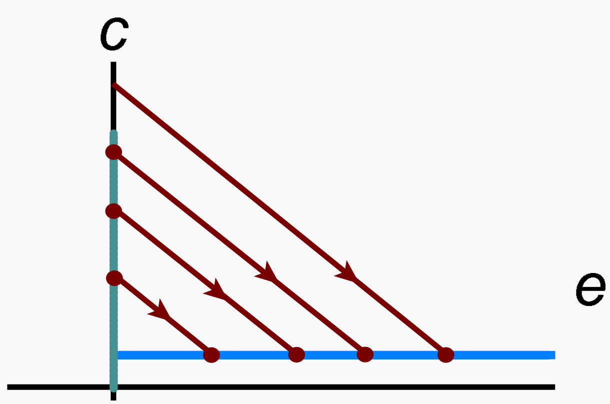
Precipitation: super-saturation vs Cahn-Hilliard

Super-saturation

$$
\begin{aligned}
c_t &= c_{xx} - f(c,e) + \mathrm{a}b \\
e_t &= \cancel{d_e e_{xx}} + f(c,e)
\end{aligned}
$$

[Ostwald 1897,Keller&Rubinow ’81]



Problems:

- **not smooth, no width law**
- **numerically subtle**
- **no exotic patterns**
- **not structurally stable**

Precipitation: super-saturation vs Cahn-Hilliard

Super-saturation

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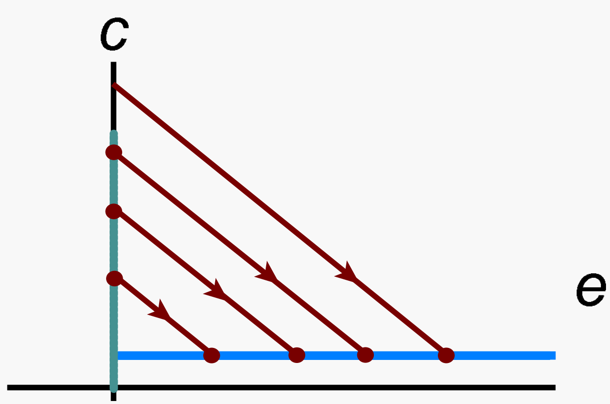
[Ostwald 1897,Keller&Rubinow ’81]



Problems:

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Cahn-Hilliard

$$u \sim e/(c+e) \in [0,1]$$

$$u_t = -\Delta(d\Delta u + g(u)) + \mathrm{a}b$$

[Cahn&Hilliard ’58, Droz 90’s]

- Phenomenological model for nucleation and growth
- limit of threshold kinetics description $\gamma \to \infty$

Problems:

- only phenomenological
- no quantitative comparisons
- no exotic patterns
- only one length scale, no d_c

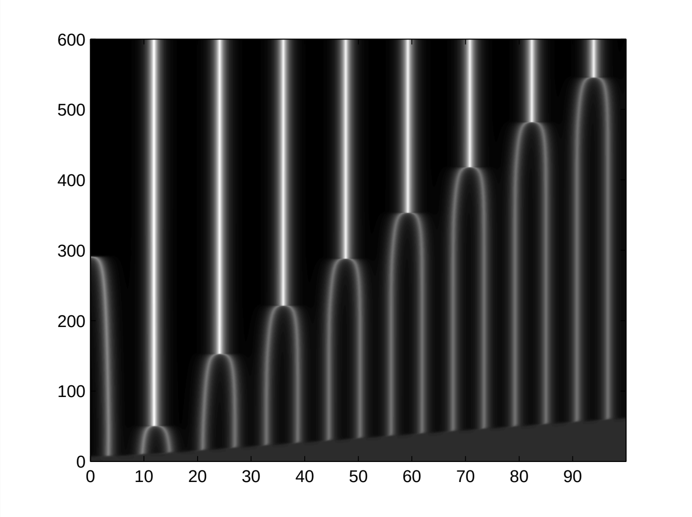
Chemotaxis

u bacteria, v chemoattractant — Keller-Segel:

$$u_t = u_{xx} - (u v_x)_x$$

$$v_t = \kappa v_{xx} - v + u$$

Instability for high concentrations:



Collective aggregation

w/ M Holzer; REU Students K Bose, T Cox, S Silvestri, P Varin

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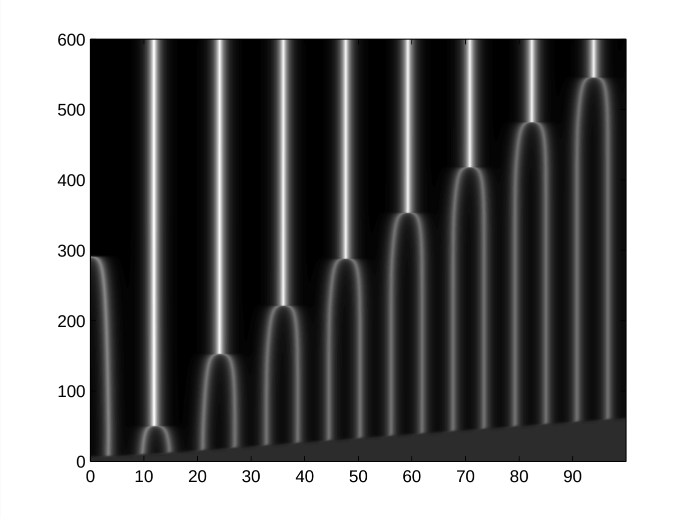

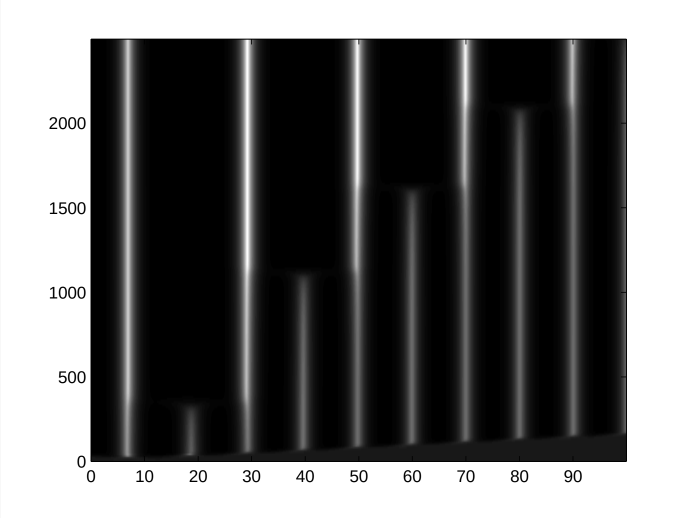

Collective aggregation **versus ripening**

w/ M Holzer; REU Students K Bose, T Cox, S Silvestri, P Varin

More timely: Opinion dynamics

Opinion dynamics: x **opinion,** u **people,** v **money**

$$u_t = u_{xx}$$

$$v_t = \kappa v_{xx} - v$$

People communicate and spend,

More timely: Opinion dynamics

Opinion dynamics: x **opinion,** u **people,** v **money**

$$u_t = u_{xx}$$

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People communicate and spend, people make money

More timely: Opinion dynamics

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$$u_t = u_{xx} - (uv_x)_x$$

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People communicate and spend, people make money , money attracts opinion. . .

More timely: Opinion dynamics

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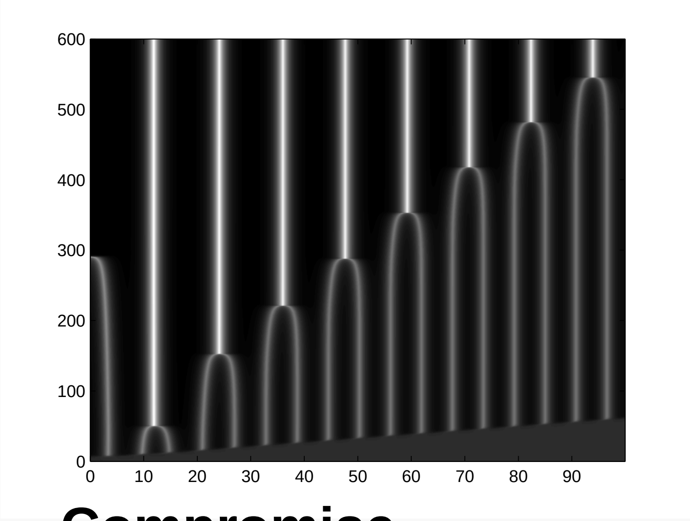
Instability when there's too much money:



Compromise

More timely: Opinion dynamics

Opinion dynamics: x opinion, u people, v money

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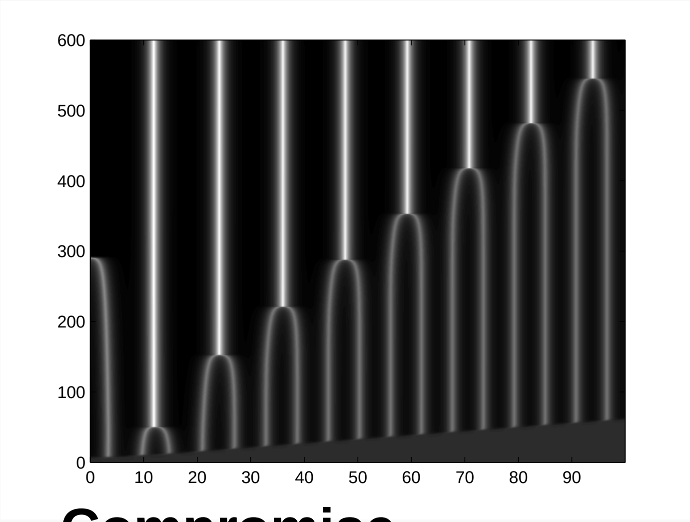

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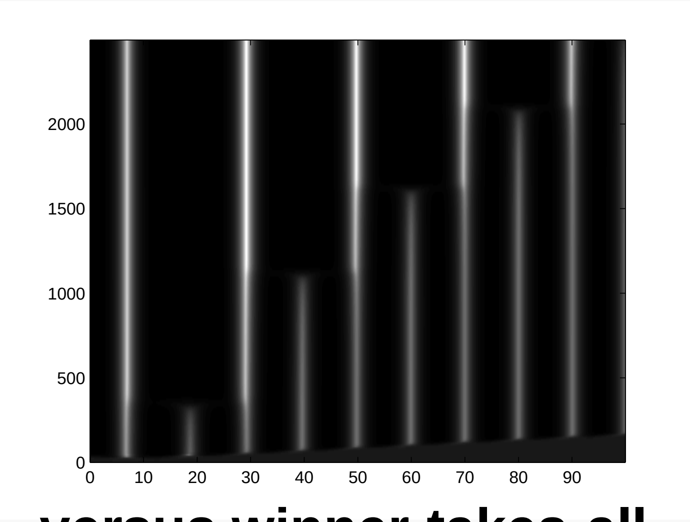
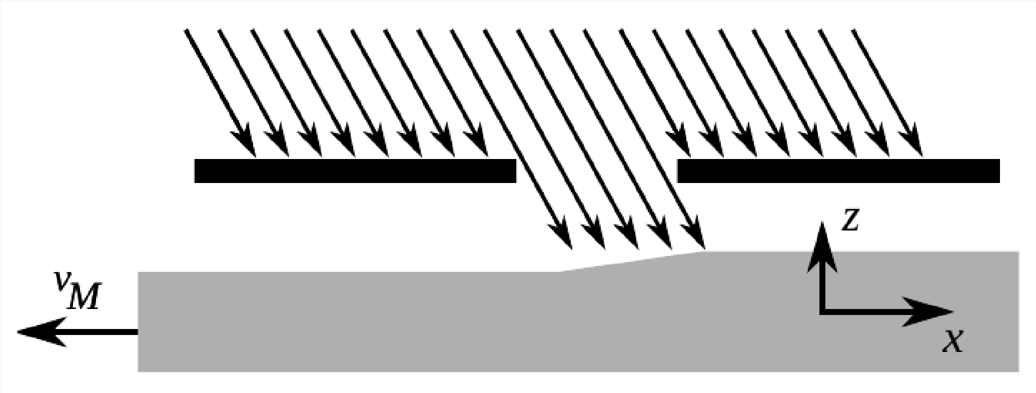
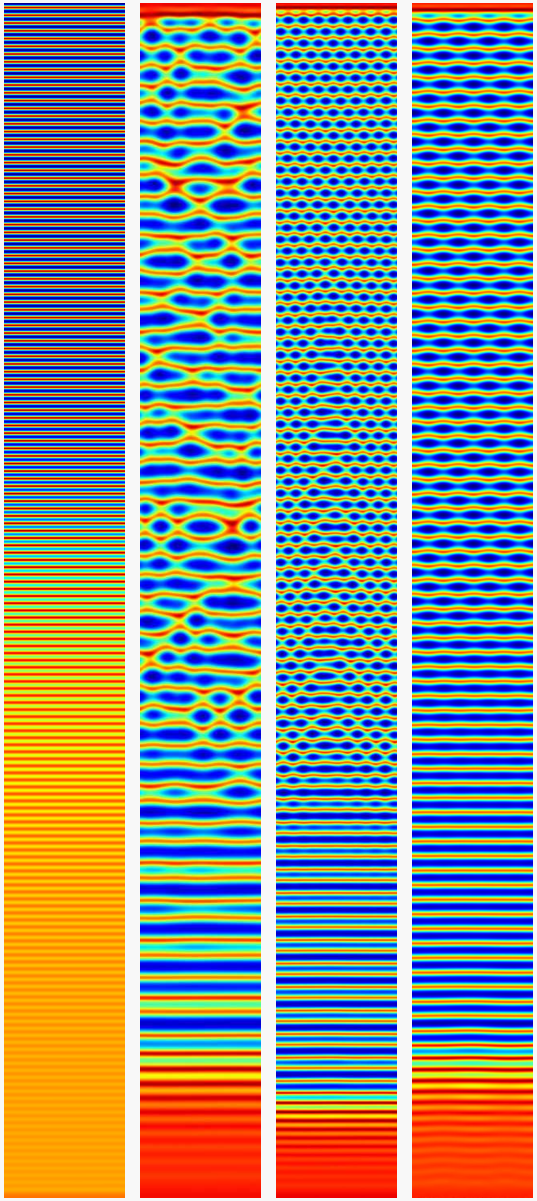


versus winner-takes-all

Different: surface-roughening and ion beams

- Surface bombardement with ion beams → instabilities
- surface roughness on nano-scales;
- highly disordered structure
- but masked fronts create highly organized surface ripples, nano-dots, . . . [Gelfand&Bradley]

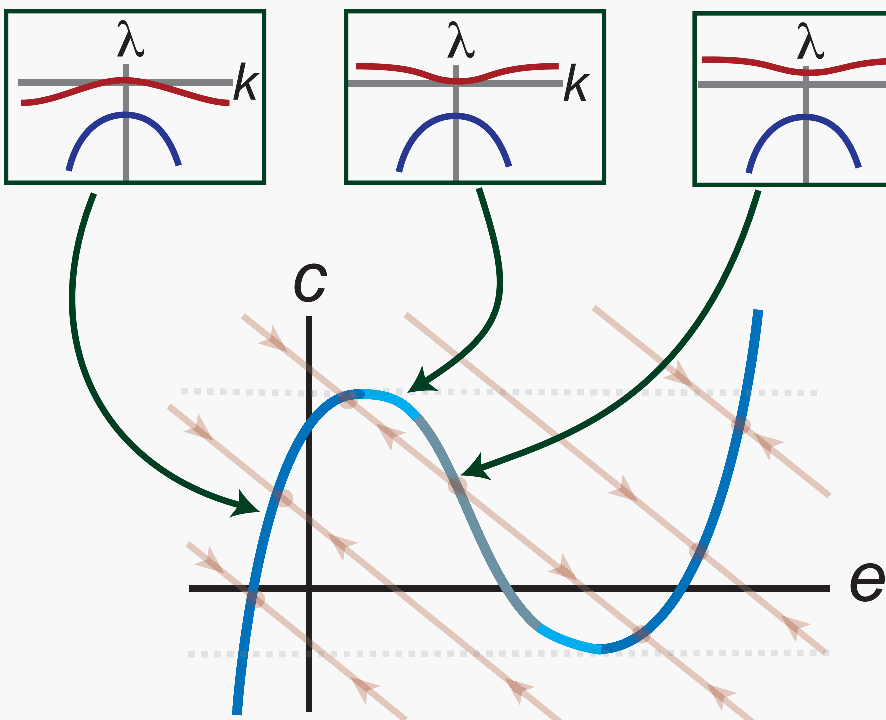




Pattern formation: fronts versus noise

$$\begin{cases} c_t &= \Delta c - e(1-e)(e-a) - \gamma c \\ e_t &= \qquad e(1-e)(e-a) + \gamma c \end{cases}$$

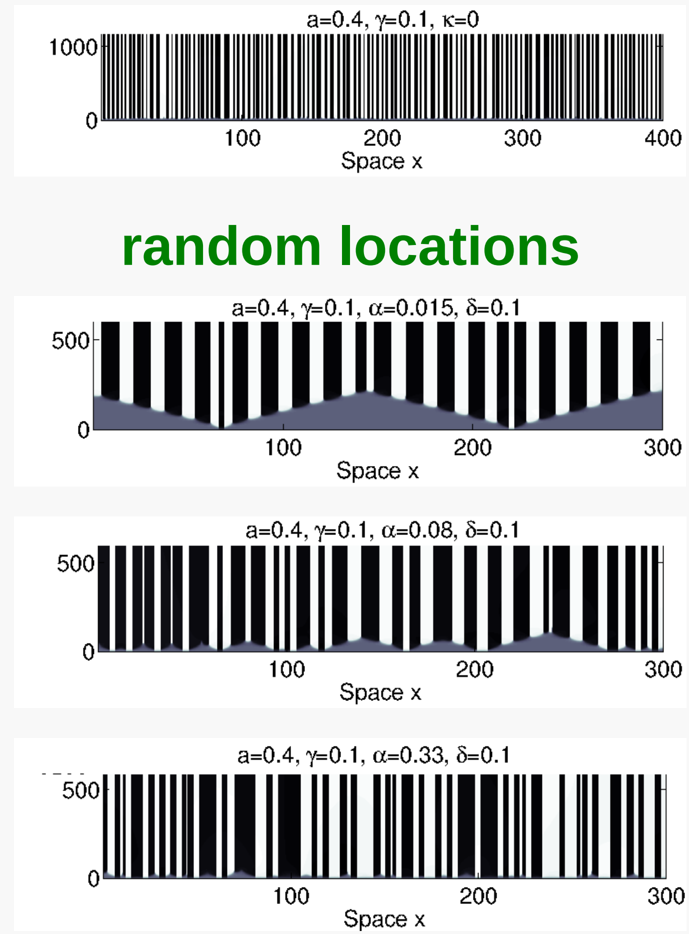
Linear Stability of equilibria



Perturbing $e = a, c = 0$

random amplitudes

random locations



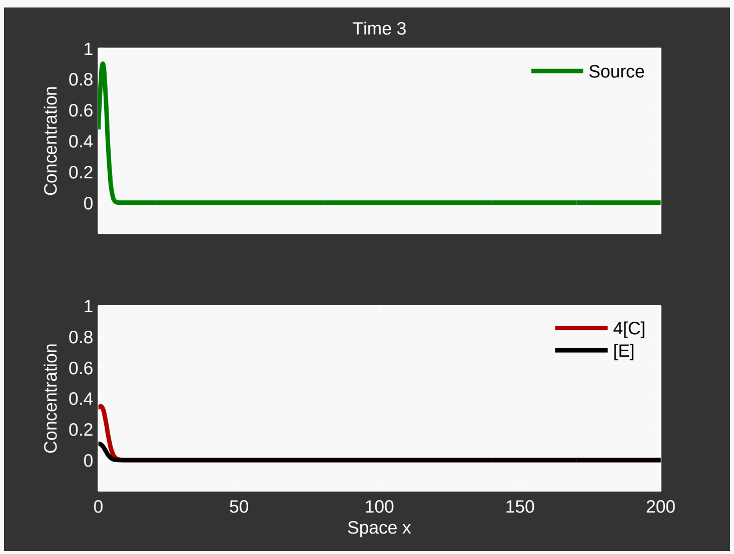
w/ Qiliang Wu, REU students M Kotzagiannidis, J Peterson, J Redford

Invasion fronts: free and triggered

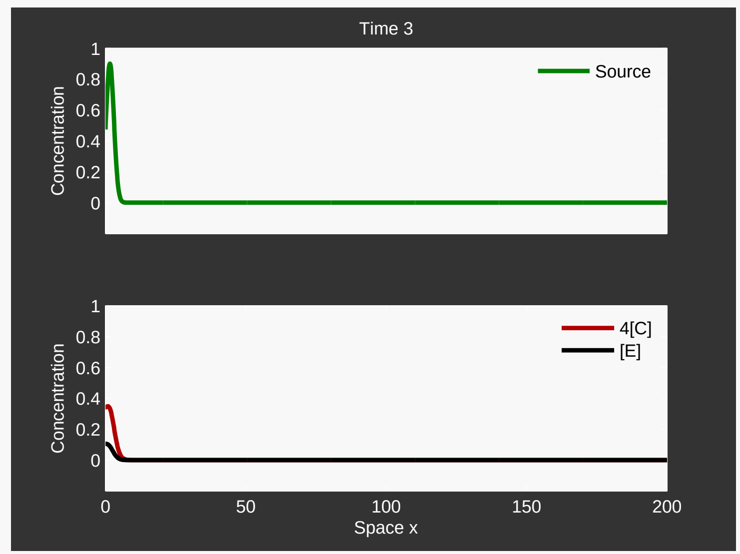
Spatio-temporal source term $h(t,x)$, depositing mass

$$\begin{cases} c_t &= \Delta c - f(c,e) + h(t,x) \\ e_t &= \kappa \Delta e + f(c,e) \end{cases}$$

Basic example: $h(t,x) = H(x - st)$, H localized



Fast source $s \sim 1$



Slow source $s \ll 1$

Invasion fronts: free and triggered

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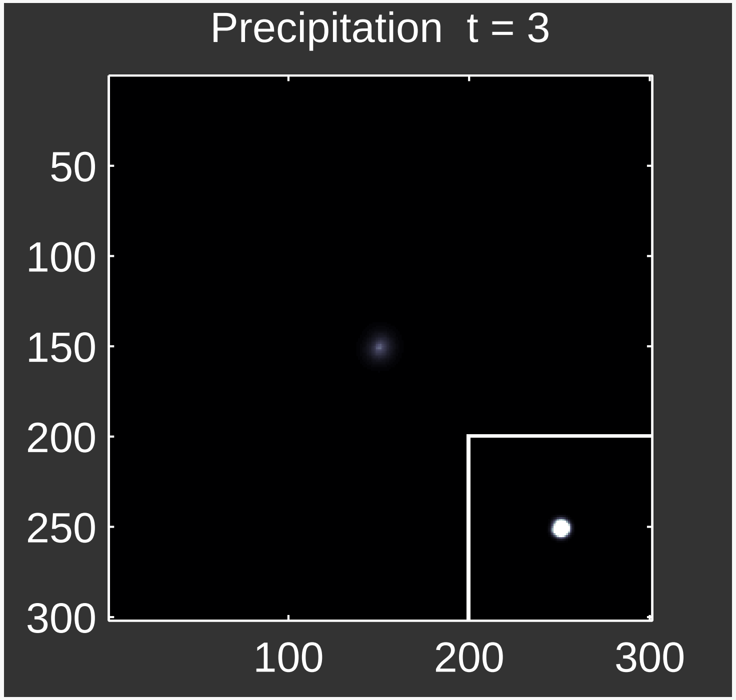
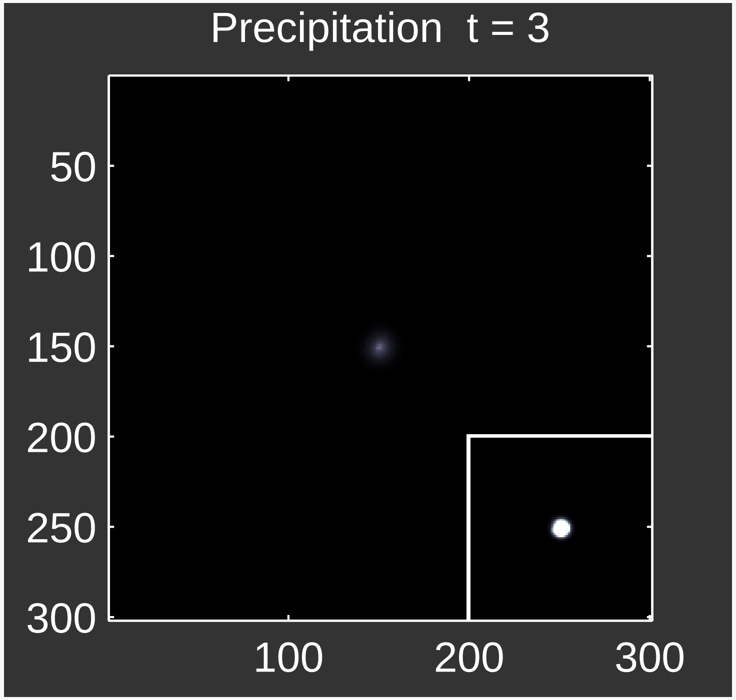
Fast source $s \sim 1$

Slow source $s \ll 1$

Multi-dimensional patterns

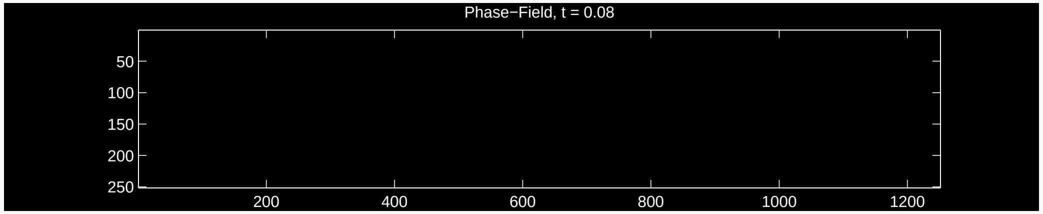
A plethora of patterns from “growth” and “threshold conversion”:





Wet stamping — isotropic

— anisotropic



trigger front and inhomogeneity

Multi-dimensional patterns

A plethora of patterns from “growth” and “threshold conversion”:

Wet stamping — isotropic

— anisotropic

trigger front and inhomogeneity

Outline

- Motivation & Models
- **Invasion fronts**
- Speed and pattern selection

Existence of invasion fronts

Two approaches:

Robustness:

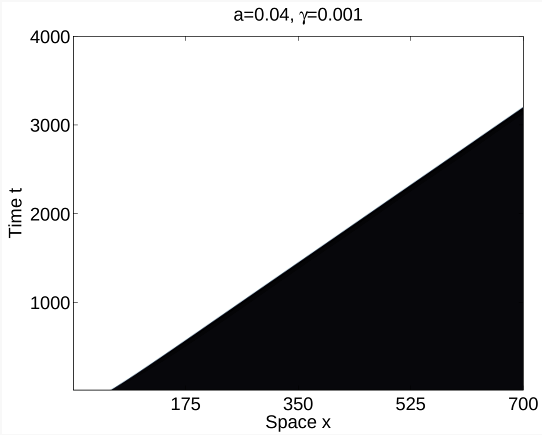
Show that linearization at a given front is Fredholm index 0 — without proving existence!

Existence:

Show that fronts exist using Conley's index!

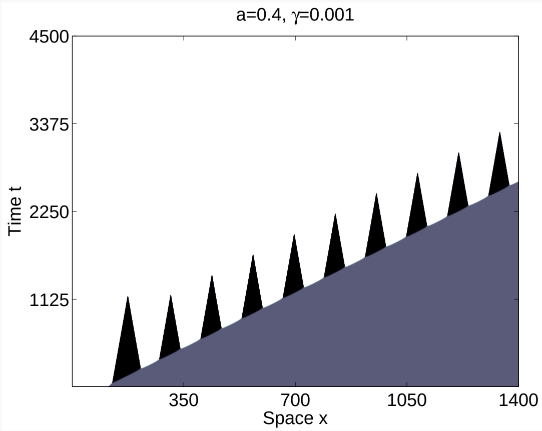
Robustness — phenomena

Initial conditions $(c \equiv 0,\ e \equiv a)$ **+ perturbation near** $x = 0$



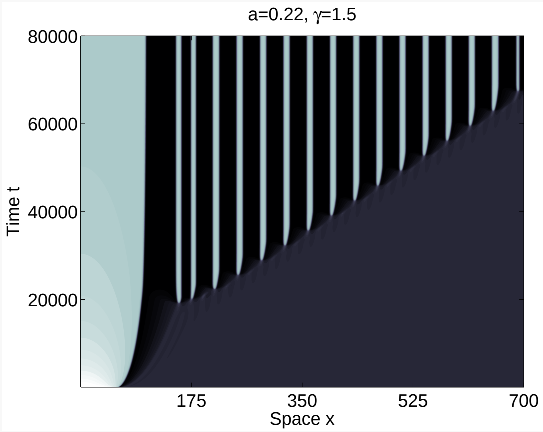
$\gamma = 0.001,\ a = 0.04$

Bulk Front



$\gamma = 0.001,\ a = 0.4$

Transient Pattern



$\gamma = 1.5,\ a = 0.22$

Persistent Pattern

$$c_t = c_{xx} - e(1-e)(e-a) - \gamma c$$

$$e_t = d e_{xx} + e(1-e)(e-a) + \gamma c$$

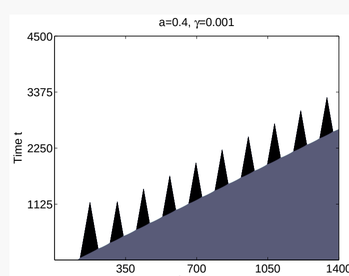
Robustness — results

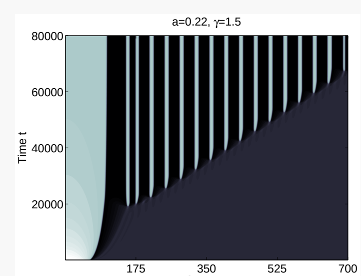
Theorem [R Goh, S Mesuro, S.]

Pattern-forming fronts are robust iff the pattern in the wake is stable with respect to co-periodic perturbations

Remarks

- **Effectively discriminate between transient and persistent patterns!**
- **All periodic patterns are unstable on $x \in \mathbb{R}$ or period 2!**
- **Bulk fronts are pushed fronts;** [van Saarloos]
- **The transition from bulk to pattern-forming is an "essential, pointwise" Hopf bifurcation at $a_*(d)$...**





Robustness — proofs

- **Traveling-wave equation for $u = (c, e)(x - st, kx)$ as dyn' sys'**

$$
\begin{aligned}
u_\xi &= -k\partial_y u + v \\
v_\xi &= -k\partial_y v - D^{-1}\left(F(u) + c(v - k\partial_y u)\right)
\end{aligned}
$$

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- **Ill-posed, pseudo-elliptic: relative Morse indices!**

Linearize at $u_*^\pm(y)$ and count dimension of W_+^s, W_-^u:

$$\iota : (u_+ \in TW_+^s, u_- \in TW_-^u) \mapsto u_+ - u_-,$$

Index theorem e.g. [Sandstede,S. '01]: Fredholm = relative Morse:

$$i_{\mathrm{F}}(\iota) = i_{\mathrm{M}}(u_*^-) - i_{\mathrm{M}}(u_*^+)$$

Resonant modes and Morse indices

How do we compute $i_{\mathrm{M}}(u_*^{\pm})$? $\longrightarrow$ **Homotopies!**

Spatial growth modes $u(y)\mathrm{e}^{\nu\xi}$ satisfy

$$-s\nu u = D(k\partial_y + \nu)^2 u + F'(u_*)u$$

Idea: Homotope $u_* = u_*^- \ldots u_*^+$ and count ν's crossing $\mathrm{i}\mathbb{R}$

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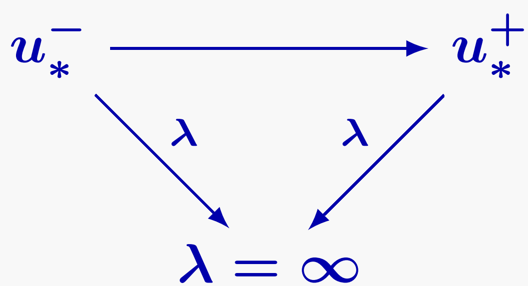
Here's how: homotope to $\lambda = +\infty$ for $u_* = u_*^{\pm}$:

$$\lambda u - s\nu u = D(k\partial_y + \nu)^2 u + F'(u_*)u$$

where we can neglect F' and hence the dependence on u:



Crossings in λ

$\Longleftrightarrow$

resonant unstable modes

Existence: The Cahn-Hilliard equation

A model for phase separation, u order parameter

$$u_t = -(u_{xx} + u - u^3)_{xx}$$

on $\mathbb{R}/L\mathbb{Z}$, say

Mass conservation

$$m(u) = \fint u$$

Energy dissipation

$$E(u) = \int u_x^2 - u^2 + \frac{1}{2}u^4$$

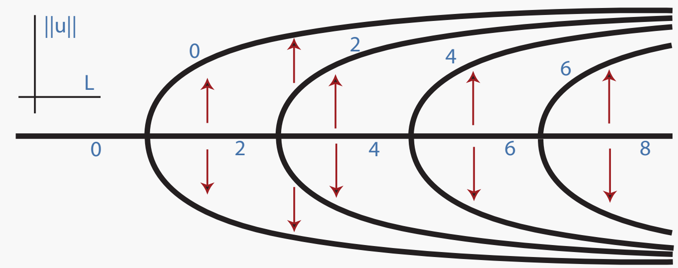
Gradient structure

$$u_t = -\nabla_{H^{-1}} E(u)$$

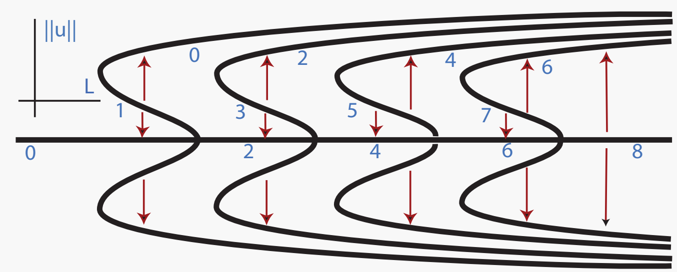
Equilibria and attractors

Dynamics on attractor: equilibria and heteroclinic connections

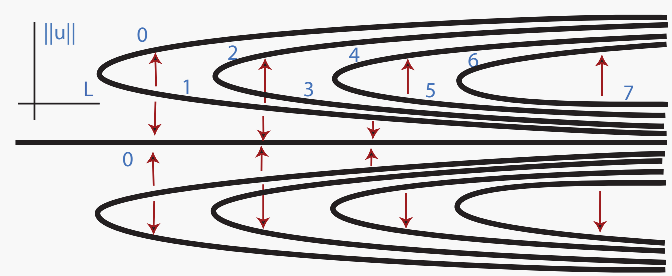
$$|m| < \frac{1}{\sqrt{5}}$$



$$\frac{1}{\sqrt{5}} < |m| < \frac{1}{\sqrt{3}}$$



$$\frac{1}{\sqrt{3}} < |m| < 1$$



[Grinfeld,Novick-Cohen]

Spinodal decomposition fronts

Main Theorem [S.]

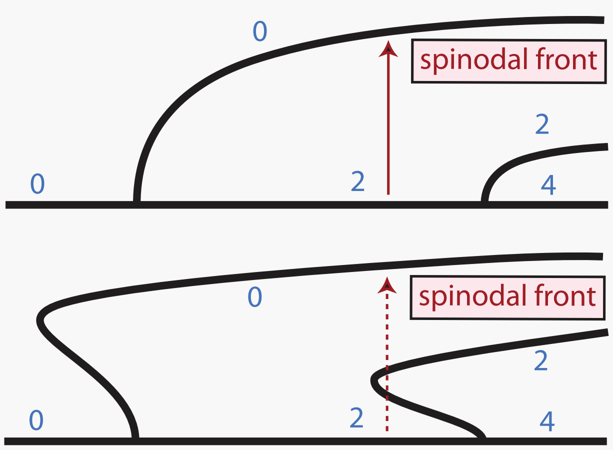
For each $|m| < 1/\sqrt{3}$, there exists a modulated front solution

$$u_*(x - s_{\rm lin}t, k_{\rm lin}x), \qquad u_*(\xi, y) = u_*(\xi, y + 2\pi)$$

with asymptotics
$$\begin{cases} u_*(\xi, y) \to m, & \xi \to +\infty, \text{ unif. in } y, \\ u_*(\xi, y) \to u_-, & \xi \to -\infty, \text{ unif. in } y, \end{cases}$$

More specifically,

- **For $|m| < 1/\sqrt{5}$, u_- has minimal period $2\pi/k_{\rm lin}$.**
- **For $1/\sqrt{5} < |m| < 1/\sqrt{3}$, exist chain of waves $u_{*,1}, \ldots, u_{*,j}$ so that the last wave connects to $u_{-,j}$ with minimal period $2\pi/k_{\rm lin}$.**



Existence — Outline

- **"Translate" Lyapunov function $\mathcal{L}$ and mass conservation $\mathcal{I}$:**

$$\mathcal{L}(u) = \int_0^{2\pi} \left(\frac{1}{2} u_\xi^2 - G(u) - k u_\xi u_\tau - \frac{1}{s} \theta \theta_\xi \right) \mathrm{d}\tau$$

$$\mathcal{I}(u) = ⨍ (su - \theta_\xi)$$

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- H^{-1}**-estimates: define** $\partial_\tau \phi = u - ⨍_\tau u$**!**

$$\int_\xi \int_\tau \left(\phi_{\xi\xi}^2 + \phi_\xi^4\right) \chi(\xi) < \infty, \qquad \int_\xi \left(⨍_\tau su - \mathcal{I} \right)^2 < \infty$$

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- **Morse indices and connection matrices [Franzosa],[Mischaikow]**

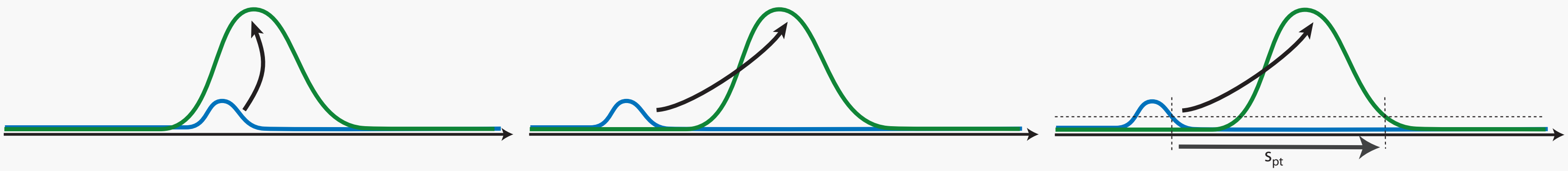
$\longrightarrow$ **heteroclinic orbits**

Outline

- Motivation & Models
- Invasion fronts
- **Speed and pattern selection**

Spreading speeds

Absolute and convective instabilities: $u_0(x)$ compactly supported



pointwise growth **pointwise decay** **spreading speed**

Linear dispersion relation

$$u = (c, e) \sim (c_0, e_0)\mathrm{e}^{\lambda t + \nu x},$$

$$D(\lambda, \nu) = 0$$

Pointwise growth modes (λ, ν):

$$D(\lambda, \nu) = 0,$$

$$\partial_\nu D(\lambda, \nu) = 0 \text{ + "pinching"}$$

Classic "Lemma": **Typically,**

pointwise instability $\Leftrightarrow$ unstable pointwise modes, $\mathrm{Re}\,\lambda > 0$

Pointwise spreading speed:

$s_{\mathrm{pt}} := \sup\{s|$ **pointwise unstable in frame** $\xi = x - st\}$

Selected wavenumber:

$k_{\mathrm{lin}} = \omega_{\mathrm{lin}}/s_{\mathrm{pt}}$ **where** $\mathrm{i}\omega_{\mathrm{lin}}$ **neutral pointwise growth mode at**

Spreading speeds — subtleties

The previous slide often gives the wrong answer [w/ M Holzer]:

- linear speeds $<$ "pinched speed"

$$u_t = u_x + u, \qquad v_t = -v_x + v$$

  has pointwise decay, yet a pinched double root at $\lambda = 1$

- linear speed $<$ nonlinear speed: $\longrightarrow$ pushed fronts

$$u_t = u_{xx} + u(1-u)(u-a), \qquad a < 1/3$$

- nonlinear speed $<$ linear speed $\longrightarrow$ Lotka-Volterra
- linear speed $<$ nonlinear speed: $\longrightarrow$ staged invasion

Related question: What happens in the wake of invasion?

Spreading speeds — multi-d

Consider isotropic system, initial conditions $u_{\mathrm{cpt}}(x)\mathrm{e}^{\mathrm{i}ky}$.

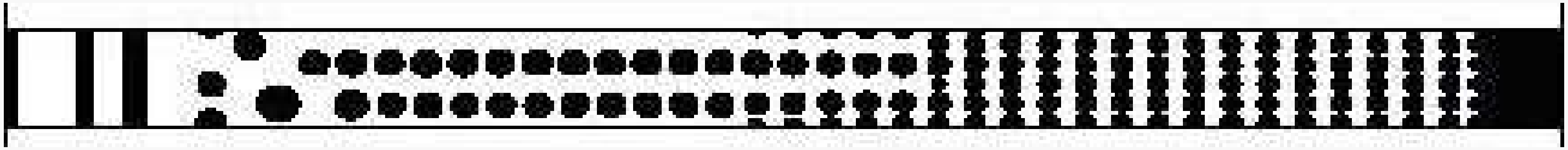
Define transverse modulated spreading speeds $s_{\mathrm{lin}}(k)$

Conjecture. . . Theorem? [R Goh, M Holzer, S.]

$$s_{\mathrm{lin}}(k) \le s_{\mathrm{lin}}(0)$$

Linear theory *always* predicts stripes in the leading edge of invasion fronts

Other patterns emerge through pushed fronts or staged invasion:



Cahn-Hilliard in strip

Coupled KPP

Toy model for staged invasion

$$u_t = u_{xx} + u(1-u)$$

$$v_t = dv_{xx} + g(u)v - v^3$$

How do compactly supported initial data evolve?

- u**-equation: convergence to critical KPP front** $s_u = 2$
- v**-equation: speeds**
  - $u = 0$: $s_v = 2\sqrt{dg(0)}$
  - $u = 1$: $s_v = 2\sqrt{dg(1)}$

Question: Determine the v**-invasion speed!**

Coupled KPP — phenomena

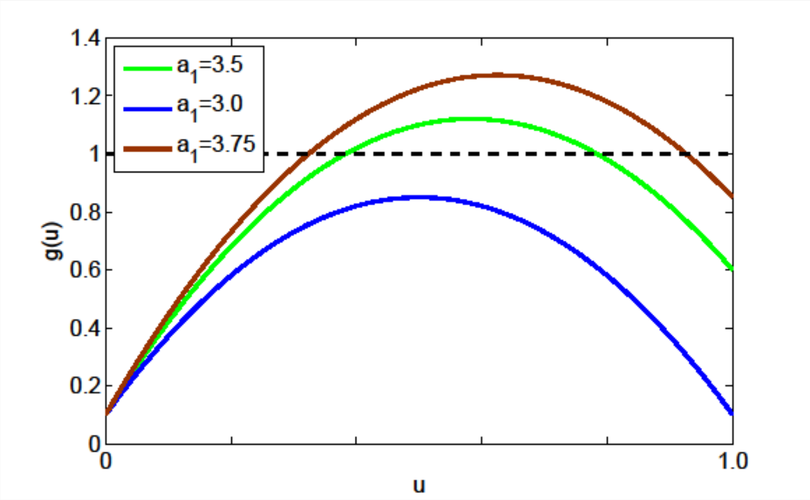
$$u_t = u_{xx} + u(1-u)$$

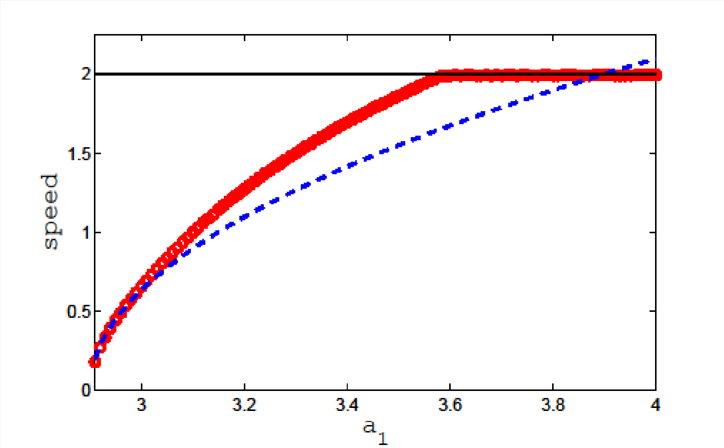
$$v_t = dv_{xx} + g(u)v - v^3$$

"Instantaneous" v-speed:

$$s_v = -2\sqrt{dg(u)}$$

E.g. $g(u) = 0.3 + a_1 u - 3u^2$





3 Regimes:

- **locked regime**
  (strong inhomogeneity)
- **accelerated regime**
  (intermediate)
- **pulled regime**
  (weak, uncoupled)

Resonance poles, locked, and accelerated fronts

Linearizing v-equation along u-front gives

$$v_t = dv_{\xi\xi} + 2v_\xi + g'(u_{KPP})v = \mathcal{L}u$$

Resonance pole λ_{rp} of $\mathcal{L}$ determine regime:

- $\lambda_{\mathrm{rp}} > 0 \quad \Longrightarrow$ **locked fronts,** $s_v = 2$
- $0 > \lambda_{\mathrm{rp}} > -\lambda_* \quad \Longrightarrow$ **accelerated fronts,** $s_v > s_v^1 = \sqrt{dg(1)}$

Acceleration since resonance mode induces spreading:

$v(\xi) \sim \mathrm{e}^{\lambda_{\mathrm{rp}} t + \nu_v^+}$ **for** $\xi \to +\infty$**,** $s_v = \lambda/\nu_+$

Note:

u-front accelerates v-front by fixed amount while separation distance goes to infinity!

$\Longrightarrow$

Interaction force *growing* exponentially with distance

Accelerated fronts — proofs

Idea:

Construct *steep* sub- and supersolutions based on the resonance pole

Similar technique: [Nolen,Roquejoffre,Ryzhik,Zlatos 2012]

$\longrightarrow$ **KPP with *steady* inhomogeneity, compact support**

$$u_t = u_{xx} + g(x)u - u^2$$

Locked fronts

Theorem

Suppose $\lambda_{\mathrm{rp}} > 0$, then there exists stable locked front, v-component has steep exponential decay

$$u = u_{\mathrm{kpp}}(x - 2t), v_{\mathrm{lock}}(x - 2t) \sim \mathrm{e}^{-d^{-1}(1+\sqrt{1+dg(1)})\xi}$$

For $\lambda_{\mathrm{rp}} > 0$, small, the bifurcation to locked fronts is supercritical and the separation distance scales with

$$\frac{1}{2\nu_v^+} \log(\lambda_{\mathrm{rp}}) \text{ if } 2\nu_v^+ - \nu_v^- > 0$$

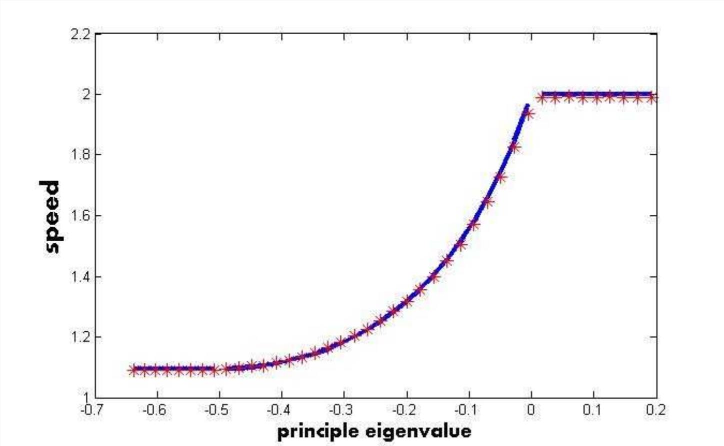
$$\frac{1}{\nu_v^- \nu_v^+} \log(\lambda_{\mathrm{rp}}) \text{ if } 2\nu_v^+ - \nu_v^- < 0$$

Proof **Heteroclinic orbit flip, Shilnikov coordinates after normal form transformations that straighten fibrations [Homburg].**

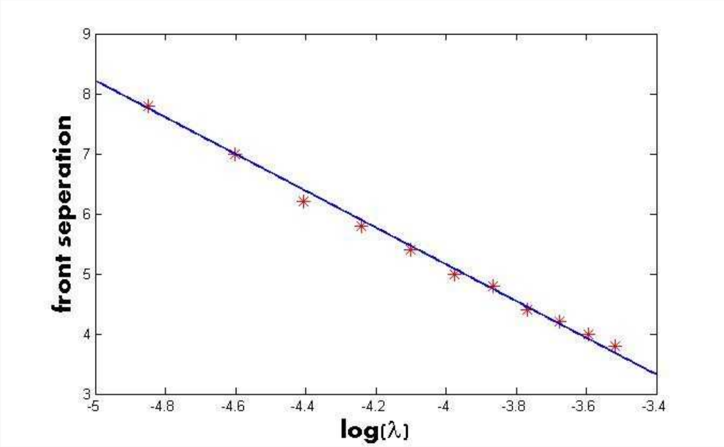
Comparison with simulations

Fix $g(u) = 0.3 + \alpha(u - u^2)$, $d = 1$.

Speed versus prediction locked, accelerated, uncoupled



Separation versus prediction, locked case



Summary and references

Pattern-forming fronts need more attention!

- **existence and robustness**
- **speeds and wavenumber predictions — 1-d**
- **some results for multi-d, staged invasion**

References

- **existence & robustness:**

  R Goh, S Mesuro, A Scheel, ***Spatial wavenumber selection in recurrent precipitation***, **SIADS 2011**

  A Scheel, ***Spinodal decomposition and coarsening fronts in the Cahn-Hilliard equation***, **very soon**

- **staged invasion and anomalous spreading:** M Holzer, A Scheel,

  A slow pushed front in a Lotka-Volterra competition model, **Nonlinearity 2012**

  Accelerated fronts in a two stage invasion process, **preprint**