CROSS CHANNELS AND KINEMATIC INVARIANTS

General Definitions in $a + b \to c + d$ processes

(I, or s) Primary channel: $1 + 2 \to 3 + 4$

II, or t channel: $1 + \bar{3} \to \bar{2} + 4$

III, or u channel: $1 + \bar{4} \to \bar{2} + 3$

It is clear that the definitions of t and u channels are somewhat arbitrary, because this naming depends on what particle is called 3, and what is called 4. Let us consider the example of the Compton scattering as the primary process,

$$e^-(p) + \gamma(k) \to e^-(p') + \gamma(k') \tag{1}$$

where the momenta of the particles are indicated in parentheses. The kinematic invariants are

$$\begin{aligned} s &= (p+k)^2 = (p'+k')^2\,, \\ t &= (p-p')^2 = (k'-k)^2\,, \\ u &= (p-k')^2 = (p'-k)^2\,. \end{aligned} \tag{2}$$

Correspondingly, the channel II (t) is

$$e^-(p) + e^+(-p') \to \gamma(-k) + \gamma(k')\,, \tag{3}$$

while the channel III (u) is

$$e^-(p) + \gamma(-k') \to e^+(-p') + \gamma(-k)\,. \tag{4}$$

The Compton scattiring is determined by two Feynman diagrams depicted in Fig. 1. The graph on the left is referred to as the s-channel pole diagram, while

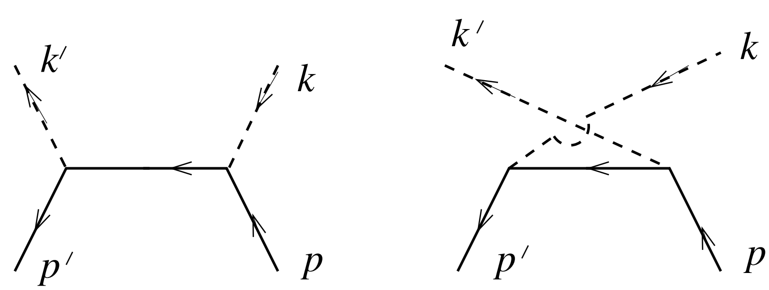

Figure 1: Two graphs describing the Compton scattering

that on the right is u-channel pole diagram. They are related to each other by the interchange $s \leftrightarrow u$ and the corresponding adjustment of polarizations. The cross section of the Compton scattering is given by the formula

$$d\sigma = \alpha^2 \, \frac{\pi}{4} \left\{ \frac{dt}{(s-m^2)^2} \right\} \Big[f(s,u) + f(u,s) + g(s,u) + g(u,s) \Big] \,. \qquad (5)$$

Here $f(s,u)$ represents the square of the graph on the left, while $f(u,s)$ represents the square of the graph on the right,

$$f(s,u) = \frac{2}{(s-m^2)^2} \Big[4m^4 - (s-m^2)(u-m^2) + 2m^2(s-m^2) \Big] \,. \qquad (6)$$

Moreover, the function g appears due to interference contribution of both graphs and is proportional to both amplitudes,

$$g(s,u) = \frac{2m^2}{(s-m^2)(u-m^2)} \Big[4m^4 + (s-m^2) + (u-m^2) \Big] \,. \qquad (7)$$

Note that, as expected, $g(s,u)$ is invariant under interchange $s \leftrightarrow u$. That's because it represents the product of both amplitudes.

What happens if we pass to the channels II and III (i.e. t and u)? The factor in the braces in Eq. (5) is channel specific, since it comes from the phase space of the final particles and the flux of the initial particles. say, in the t channel both initial particles are massive, while both final particles are massless. This factor must be written individually for each channel.

However, the expression in the square brackets, representing the amplitudes, remains the same for all three channels. The only change is the values of the momenta ascribed to the particles (say, in the t channel the momentum of the *incoming* positron is $-p'$. In the u channel the momentum of the incoming photon is $-k'$ while the momentum of the outgoing photon is $-k$, see Eqs. (3) and (4). Physically, the s and u channels are the same in the process at hand, only the definitions of the photon momenta change.

In passing from the channel I to channels II or III the values of s, t, u changes. The boundaries of all three channels are described by the lines

$$s + t + u = 2m^2 \,, \qquad sut \geq m^4 t \,. \qquad (8)$$

In the class we considered the case with all four masses equal. In the general unequal mass case $s + t + u = \sum_{i=1}^{4} m_i^2$. The second inequality in (8) was not derived. The second inequality (which is referred to as the Kibble inequality) was not derived. Its derivation can be found e.g. in Landau and Lifshitz, *Course of Theoretical Physics*, Vol. 4, Sec. 67. We discussed it in the case with all masses equal, when it takes the form $stu \geq 0$, implying that either s is positive and two other invariants are negative (s channel), or t is positive while s and u are negative (t channel), etc.

Let us discuss a convenient graphic representation of the domains of variation of s, t, and u. The first constraint in (8) allows one to eliminate, say, u in favor of s and t, but this is inconvenient because then one loses the symmetry between all these three kinematic invariants. S. Mandelstam suggested (in 1958) to use the so-called triangular coordinates in a plane (called the Mandelstam plane), see Fig. 2.

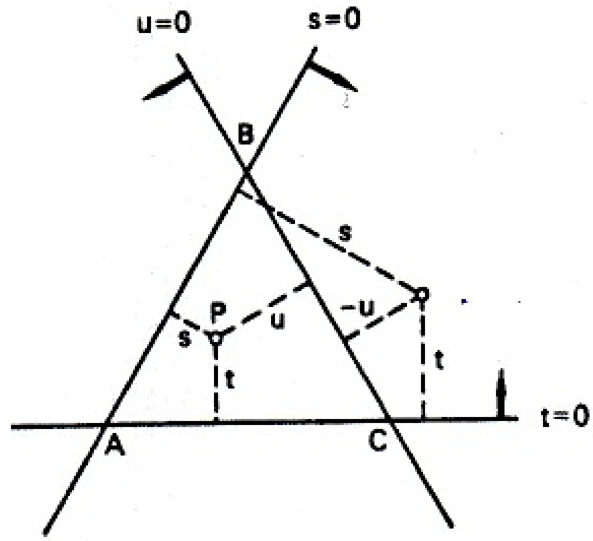

Figure 2: The Mandelstam plane.

The coordinate axis on the Mandelstam plane are three straight lines which intersect to form an equilateral triangle. The coordinates s, t, u are measured along directions perpendicular to these three lines; the directions towards the interior of the triangle are reckoned positive, as shown by arrows in Fig. 2. Each point on the plane has corresponding values of s, t, u which are represented (with the appropriate signs) by the lengths of the perpendiculars to the three axes. The condition $s+t+u = \sum_{i=1}^{4} m_i^2$ is satisfied on account of a known theorem of geometry, $\sum_{i=1}^{4} m_i^2$ being equal to the altitude of the triangle.

The physical domains for (1), (3) and (4) are shown in Fig. 3.

Because the photon and electron masses are different, in the primary channel (channel I) s is positive, t is negative, but u can be both positive and negative. The boundary $u \geq 0$ is only asymptotic, it is approached at $s \to \infty$ (i.e. $m \to 0$).

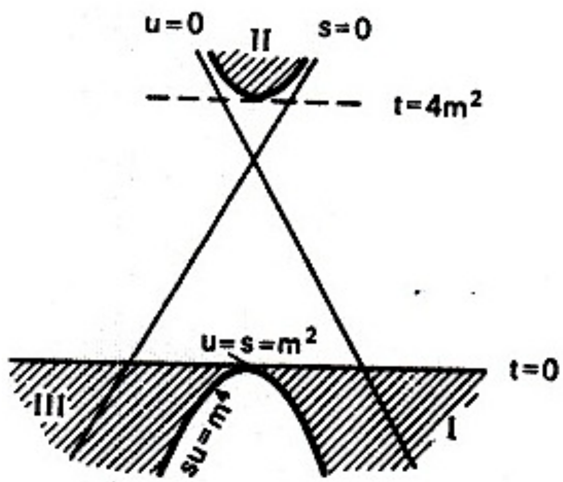

Figure 3: Three domains I, II, and III in the Comptom scattering (see also (3) and (4)).