Around SuSy 1970

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Let me turn back to look at the past, and tell you about the history of construction of the first supersymmetric model. The papers I will shortly review have been for the most part rather inaccessible. A part of the results obtained is presented only in my Ph.D. thesis. I will also mention some lost opportunities and delusions.

I graduated from the Physical Department of Moscow State University obtaining the diploma with honors in 1968. At that time, I was a co-author of two publications on some plausible charge-parity breaking effects in atomic nuclei. I dreamed of solving more ambitious problems, but this required becoming a postgraduate student. However my diploma supervisor, Yuri Shirokov, could not arrange my postgraduate studies — neither at the Physics Department of Moscow State University, where he was a Professor, nor at the Mathematics Institute of the Academy of Science, where he was a staff member. Therefore, he recommended me to Yuri Golfand who was working in the Theory Department of the Lebedev Physical Institute of the Academy of Sciences.

In the late 1960's — early 70's, the staff members of the Theoretical Department were famous Soviet physicists such as Igor Tamm, Vitaliy Ginzburg,and Andrey Sakharov. At that time Golfand was still preparing his doctoral thesis [1]. A moderate height, a rapid gait and a charming well-wishing smile were to level down the differences between our ages and positions. He showed me several commutation and anticommutation relations between the operators of momentum, angular momentum and some spinors and explained to me that their consistency was verified by Jacobi identities. These commutation relations provided the basis for the science, which in six years was called the SuperSymmetry.

At the first stage I had to establish whether the proposed algebra was unique or whether there were some alternatives. In order to do this, it was necessary to solve a system of equations for the algebra structure constants, which follow from the Jacobi identities. I restricted myself to the set of four complex spinor charges and found that there were four versions of such algebras: two of them are now known as $\mathcal{N}=1$ and $\mathcal{N}=2$ superalgebras, in two others the momentum did not commute with spinor charges (à la de-Sitter algebra). These results were obtained in 1968 but were published later only in 1972, because their relation to physics was not immediately clear. We chose the simplest ($\mathcal{N}=1$) supersymmetric algebra for further investigations.

Twice a week, before seminars of the Theoretical Department, I showed the result of my calculations to Golfand, and we discussed them. At that time I used text-books on different aspects of applying group theory to physics, but I did not read any new publications. Later, when I was preparing my Ph.D. thesis, I had to write a review of modern literature on the subject, and I faced difficulties again. Just recently I have read the historical review "Revealing the path to the superworld" by Marinov in the Golfand Memorial Volume [2] with great interest. I don't know whether Golfand knew about Felix Berezin's papers or any other publications on the subject. Maybe, he believed, that he had already informed me about all that was necessary for our work. By all his visual appearance Golfand gave me, and probably not only me, an idea that, contrary to others, we worked on a very serious and important problem. At the same time, he seemed to have good relations with all staff members of the Theoretical Department. He liked jokes and banters very much, in particular about gaps in my education.

The main problem to be solved was to relate the constructed algebra to quantum field theory. Nobody knew whether such a relation existed, and if yes, whether its representations were finite dimensional. I use here the word "algebra" instead of "group" for the following reason. We introduced the notion of supercoordinates and considered the group with Grassmann parameters and established the supercoordinate transformations under spinor translations. But we didn't guess to expand the superfield with respect to the Grassmann variables and to establish the relation between the superfield and a set of the usual fields forming the supermultiplet. The notion of the Superfield was not presented in the published papers [3,4], because for us it did not provide a mechanism for constructing a superinteraction; it was presented in my Ph.D. thesis only. The way to success, as it seems to be now, was not so elegant and, therefore, more laborious. I began to seek the representations of the algebra in terms of creation and annihilation operators.

As long as we had to do with the algebra, the main problem was to properly handle the gamma-matrices and not to forget to change the signs in certain places of the Jacobi identities while dealing with anticommutators. When we turned to field theory, we found it very difficult to get accustomed to the fact that both bosons and fermions are in the same multiplet and even more difficult to imagine how they transform into each other under spinor translation. There was no way to get an answer from a crib as I used to do at my exams on the Marx-Lenin philosophy. Golfand encouraged me: "A cat may look at a king." The only thing that was clear from the very beginning was that all particles in the multiplet have equal masses, but this fact didn't give optimism.

At last, in 1969 the superspin operator and two irreducible representations of algebra (chiral multiplet of spins zero and one half and vector multiplet of spins zero and one half and one) were constructed and following general properties of irreducible representations were established:

- First, the maximum spin in any irreducible representation differs from minimum one by not more than one. This conclusion was obtained by expanding the group operator (not the superfield) in Taylor series with respect to Grassmann parameters of spinor translations. The expansion turned out to have a cut-off and therefore the irreducible representations were finite!

- The second discovered property was that the number of bosonic and fermionic degrees of freedom in every multiplet was the same, and, as a consequence, the total vacuum energy of bosons and fermions was equal to zero.

I cite here the Lebedev Physical Institute preprint #41, 1971 (see Appendix), of which the only reader was probably Misha Shifman. I wanted very much to publish the obtained results immediately, in 1969, but Golfand believed that they would attract no attention. Moreover, he didn't want to lose time on the manuscript preparation. Therefore, I began constructing an interaction of the found multiplet.

At that time the Academy of Science was receiving letters from inventors with requests to make examination of their projects. Post graduate students had to make sense of them and to give a conclusion. I was given a project of a rocket, where liquid moved inside the rocket in a closed tube: straight in one direction and in the zigzag fashion in the opposite direction. The author of the project believed that due to the relativistic effects a force has to arise and push the rocket forward. I could not point out to the author that this contradicted the momentum conservation law, because the author, evidently, did not know about it. I had to spend a lot of time in order to find mistakes in his considerations. After this case I was afraid to find myself in a similar situation, in the position of luckless inventor.

The psychological barrier associated with the fermi-bose mixture was broken, but some technical difficulties arose. The method of constructing interactions was to write down a general form of spinor generators of the algebra not only in terms of the second but also the third powers of the fields. Consequently, the number of unknown constants in front of different combinations of fields was about a dozen. Today I solve about eight hundred of nonlinear equations with ease and have problems with the computer only if the number of equations exceeds a thousand. To solve the equations in 1970 I used pen and the reverse side of blueprints of old drawings. It is easy to solve a problem from a text-book, when you know, that the answer does exist and the only thing to do is to find it. When one of equations contradicted the others I didn't know whether this was the result of arithmetic mistakes or whether the problem had no solution at all. My postgraduate term went to finish and I had to think about my Ph.D. thesis and future job. Finally, when constants obtained from one of the equations appeared to satisfy the others, I took it as a miracle. Moreover, the unknown constants at the fourth powers of fields in the spin translation operators could be set to zero. The system was solved, the first superinteraction was constructed. Now it is known as massive supersymmetric electrodynamics.

The time came to draw the results up. I was going to write down one big consistent paper, but Golfand took another decision. Time of publication in our journals was very large, and he decided to write down a short paper for JETP Letters [3]. Golfand cut down my manuscript without any pity to fit the required volume. Deleted fragments were scattered around in other publications. At the same time I constructed the selfinteraction for the vector multiplet. I succeeded in constructing only the trilinear part of the interaction, therefore the result was presented only in my Ph.D. thesis. The title of the thesis was so complicated, that during the defense of my Ph.D. thesis in September 1971, the scientific secretary of our institute faced difficulties in reading it aloud. The title seemed to reflect our understanding of the problem. We erroneously believed, that constructing supersymmetric interaction through the power series expansion of group generators in the coupling constant was related to specific features of supersymmetry. Nevertheless, summarizing the results of three years' work I conclude that under the continual support by Golfand, I had solved the problem formulated by him, that is I demonstrated the quantum field realization of supersymmetry by a specific example.

What stimulated Golfand to formulate the problem? It is clear that it was not any specific experimental result such as constant speed of light or the approximate equality of the neutron and proton masses. There were also no questions of overcoming some internal contradictions in field theory as it took place in constructing general relativity or the Weinberg-Salam model. I think that he was inspired by numerous positive results which were obtained by applying one or another symmetry in physics during the 20th century.

At the end of 1971 I got a job at the Physics Department of All-Union Institute of Scientific and Technical Information and hardly found time to continue my work on fermi-bose symmetry. Golfand's position was even worse: he lost his job and became unemployed.

I really hated divergences in quantum field theory, and the hypothesis came to me that the cancelation of the vacuum energies divergences for free fields, that I found, would take place for interacting fields as well. Knowing coupling constants it was easy to establish that the one-loop mass divergences of bosonic fields in constructed model will not be quadratic, but logarithmic as for fermionic field. Golfand was not excited by this result because he believed it happened just by chance. Maybe his intuition left him, but maybe his mind was occupied by completely different non-scientific problems. Considering the action of spinor transforms on S-matrix, I came to the conclusion, that the cancelation of the divergences was not accidental and had to take place in higher orders. However, the divergences did not disappear completely. What if the nonrenormalizable theory became renormalizable one? Unfortunately, with my methods for constructing models there were no chances to treat nonrenormalizable models.

In 1974 Igor Tyutin told me about the Wess-Zumino paper which I have read with great interest. The linear realization of representations owing to the presence of auxiliary fields, covariant derivatives with respect on Grassmann variables — it was very beautiful, and allowed one to construct the superinteraction rather easily. Integration over Grassmann variables, used in the paper by Salam-Strathdee, finally set both types of arguments of superfields on equal footing. The postulates of this integration were formulated by Felix Berezin in 1965. Using these techniques I constructed two models with the Abelian and non-Abelian massive vector fields and proved their renormalizability. At the same time I also noted that the proof of renormalizability could be carried out in another way, using the standard mechanism of spontaneous gauge symmetry breaking. The question of any relation of supersymmetry to high energy physics remained open.

After the publication of these two papers [5], I practically stopped my work on supersymmetry because of strong competition and the absence of any support. I began to seek my own unique topic. A small number of assumptions and the simplicity of constructions, together with nontrivial and maybe not very realistic (at first glance) results — that was the main impression, which I gained from my collaboration with Golfand. On the other hand, my ability to operate the personal computers allows me to try to solve problems, while I do not even know about the existence of their solutions. I heard that to seek a black cat in a dark room is very risky business, especially in its absence. Nevertheless, today I am trying to study numerically the model of Born-Infeld electromagnetic field interacting with a membrane of finite size and with a string boundary, where it is beforehand known that the divergences of electric and magnetic energies cancel each other. But this cancelation is not the final aim. It appears that some combination of mass, electric charge and magnetic momentum is independent on two unknown dimensional constants of the model on the one hand and, on the other hand, is related to well-known observable value — the fine structure constant. At the present time, the solutions obtained slowly converge with the growth of the number of the lattice cells. Maybe this points to the absence of solutions, or maybe it is necessary to arrange the lattice better, but maybe it is necessary to seek the unknown symmetry of the equations of motion to use it for preliminary analytical analysis.

In conclusion I would like to thank the organizers of the conference *Thirty Years of Supersymmetry* for giving me the opportunity to share my vision of thirty-year old events, and for financial support. I would like to express my gratitude to Misha Shifman for comments to my notes in the Golfand Memorial Volume [2]. These comments made some aspects of the Soviet life in the 1970's more palpable for the Western readers. And, the last but not the least, Yuri Golfand remains for me not only the man who taught me a trade, but the man who made me love the risky business — not to be afraid to go deep in the forest by the paths nobody walked on before. Thank you for your attention.

REFERENCES

1. The academic hierarchy in Russia follows the German rather than the Anglo-American pattern. An approximate equivalent of Ph.D. in the US is the so called *candidate* degree. The highest academic degree, doctoral, is analogous to the German *Habilitation*. The doctoral dissertation is usually presented at a mature stage of the academic career; only a fraction of the *candidate* degree holders make it to the doctoral level.
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APPENDIX

At the suggestion of Misha Shifman I present below a *verbatim* translation of my paper written in 1970 which circulated as FIAN preprint No. 41 (1971). The final authorization for issuing this preprint was obtained on April 12, 1971.

LEBEDEV PHYSICS INSTITUTE OF THE USSR ACADEMY OF SCIENCES

Theory Department

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IRREDUCIBLE REPRESENTATIONS OF THE EXTENSION OF THE ALGEBRA OF GENERATORS OF THE POINCARÉ GROUP BY BISPINOR GENERATORS

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1. INTRODUCTION

In Ref. 1 a special extension of the algebra $\mathcal{P}$ of the generators of the Poincaré group was considered. The extension was performed by virtue of introduction of the generators of the spinorial translations W_α and $\bar{W}_\beta$,

$$\begin{aligned}
&[M_{\mu\nu}, M_{\sigma\lambda}]_- = i(\delta_{\mu\sigma}M_{\nu\lambda} + \delta_{\nu\lambda}M_{\mu\sigma} \\
&\qquad - \delta_{\mu\lambda}M_{\nu\sigma} - \delta_{\nu\sigma}M_{\mu\lambda});\\
&[P_\mu, P_\nu]_- = 0;\\
&[M_{\mu\nu}, P_\lambda]_- = i(\delta_{\mu\lambda}P_\nu - \delta_{\nu\lambda}P_\mu);\\
&[M_{\mu\nu}, W]_- = \frac{i}{4}[\gamma_\mu, \gamma_\nu]W; \qquad\qquad \text{(1a)}
\end{aligned}$$

$$\begin{aligned}
&\bar{W} = W^+\gamma_0;\\
&[W, \bar{W}]_+ = \gamma_\mu^+ P_\mu;\\
&[W, W]_+ = 0; \qquad [P_\mu, W]_- = 0; \qquad\qquad \text{(1b)}\\
&\gamma_\mu^\pm = s^\pm\gamma_\mu; \qquad s^\pm = \frac{1}{2}(1 \pm \gamma_5), \qquad \gamma_5^2 = 1.
\end{aligned}$$

The spinorial indices are omitted, while the expressions of the type $d_\mu d_\nu$ are to be understood as $d_0d_0 - d_1d_1 - d_2d_2 - d_3d_3$ hereafter. In the same work a realization of this algebra was constructed in which a Hamiltonian describes interactions of quantum fields. This example shows that algebra (1) imposes rigid constraints on the form of the quantum field interaction. In constructing this example we used two linear irreducible representations of the algebra (1). Their definition was not given in Ref. 1. In this work I present the definition of these representations of algebra (1), and build other representations. The properties of these representations are investigated.

2. THE SPACE OF STATES AND INVARIANT SUBSPACES

In order to determine in which space the representations of algebra (1) act note that the algebra $\mathcal{P}$ is its subalgebra. Therefore, any representation of algebra (1) is also a representation of the algebra $\mathcal{P}$. The spaces in which these representations act coincide. However, irreducible representations of algebra (1) are reducible with respect to $\mathcal{P}$. This reducible representation splits in several irreducible representations of $\mathcal{P}$ and one and the same irreducible representation can be involved several times. To distinguish them we will introduce the number χ of the irreducible representations.

Of physical interest are those representations of (1) which can be reduced to (several) representations of $\mathcal{P}$ characterized by mass and spin. therefore, the basis vector of the space in which we build representations of (1) can be written as follows:

$$|\kappa, p_\lambda, j, m, \chi\rangle, \tag{2}$$

where κ is the mass, p_λ is the spatial momentum, j stands for the spin, m is its projection on the z axis, and, finally, χ is the number of the irreducible representation of $\mathcal{P}$.

In the space with the basis vectors (2) there are subspaces invariant under the action of the operators from the algebra in Eq. (1). According to Schur's lemma [2], in order to find invariant subspaces it is necessary to find invariant operators which, by definition, commute with all operations of algebra (1). It is easy to observe that the operator P_μ^2 does have this property. Therefore, the space of which the basis vectors correspond to particles of one and the same mass κ will be the invariant subspace. The spins of the vectors of this invariant subspace cannot be all the same, since the square of the spin operator is not an invariant operator of algebra (1),

$$[\Gamma_\mu^2, W]_- \neq 0; \qquad (\Gamma_\mu = \frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma} M_{\nu\lambda} P_\sigma).$$

Instead, the invariant operator of algebra (1) is D_μ^2 where

$$D_\mu = \Gamma_\mu + \frac{1}{2}(\bar{W}\gamma_\mu W - \frac{P_\mu P_\nu}{P_\sigma^2}\bar{W}\gamma_\nu W),$$
$$P_\sigma^2 = \kappa^2 > 0.$$

Other invariant operators in algebra (1) are likely to be absent.

A priori it is not exactly known which vectors (2) form the basis of the irreducible representation of (1), since the properties of the operator D_μ^2 are not known. Besides the fact that the mass of all states is the same, one can assert that the difference between the maximal and minimal spins is the irreducible representation of (1) does not exceed 1. In other words, irreducible representations of (1) contain no more than 3 distinct spins. Otherwise, by consecutively applying the operators from algebra (1) to the state vectors with spin j_1 one could obtain a vector with a nonvanishing projection on the state vectors with spin j_2, $|j_2 - j_1| > 1$. To see that this is impossible note that the consecutive action of the operators from algebra (1) in the most general case can be represented as a polynomial in these operators. Due to the (anti)commutation relations (1) the form of this polynomial is

$$\sum C^{\alpha,\ldots;\beta,\ldots}_{\mu\nu,\ldots;\lambda\ldots}$$
$$\times\ M_{\mu\nu}\times\ldots\times P_\lambda\times\ldots\times W_\alpha\times\ldots\times\bar{W}_\beta\times\ldots,$$

where $C^{\alpha,\ldots;\beta,\ldots}_{\mu\nu,\ldots;\lambda\ldots}$ are numerical coefficients.

The product of any number of operators $M_{\mu\nu}$ and P_λ does not change the spin of the state. The number of operators W (and $\bar{W}$) in each term can be equal to one or two since if it is larger than 2 then the product vanishes because of the anticommutation relation in Eq. (1). By the same reason the product $W_\alpha W_\beta \neq 0$ only provided that $\alpha \neq \beta$. Such operator does not change the spin of the state as well. Thus, an invariant subspace can contain only the spins j, $j+\frac{1}{2}$, $j+1$.

In order to build the irreducible representation of algebra (1) it is necessary to find the matrix elements of the operators from algebra (1) between these states.

3. EQUATIONS FOR THE REDUCED MATRIX STATES

The matrix element of the operator of conventional translations can obviously be written as

$$\begin{aligned}\langle\kappa, p_\lambda, j, m, \chi|P_\mu|\kappa, p'_\lambda, j', m', \chi'\rangle \\ = \quad (\delta_{\mu 0}\sqrt{\kappa^2 + p_\lambda^2} + \delta_{\mu\lambda}p_\lambda) \\ \times \quad \delta(p_\lambda - p'_\lambda)\delta_{jj'}\delta_{mm'}\delta_{\chi\chi'},\end{aligned} \tag{3}$$

where $\lambda = 1, 2, 3$. The operator $M_{\mu\nu}$ will have its conventional form as well. We will be interested in the matrix elements of the operators W_α and $\bar{W}_\beta$, which, due to Eq. (1), are diagonal with respect to κ and p_λ. One can readily convince oneself that the operators $s^- W$ and $\bar{W}s^+$ satisfy the trivial commutation relations. Therefore, we will limit ourselves to investigating such representations in which [1]

$$s^- W = \bar{s}^+ = 0.$$

Without loss of generality let us choose the representation of the γ matrices with a diagonal γ_5. Then the operators W_α and $\bar{W}_\beta$ become two-component. Moreover, knowing the transformation law of spinors under the Lorentz transformations, and assuming that $\kappa > 0$, let us pass to the reference frame with $p_\lambda = p'_\lambda = 0$. In this reference frame one could apply the Wigner-Eckart theorem [4] according to which

$$\begin{aligned}
&\langle \kappa, 0, j, m, \chi | s^+ W_\alpha | \kappa, 0, j', m', \chi' \rangle \\
&\quad = \begin{pmatrix} j & \frac{1}{2} & j' \\ m & \alpha & -m' \end{pmatrix} (-1)^{j'-m'} \\
&\quad \times \sqrt{(2j+1)(2j'+1)} \langle j\chi | f | j'\chi' \rangle; \\
&\langle \kappa, 0, j, m, \chi | \bar{W} s^-_\beta | \kappa, 0, j', m', \chi' \rangle \\
&\quad = \begin{pmatrix} j' & \frac{1}{2} & j \\ m' & \beta & -m \end{pmatrix} (-1)^{j-m} \\
&\quad \times \sqrt{(2j+1)(2j'+1)} \langle j\chi | f^+ | j'\chi' \rangle, \qquad (4)
\end{aligned}$$

where $\begin{pmatrix} j & \frac{1}{2} & j' \\ m & \alpha & -m' \end{pmatrix}$ are the Wigner symbols, $|j - j'| = \frac{1}{2}$, $\langle j\chi | f | j'\chi' \rangle$ are the reduced matrix elements, $\sqrt{(2j+1)(2j'+1)}$ is a convenient normalization factor. The representation (4) ensures the correct commutation relation with the momentum operator and the operator of the spatial rotations. In order to satisfy other commutation relations of algebra (1), we substitute (3) and (4) in (1b) and exploit the formulae of summation of the $3j$ symbols in the spin projections [4],

$$\begin{aligned}
&\sum_{m'} (-1)^{j'+m} \\
&\quad \times \begin{pmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m' \end{pmatrix} \begin{pmatrix} j_3 & j_4 & j' \\ m_3 & m_4 & -m' \end{pmatrix} \\
&\quad = \sum_{JM} (-1)^{2j_4+J+M} (2J+1) \\
&\quad \times \begin{Bmatrix} j_1 & j_2 & j' \\ j_3 & j_4 & J \end{Bmatrix} \begin{pmatrix} j_3 & j_2 & J \\ m_3 & m_2 & M \end{pmatrix} \\
&\quad \times \begin{pmatrix} j_1 & j_4 & J \\ m_1 & m_4 & -M \end{pmatrix},
\end{aligned}$$

[1]Relaxing this requirement would require, of necessity, the introduction of an indefinite metric [3].

where $\begin{Bmatrix} j_1 & j_2 & j' \\ j_3 & j_4 & J \end{Bmatrix}$ is the $6j$ symbol. As a result of this substitution we obtain, after performing the summation,

$$\begin{aligned}
&\sum_{JMj'\chi'} (-1)^{2j''+J+M} (2J+1) \\
&\quad \times \begin{Bmatrix} j & j' & J \\ \frac{1}{2} & \frac{1}{2} & j' \end{Bmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & J \\ -\beta & \alpha & M \end{pmatrix} \\
&\quad \times \begin{pmatrix} j & j'' & J \\ m & -m'' & -M \end{pmatrix} \\
&\quad \times (-1)^{m+\alpha-j'} \sqrt{(2j+1)(2j''+1)(2j'+1)^2} \\
&\quad \times \Big(\langle j\chi | f | j'\chi' \rangle \langle j'\chi' | f^+ | j''\chi'' \rangle \\
&\quad + (-1)^{J-j+j''} \langle j\chi | f^+ | j'\chi' \rangle \langle j'\chi' | f | j''\chi'' \rangle \Big) \\
&\quad = \kappa \delta_{jj''} \delta_{mm''} \delta_{\chi\chi''}. \qquad (5)
\end{aligned}$$

(An analogous formula is obtained upon substitution of Eq. (4) in the anticommutation relation $[W, W]_+ = 0$.) Next we use the values of the $6j$ symbols [4],

$$\begin{aligned}
&\begin{Bmatrix} j_1 & j_2 & j_3 \\ \frac{1}{2} & j_3 - \frac{1}{2} & j_2 + \frac{1}{2} \end{Bmatrix} = (-1)^{j_1+j_2+j_3} \\
&\quad \times \left[\frac{(j_1 + j_3 - j_2)(j_1 + j_2 - j_3 + 1)}{(2j_2+1)(2j_2+2)(2j_3)(2j_3+1)} \right]^{\frac{1}{2}}; \\
&\begin{Bmatrix} j_1 & j_2 & j_3 \\ \frac{1}{2} & j_3 - \frac{1}{2} & j_2 - \frac{1}{2} \end{Bmatrix} = (-1)^{j_1+j_2+j_3} \\
&\quad \times \left[\frac{(j_1 + j_2 + j_3 + 1)(j_2 + j_3 - j_1)}{(2j_2)(2j_2+1)(2j_3)(2j_3+1)} \right]^{\frac{1}{2}}.
\end{aligned}$$

Then Eq. (5) takes the form

$$\begin{aligned}
&\sum_{j'\chi'} \left(\frac{2j'+1}{2} \right) \Big(\langle j\chi | f | j'\chi' \rangle \langle j'\chi' | f^+ | j''\chi'' \rangle \\
&\quad + \langle j\chi | f^+ | j'\chi' \rangle \langle j'\chi' | f | j''\chi'' \rangle \Big) \\
&\quad = \kappa \delta_{jj''} \delta_{\chi\chi''}; \qquad (6a) \\
&\sum_{j'\chi'} (-1)^{j'} \Big(\langle j\chi | f | j'\chi' \rangle \langle j'\chi' | f^+ | j''\chi'' \rangle \\
&\quad - \langle j\chi | f^+ | j'\chi' \rangle \langle j'\chi' | f | j''\chi'' \rangle \Big) \\
&\quad = 0, \quad \text{for } j \neq 0; \qquad (6b) \\
&\sum_{j'\chi'} \langle j\chi | f | j'\chi' \rangle \langle j'\chi' | f^+ | j''\chi'' \rangle \\
&\quad = 0, \quad \text{except for } j = j'' = 0. \qquad (6c)
\end{aligned}$$

These are the equations that were our goal. Their solution will allow us to find the explicit form of s^+W and $\bar{W}s^-$. Note that Eq. (6) describes the representation in which the invariant operator D_μ^2 need not necessarily be proportional to the unit operator, i.e. we get reducible representations of algebra (1), generally speaking.

4. THE NUMBER OF PARTICLES IN THE REPRESENTATION OF ALGEBRA (1) AND SOME SOLUTIONS OF EQ. (6)

First of all, starting from Eq. (6) I will deduce constraints on the number of particles in the representation of algebra (1). To this end let us multiply Eq. (6a) by $(-1)^{2j}(2j+1)/2$ and sum over $j = j''$ and $\chi = \chi''$. After this operation the left-hand side will vanish. To see that this is the case it is sufficient to transpose two factors in the second term (which is justified since this is inside the trace) and to use the fact that $(-1)^{2j} = (-1)^{2j'+1}$ (see Eq. (4)). Then the second term will differ from the first one by the sign only. The right-hand side of Eq. (6a) must also vanish, and we obtain the constraint on the number n of the particles with spin j in the representation of algebra (1),

$$\sum_j (-1)^{2j}(2j+1)n_j = 0. \tag{7}$$

As is known [5], in relativistic quantum field theory, the particle energy operator, being transformed to the normal form, contains an infinite term which is interpreted as the vacuum energy. It is also known that the sign of this term is different for particles subject to the Bose and Fermi statistics. According to Eq. (7), the representations of algebra (1) include particles with different statistics, so that the number of the boson states is always equal to that of the fermion states. From this it follows that the infinite positive energy of the boson states is canceled by the infinite negative energy of the fermion states in any representation of algebra (1).

After these preliminary remarks we proceed directly to solving Eq. (6). Let us try to find representation in which only particles with two distinct spin values enter. In this case j' in Eq. (6) takes only one value, and the summation over j' is, in fact, absent. Using this fact, let us multiply Eq. (6b) by $(-1)^{j'}(2j'+1)/2$ and add up the result with Eq. (6a). Then on the right-hand side one will find a nonsingular matrix acting in the space of the variable χ (at fixed j, j', and j''). On the left-hand side the matrix will be nonsingular only if $j \neq 0$. (Cf. Eq. (6c)). Therefore, the representation of algebra (1) with two spins can contain only spin-0 and spin-1/2 states. It is easy to check that in this case the simplest solution of the system with $n_0 = 2$ ($\chi = 1, 2$) and $n_{1/2} = 1$ ($\chi = 1$) has the form

$$\langle j\chi|f|j'\chi'\rangle = \sqrt{\kappa}\left(\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \hline 0 & 1 & 0 \end{array}\right), \tag{8}$$

where the matrix on the right-hand side acts on the state

$$\left(\begin{array}{c} a \\ b \\ \hline c \end{array}\right).$$

The amplitudes a and b describe the spin-0 particles while c describes the spin-1/2 particle.

In the case of the three-spin representations, j, $j+1/2$, and $j+1$, the lowest spin j can be arbitrary. The simplest solution with $n_j = 1$ ($\chi = 1$), $n_{j+1/2} = 2$ ($\chi = 1, 2$), and $n_{j+1} = 1$ ($\chi = 1$) can be written in a form analogous to (8),

$$\langle j\chi|f|j'\chi'\rangle = \sqrt{\kappa}\left(\begin{array}{c|cc|c} 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \end{array}\right). \tag{9}$$

The representations (8) and (9) are irreducible. One can readily convince oneself that this is the case even without calculating the eigenvalues of the operator D_μ^2. It is enough to observe that there exist no representations with a lesser number of particles satisfying the necessary condition (7). The question of existence/nonexistence of other irreducible representations of (1), besides those found here, requires further investigation.

5. SECOND-QUANTIZED RELATIVISTIC REPRESENTATIONS OF ALGEBRA (1)

In this section we will show how one can pass from the representations (8), (9) acting in the space with the basis vectors (2) to the relativistic-covariant form of these representations, acting in the space of the occupation numbers. The operators of the algebra in such representations have to be expressible in terms of the second-quantized free fields with equal masses but distinct spins. Since relativistic equation for spin-1/2 particles describes both particles and antiparticles, it is necessary to introduce antiparticles in the representation (7) . Then the operators of the algebra will be expressible in terms of non-Hermitian free scalar fields $\varphi(x)$, $\omega(x)$, and a spinor field $\psi_1(x)$. Let us show that the operator

$$W^o = s^+W^o = \frac{1}{i}\int\Big(\varphi^*(x)\overset{\leftrightarrow}{\partial_0} s^+\psi_1(x) + \omega(x)\overset{\leftrightarrow}{\partial_0} s^+\psi_1^c(x)\Big)d^3x, \qquad (10)$$

(where the superscript o means that the operator is bilinear in the field operators while the superscript c means the charge conjugation) satisfies [2] (anti)commutation relations (1). For instance,

$$\begin{aligned}
[W^o,\bar W^o]_+ &= i\iint d^3x\, d^3y \\
&\times \Big(\varphi^*(x)\overset{\leftrightarrow}{\partial_{x_0}}\gamma_\mu^+ i\partial_{x_\mu}D(x-y)\overset{\leftrightarrow}{\partial_{y_0}}\varphi(y)\Big)\\
&+ i\iint d^3x\, d^3y\\
&\times \Big(\omega(x)\overset{\leftrightarrow}{\partial_{x_0}}\gamma_\mu^+ i\partial_{x_\mu}D(x-y)\overset{\leftrightarrow}{\partial_{y_0}}\omega^*(y)\Big)\\
&+ i\iint d^3x\, d^3y\\
&\times \Big(\bar\psi_1(y)s^-\times\overset{\leftrightarrow}{\partial_{y_0}}D(y-x)\overset{\leftrightarrow}{\partial_{x_0}}s^+\psi_1(x)\Big)\\
&+ i\iint d^3x\, d^3y\\
&\times \Big(\bar\psi_1^c(y)s^-\times\overset{\leftrightarrow}{\partial_{y_0}}D(y-x)\overset{\leftrightarrow}{\partial_{x_0}}s^+\psi_1^c(x)\Big)\\
&= -\int d^3x\Big(\varphi^*(x)\partial_\mu\overset{\leftrightarrow}{\partial_0}\varphi(x)\Big)\times\gamma_\mu^+\\
&- \int d^3x\Big(\omega^*(x)\partial_\mu\overset{\leftrightarrow}{\partial_0}\omega(x)\Big)\times\gamma_\mu^+\\
&+ \frac{i}{2}\int d^3x\Big(\bar\psi_1(x)\overset{\leftrightarrow}{\partial_0}\gamma_\mu^-\psi_1(x)\Big)\times\gamma_\mu^+\\
&+ \frac{i}{2}\int d^3x\Big(\bar\psi_1^c(x)\overset{\leftrightarrow}{\partial_0}\gamma_\mu^-\psi_1^c(x)\Big)\times\gamma_\mu^+\\
&= \int d^3x\, T_{\mu 0}(x)\times\gamma_\mu^+. \qquad (11)
\end{aligned}$$

[2]The operator W is defined up to a phase factor, see Ref. 3.

In this calculation I used the Fierz identity, the equations of motion, and the commutation relations for free fields. Let us further note that the energy-momentum tensor $T_{\alpha\beta}$ is nonsymmetric in the case of the spinor field, generally speaking. However, if one of the indices is zero, then

$$T_{\mu 0} = T_{0\mu},$$

and the integral on the right-hand side of Eq. (11) becomes the energy-momentum tensor of the fields $\varphi(x)$, $\omega(x)$, and $\psi_1(x)$. All other relations in (1) can be checked in a similar manner. The action of the operator W^o on the field operators is a linear transformation of these fields. Schematically, one can write it as follows:

$$\varphi(x)\to\psi(x)\to\omega(x)\to 0,$$
$$\omega^*(x)\to\psi^c(x)\to\varphi^*(x)\to 0.$$

Let us pass now to generalization of the representation (9) to cover the case of the quantized fields. We will limit ourselves to the option that the lowest spin is zero, while two spin 1/2 particles may be considered related by the operation of the charge conjugation. Then the operators of algebra (1) in this representation will be expressed in terms of a Hermitian scalar field $\chi(x)$, Hermitian vector transverse field $A_\mu(x)$, and a spinor field $\psi_2(x)$. This irreducible representation can differ from that in Eq. (10) by the mass of the particles and must differ by the eigenvalues of the invariant operator D_μ^2. The operator W^o in this representation has the form

$$W^o = s^+W^o = \frac{1}{i\sqrt2}\int\Big(\chi(x)\overset{\leftrightarrow}{\partial_0} s^+\psi_2(x) + A_\mu\overset{\leftrightarrow}{\partial_0}\gamma_\mu^+\psi_2(x)\Big)d^3x. \qquad (12)$$

One can verify Eq. (12) in the same manner as

Eq. (10),

$$
\begin{aligned}
[s^+W^o, \bar{W}^o s^-]_+ = &-\frac{1}{2i}\iint d^3x\, d^3y \\
\times & \left(\bar{\psi}_2(y)s^- \times \overset{\leftrightarrow}{\partial}_{y_0} D(y-x)\overset{\leftrightarrow}{\partial}_{x_0} s^+\psi_2(x)\right) \\
+ & \frac{1}{2i}\iint d^3x\, d^3y \Big(\psi_2(y)\gamma^+_\mu \times \overset{\leftrightarrow}{\partial}_{y_0} \\
\times & (\delta_{\mu\nu} + \frac{1}{\mu^2}\partial_{y_\mu}\partial_{y_\nu}) D(y-x)\overset{\leftrightarrow}{\partial}_{x_0} \gamma^+_\nu \psi_2(x)\Big) \\
- & \frac{1}{2}\int \left(\chi(x)\overset{\leftrightarrow}{\partial}_0 \partial_\mu \chi(x)\right) d^3x \times \gamma^+_\mu \\
- & \frac{1}{2}\int \left(A_\mu(x)\overset{\leftrightarrow}{\partial}_0 \partial_\alpha A_\nu(x)\right) d^3x \times \gamma^+_\mu \gamma^-_\alpha \gamma^+_\nu \\
= & \gamma^+_\mu P_\mu, \qquad (13)
\end{aligned}
$$

where $\mu \neq 0$ is the mass of the fields $\chi(x)$, $A_\mu(x)$, and $\psi_2(x)$. In this representation the action of the operator W^o on free fields can be schematically depicted as

$$
\psi^c_2(x) \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \chi(x) \\ \\ A_\mu(x) \end{array} \begin{array}{c} \searrow \\ \nearrow \end{array} \psi_2(x) \to 0.
$$

At intermediate stages the mass μ enters in the denominator in Eq. (13). Therefore, it cannot be set to zero. The limit $\mu \to 0$ can be realized by abandoning the condition of transversality of the vector field and passing to the diagonal pairing

$$
[A_\mu(x), A_\nu(y)]_- = -\frac{1}{2}\delta_{\mu\nu} D(x-y).
$$

In this case the field $\psi_2(x)$ becomes two-component ($s^+\psi_2 = 0$). The first term in intermediate calculations in (13) becomes unnecessary and disappears; therefore, there is no need in the introduction of the scalar field $\chi(x)$. At $\mu = 0$ the operator W^o has the following form

$$
\begin{aligned}
W^o &= s^+W^o \\
&= \frac{1}{i\sqrt{2}}\int \left(A_\mu(x)\overset{\leftrightarrow}{\partial}_0 \gamma^+_\mu \psi_2(x)\right) d^3x. \qquad (14)
\end{aligned}
$$

In Secs. 2–4 where I investigated the properties of the representations with nonvanishing mass, it was shown that the numbers of the fermion and boson states in the representations of algebra (1) coincide. Therefore, the operator P^o_μ is automatically representable in the normal form. This can be seen also from the fact that the action of the operators W^o and $\bar{W}^o$ on the vacuum always yields zero, and

$$
P_\mu = \mathrm{Tr}(\gamma^-_\mu [W^o, \bar{W}^o]_+).
$$

Therefore, the representation (14) also possesses this property, while the vector and spinor massless particles can only be in two states with the opposite chiralities.

6. CONCLUSION

Summarizing, we found several irreducible representations of algebra (1). In these representations conventional fields are united in certain multiplets. A question arises whether one can identify these multiplets with some observed particles. In answering this question the main difficulty lies in the fact that masses of all particles in the multiplet are identical, while spins are different. Therefore, at present algebra (1) and its realizations must be considered as just a model of a Hamiltonian formulation of quantum field theory.

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