THE BOSONIC-FERMIONIC DIAGONAL COINVARIANT MODULES CONJECTURE

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Abstract. We describe a general conjecture on how one may derive from the generic bosonic case all structural properties of multivariate diagonal coinvariant modules in any number (say k) of sets of n commuting variables (bosons), and any number (say j) of sets of n anticommuting variables (fermions).

1. Introduction

Much interesting work has been done recently on diagonal coinvariant spaces spaces in both commuting and anticommuting variables. See for instance [9, 10, 11, 15]. The purpose of this short note is to present a general conjecture expressing the fact that one can simply calculate all cases of multivariate diagonal coinvariant modules in k sets of n commuting variables (bosons), and j sets of n anticommuting variables (fermions), just from the generic case of multivariate diagonal coinvariant spaces.

2. Global setup

Let $\boldsymbol{x}$ and $\boldsymbol{\theta}$ respectively be matrices of variables; with $\boldsymbol{x} = (x_{ab})$ a $k \times n$ matrix, and $\boldsymbol{\theta} = (\theta_{cd})$ a $j \times n$ matrix. One may even assume that k and j are infinite. The variables in $\boldsymbol{x}$ commute with all variables (both in $\boldsymbol{x}$ and $\boldsymbol{\theta}$), whereas the variables in $\boldsymbol{\theta}$ are anticommuting among themselves, *i.e.* for θ and θ' in $\boldsymbol{\theta}$ one has $\theta\theta' = -\theta'\theta$. The $\boldsymbol{x}$-variables are said to be **bosomic**, and those in $\boldsymbol{\theta}$ to be **fermionic**. We consider that the ring of polynomials $\mathcal{R} = \mathcal{R}_n = \mathcal{R}_{k,j;n} := \mathbb{R}[\boldsymbol{x};\boldsymbol{\theta}]$ comes equipped with the group action (expressed here with matrix multiplication)

$$f(\boldsymbol{x};\boldsymbol{\theta}) \longmapsto f(P\,\boldsymbol{x}\,\sigma; Q\,\boldsymbol{\theta}\,\sigma),$$

with P and Q lying respectively in GL_k and GL_j, and $\sigma \in \mathbb{S}_n$ considered here as a $n \times n$ permutation matrix. Observe that σ acts similarly on both the $\boldsymbol{x}$ and $\boldsymbol{\theta}$ variables, by permuting columns. It has become usual to say that this is a diagonal action of $\mathbb{S}_n$. It is worth underlining that the three single actions commute.

As is often done, we denote by $\mathcal{R}^{\mathbb{S}_n}$ the subring of **invariants** of $\mathcal{R}$, *i.e.* the polynomials that are invariant under the (single) action of $\mathbb{S}_n$. The associated **coinvariant** module (often denoted by $\mathcal{R}_{\mathbb{S}_n}$, but here denoted otherwise for reasons that will become clear in the sequel) is then set to be the quotient

$$\mathbb{DBF}_{k,j;n} := \mathcal{R}_{k,j;n}/\langle \mathcal{R}_+^{\mathbb{S}_n} \rangle,$$

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with $\mathcal{R}_+^{\mathbb{S}_n}$ standing for the constant term free portion of $\mathcal{R}^{\mathbb{S}_n}$. Since $\mathcal{R}^{\mathbb{S}_n}$ is globally invariant under the action of $\mathcal{G} = \mathrm{GL}_k \times \mathrm{GL}_j \times \mathbb{S}_n$, there is an induced action of $\mathcal{G}$ on $\mathbb{DBF}_{k,j;n}$.

Consider any $\mathcal{V}$, which is a $\mathcal{G}$-submodule (or stable quotient module) of $\mathcal{R} = \mathcal{R}_{k,j;n}$. As is well known, the decomposition of $\mathcal{V}$ into irreducibles of is entirely encoded in the symmetric function expression[1]

$$\mathcal{V}(\boldsymbol{q};\boldsymbol{u};\boldsymbol{z}) := \sum_{\mu\vdash n}\sum_{\lambda,\rho} v_{\lambda\rho\mu} s_\lambda(\boldsymbol{q}) s_\rho(\boldsymbol{u}) s_\mu(\boldsymbol{z}), \tag{2.1}$$

where $s_\lambda(\boldsymbol{q})$ and $s_\rho(\boldsymbol{u})$ are respectively characters for (polynomial) irreducible representations of GL_k and GL_j expressed as functions of $\boldsymbol{q} = q_1,\ldots,q_k$ and $\boldsymbol{u} = u_1,\ldots,u_j$; and $s_\mu(\boldsymbol{z})$ is the Frobenius transform of a $\mathbb{S}_n$-irreducible, with $\boldsymbol{z} = z_1, z_2, \ldots$. The **hilbert series**[2] (or **graded character**) of $\mathcal{V}$ is:

$$\mathcal{V}(\boldsymbol{q};\boldsymbol{u}) := \sum_{\mu\vdash n}\sum_{\lambda,\rho} v_{\lambda\rho\mu} s_\lambda(\boldsymbol{q}) s_\rho(\boldsymbol{u}) f^\mu, \tag{2.2}$$

where f^μ is the dimension of the irreducible associated to s_μ, which is well known to be equal to the number of standard tableaux of shape μ. Observe that the passage to the Hilbert series $\mathcal{V}(\boldsymbol{q};\boldsymbol{u})$ is obtained by removing $\boldsymbol{z}$ in $\mathcal{V}(\boldsymbol{q};\boldsymbol{u};\boldsymbol{z})$. For the modules that we will consider, the coefficients $c_{\lambda\rho\mu}$ do not depend on k and j. The dependence on k and j is rather reflected in the fact that some of the functions $s_\lambda(\boldsymbol{q})$ and $s_\rho(\boldsymbol{u})$ when the number of variables is too small, *i.e.* k (resp. j) is less than the number of parts of λ (resp. ρ). In other words, the stable expression for $\mathcal{V}(\boldsymbol{q};\boldsymbol{u};\boldsymbol{z})$ is obtained whenever k and j become large enough[3]. Such modules are said to be **coefficient stable**.

When this is the case, it is often useful to write (2.1) in the form of a “variable free” expression:

$$\mathcal{V} := \sum_{\mu\vdash n}\sum_{\lambda,\rho} v_{\lambda\rho\mu}\, s_\lambda \otimes s_\rho \otimes s_\mu.$$

Using plethystic notation[4],

Proposition 1. *One has*

$$\begin{aligned} \mathcal{R}_n(\boldsymbol{q};\boldsymbol{u};\boldsymbol{z}) &= h_n[\Omega[\boldsymbol{q} - \varepsilon\,\boldsymbol{u}]\,\boldsymbol{z}] &(2.3)\\ &= \sum_{\mu\vdash n} s_\mu[\Omega[\boldsymbol{q} - \varepsilon\,\boldsymbol{u}]]\, s_\mu(\boldsymbol{z}), &(2.4)\end{aligned}$$

with $\Omega = \sum_{i\geq 0} h_i$.

For the above plethystic calculation, ε is defined to be such that $p_i[\varepsilon] = (-1)^i$, so that $p_i[-\varepsilon\,\boldsymbol{u}] = \omega\, p_i(\boldsymbol{u})$. In particular, by the classical summation formula for Schur functions, one gets

$$s_\theta[\boldsymbol{q} - \varepsilon\,\boldsymbol{u}] = \sum_{\nu\subseteq\theta} s_\nu(\boldsymbol{q}) s_{\theta'/\nu'}(\boldsymbol{u}).$$

[1] Observe that various sets of variables are separated by semi-colons.

[2] Observe that, in our notation for the passage to the Hilbert series, we simply "drop" the variables $\boldsymbol{z}$.

[3] It is sufficient to take k larger or equal to n, since this holds for the whole space of polynomials.

[4] See [2] for notions not described here.

Exploiting the Hall scalar product on symmetric function[5] in $\boldsymbol{z}$, we may consider the **coefficient** $R_\mu(\boldsymbol{q};\boldsymbol{u}) := \langle \mathcal{R}_n(\boldsymbol{q};\boldsymbol{u};\boldsymbol{z}), s_\mu(\boldsymbol{z})\rangle$ of s_μ in $\mathcal{R}_n$. As a variable free expression, we thus set

$$R_\mu := \langle \mathcal{R}_n, s_\mu\rangle = \sum_{\lambda,\rho} d_{\lambda\rho\mu}\, s_\lambda \otimes s_\rho,$$

and likewise for any coefficient stable $\mathcal{G}$-module $\mathcal{V}$:

$$V_\mu := \langle \mathcal{V}_n, s_\mu\rangle = \sum_{\lambda,\rho} v_{\lambda\rho\mu}\, s_\lambda \otimes s_\rho,$$

It is clear that the R_μ form an upper bound for all V_μ, so that

$$0 \le v_{\lambda\rho\mu} \le d_{\lambda\rho\mu}, \qquad \text{for all } \lambda, \rho, \text{ and } \mu.$$

Thus it is interesting to observe that the Cauchy kernel formula implies that

Corollary 2.1. *When $j = 0$, the coefficient $R_\mu(\boldsymbol{q}) = \sum_\lambda d_{\lambda\mu} s_\lambda(\boldsymbol{q})$ of $s_\mu(\boldsymbol{z})$ in $\mathcal{R}_n(\boldsymbol{q};\boldsymbol{z})$ is given by the formula*

$$\begin{aligned} R_\mu(\boldsymbol{q}) &= s_\mu[\Omega(\boldsymbol{q})] \\ &= s_\mu[\textstyle\sum_{i\ge 0} h_i(\boldsymbol{q})]. \end{aligned} \tag{2.5}$$

3. The boson-fermion modules

As show in [1], there exist a coefficient stable expression for the pure bosonic (commuting variables) multivariate coinvariant module, which we denote by

$$\mathcal{E}_n = \sum_{\mu\vdash n} \mathcal{C}_\mu \otimes s_\mu, \qquad \text{with} \qquad \mathcal{C}_\mu := \sum_\lambda c_{\lambda\mu}\, s_\lambda.$$

The integers $c_{\lambda\mu}$ are non-vanishing only for partitions λ of size at most $\binom{n}{2} - \eta(\mu')$, and having at most $n - \mu_1$ parts. Recall that $\eta(\mu) := \sum_i \mu_i\,(i-1)$. Expressed in terms of variables, the above expression takes the form

$$\mathcal{E}_n(\boldsymbol{q};\boldsymbol{z}) = \sum_{\mu\vdash n}\sum_\lambda c_{\lambda\mu}\, s_\lambda(\boldsymbol{q}) s_\mu(\boldsymbol{z}). \tag{3.1}$$

Our main conjecture is that

Conjecture 1 (Diagonal Supersymmetry)**.** *The multigraded Frobenius characteristic of the boson-fermion diagonal modules may be calculated from the generic Frobenius characteristic for bosons modules via the universal formula*

$$\mathbb{DBF}_{k,j;n}(\boldsymbol{q};\boldsymbol{u};\boldsymbol{z}) = \mathcal{E}_n(\boldsymbol{q} - \varepsilon\,\boldsymbol{u};\boldsymbol{z}) \tag{3.2}$$

$$= \sum_{\mu\vdash n} \mathcal{C}_\mu[\boldsymbol{q} - \varepsilon\,\boldsymbol{u}] s_\mu(\boldsymbol{z}) \tag{3.3}$$

$$= \sum_{\mu\vdash n}\sum_\lambda c_{\lambda\mu}\, s_\lambda[\boldsymbol{q} - \varepsilon\,\boldsymbol{u}] s_\mu(\boldsymbol{z}). \tag{3.4}$$

[5] For which the Schur functions s_μ form an orthonormal basis.

Thus, the (k,j)-multi-degree enumeration (or $\mathrm{GL}_k\times\mathrm{GL}_j$-character) of the $\mathbb{S}_n$-irreducible component of type μ in $\mathbb{DBF}_n$ is obtained as

$$\mathcal{C}_\mu[\boldsymbol{q}-\varepsilon\boldsymbol{u}] = \sum_\lambda c_{\lambda\mu}\, s_\lambda[\boldsymbol{q}-\varepsilon\,\boldsymbol{u}] \tag{3.5}$$

$$= \sum_\lambda c_{\lambda\mu} \sum_{\nu\subseteq\lambda} s_\nu(q_1,\ldots,q_k) s_{\lambda'/\nu'}(u_1,\ldots,u_j). \tag{3.6}$$

Observe that the specification of k and j in $\mathbb{DBF}_{k,j;n}(\boldsymbol{q};\boldsymbol{u};\boldsymbol{z})$ is redundant once the parameters $\boldsymbol{q}=q_1,\ldots,q_k$ and $\boldsymbol{u}=u_1,\ldots,u_j$ are specified. We may thus omit them when this is the case, and also formaly write

$$\mathbb{DBF}_n = \sum_{\mu\vdash n}\sum_\lambda\sum_{\nu\subseteq\lambda} c_{\lambda\mu}\, s_\nu\otimes s_{\lambda'/\nu'}\otimes s_\mu, \tag{3.7}$$

for the **generic diagonal Boson-Fermion Frobenius**. Clearly, we have

$$\mathbb{DBF}_n(q;0;\boldsymbol{z}) = h_n^*(\boldsymbol{z})/h_n^*(1), \qquad \text{with} \qquad f^*(\boldsymbol{z}) := f[\boldsymbol{z}/(1-q)], \tag{3.8}$$

$$\mathbb{DBF}_n(0;u;\boldsymbol{z}) = \sum_{a=0}^{n-1} u^a s_{(n-a,1^a)}(\boldsymbol{z}), \tag{3.9}$$

$$\mathbb{DBF}_n(q,t;0;\boldsymbol{z}) = \nabla(e_n)(q,t;\boldsymbol{z}), \tag{3.10}$$

$$\mathbb{DBF}_n(q_1,\ldots,q_k;0;\boldsymbol{z}) = \mathcal{E}_n(q_1,\ldots,q_k\boldsymbol{z}), \tag{3.11}$$

$$\mathbb{DBF}_n(0;u_1,\ldots,u_j;\boldsymbol{z}) = \sum_{\mu\vdash n}\sum_\lambda c_{\lambda\mu}\, s_{\lambda'}(u_1,\ldots,u_j) s_\mu(\boldsymbol{z}) \tag{3.12}$$

The ∇ operator occurring in (3.10) is a Macdonald "eigenoperator" (introduced in [4]). This is to say it affords as eigenfunctions the (modified) Macdonald operators, usually denoted $\widetilde{H}_\mu$.

It has been conjectured[6] in [5] that

$$\mathbb{DBF}_n(q,t,1;0;\boldsymbol{z}) = \sum_{\alpha\preceq\beta} q^{\mathrm{dist}(\alpha,\beta)}\, \mathbb{L}_\beta(t;\boldsymbol{z}), \tag{3.13}$$

where the sum is over pairs of elements of the Tamari lattice, and $\mathrm{dist}(\alpha,\beta)$ is the length of the longest chain going from α to β. Here, $\mathbb{L}_\beta(t;\boldsymbol{z})$ stands for the LLT-polynomial associated to the Dyck-path β (see [3] for more details). Furthermore, it has been conjectured by N. Bergeron-Machacek-Zabrocki[7] that

$$\mathbb{DBF}_n(q;u;\boldsymbol{z}) = \sum_{k=0}^{n-1}\sum_{\lambda\vdash n}\sum_{\tau\in\mathrm{SYT}(\lambda)} q^{\alpha(\tau)} \begin{bmatrix}\mathrm{des}(\tau)\\ k\end{bmatrix}_q u^k\, s_\lambda(\boldsymbol{z}), \tag{3.14}$$

where, for a standard tableau τ of shape λ, one sets

$$\alpha(\tau) := \mathrm{maj}(\tau) - k\,\mathrm{des}(\tau) + \binom{k}{2}.$$

[6]Notice that one of the parameters is equal to 1. This is because the lacking "statistic" on Dyck-path pairs (α,β) is not yet known.

[7]In fact, this follows from (3.17), via a formula of Haglund, Rhoades and Shimozono (see [7]).

In [8], Kim and Rhoades show that

$$\mathbb{DBF}_n(0;u,v;\boldsymbol{z}) = \sum_{a+b\leq n-1} u^a\, v^b \left(s_{(n-a,1^a)} \star s_{(n-b,1^b)} - s_{(n-(a-1),1^{a-1})} \star s_{(n-(b-1),1^{b-1})}\right), \quad (3.15)$$

with "$\star$" standing for the Kronecker product. Denoting by $g^{\mu}_{\alpha,\beta}$ the **Kronecker coefficients**:

$$g^{\mu}_{\alpha,\beta} := \langle s_\alpha \star s_\beta, s_\mu\rangle,$$

one may reformulate the above as

$$\mathbb{DBF}_n(0;u,v;\boldsymbol{z}) = \sum_{\mu\vdash n}\left(\sum_{b+d\leq n-1} u^b\, v^d\, (g^{\mu}_{(a\,|\,b),(c\,|\,d)} - g^{\mu}_{(a+1\,|\,b-1),(c+1\,|\,d-1)})\right) s_\mu(\boldsymbol{z}), \qquad (3.16)$$

using the Frobenius notation $(a\,|\,b) = (a+1,1^b)$ for hook-shaped partitions. For each term in the inner sum above, we assume that $a+b=n-1$ (likewise for c and d). The differences are know to be positive (see [13]). The various results (see [14]) on the stability of Kronecker coefficients certainly have a bearing here, since they imply corresponding stabilities for the coefficients of the s_μ.

Zabrocki has conjectured (see [16]) that

$$\mathbb{DBF}_n(q,t;u;\boldsymbol{z}) = \sum_{a=0}^{n-1} u^a \Delta'_{e_{n-a-1}}(e_n(\boldsymbol{z})). \qquad (3.17)$$

The parameters q and t arise from the application of the operators Δ'_{e_k}. It follows that (3.14) may also be written as

$$\mathbb{DBF}_n(q;u;\boldsymbol{z}) = \sum_{a=0}^{n-1} u^a \Delta'_{e_{n-a-1}}(e_n(\boldsymbol{z}))\Big|_{t=0}. \qquad (3.18)$$

Finally, we have

$$\mathbb{DBF}_n(1;2;\boldsymbol{z}) = \frac{1}{2}\sum_{\mu\vdash n} 2^{\ell(\mu)}(-1)^{n-\ell(\mu)}\binom{\ell(\mu)}{d_1,\dots,d_n} p_\mu(\boldsymbol{z}), \qquad (3.19)$$

where $d_i = d_i(\mu)$ stands for the number of parts of of size i in μ. Finally, D'Adderio-Iraci-Wyngaerd conjecture in [6, Conj. 8.2.] the more inclusive identity:

$$\mathbb{DBF}_n(q,t;u,v;\boldsymbol{z}) = \sum_{k=1}^{n-1}\sum_{i+j=k} u^i v^j \Theta_{e_i e_j}\nabla(e_{n-k}), \qquad (3.20)$$

where, for any symmetric functions g and f, $\Theta_g f$ is defined as

$$\Theta_g f(\boldsymbol{z}) := \Pi\, g^*\, \Pi^{-1} f(\boldsymbol{z}), \qquad \text{setting} \qquad g^*(\boldsymbol{z}) := g[\boldsymbol{z}/(1-t)(1-q)].$$

Here, Π stands for the Macdonald eigenoperator having as eigenvalues for $\widetilde{H}_\mu$ the product $\prod_{(i,j)\in\mu/(1)}(1-q^it^j)$, for (i,j) running over cartesian coordinates of cells in μ (omitting the cell $(0,0)$).

$$\mathcal{G}(q,t;\underline{z};\underline{x}) := \sum_{k=0}^{n-1}\sum_{\nu\vdash k} m_\nu(\underline{z})\,\Theta_{e_\nu}\nabla(e_{n-k})(q,t;\underline{x}).$$

Table 1 summarizes the overall situation[8]. Conjecture 1 essentially states all entries may

$k \setminus j$	0	1	2	$\cdots$	j
0	1	(3.9)	(3.15)	$\cdots$	(3.12)
1	(3.8)	(3.14)	(3.19)	$\cdots$	
2	(3.10)	(3.17)	(3.20)	$\cdots$	
3	(3.13)			$\cdots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
k	(3.11)			$\cdots$	(3.7)

Table 1. Overall situation for the various formulas.

be obtained from (3.11) (or equivalently from (3.12)). It is interesting to observe that one obtains polynomial expressions in k and j, when setting all parameters $q_i = 1$ and $u_j = 1$. More precisely, writing $\mathbb{DBF}_n(k; j; \boldsymbol{z})$ for the resulting expression, we have the following.

Proposition 2. *The coefficients of each $s_\mu(\boldsymbol{z})$, in the Schur expansion of $\mathbb{DBF}_n(k; j; \boldsymbol{z})$, is a polynomial in k and j, with coefficients in $\mathbb{Q}$. Hence, this is also the case for the associated dimension*[9] *$\mathbb{DBF}_n(k; j)$.*

4. Links with the main conjecture

Conjecture (3.17) directly led to our main conjecture, in view of an elegant link (first stated in 2017, but only recently published) between the generic expression for $\mathcal{E}_n$ and the effect of the Δ'_{e_k} operators on e_n. The precise relevant statement (see [3, Conj. 1]) says that

Conjecture 2 (Delta via skew)**.** *For all k,*

$$\begin{aligned}(e_k \otimes \mathrm{Id})\, \mathcal{E}_n &= \sum_{\mu \vdash n} (e_k^\perp \mathcal{C}_\mu)(q,t)\, s_\mu(\boldsymbol{z}) \\ &= \Delta'_{e_{n-k-1}}(e_n(\boldsymbol{z})). \end{aligned} \tag{4.1}$$

In other words, we get $\Delta'_{e_{n-k-1}}(e_n(\boldsymbol{z}))$ from $\mathcal{E}_n$, first by applying the skew operator $e_k^\perp$ to the various s_λ, and then by evaluation of the resulting expression in q, t. To see how this relates to our general conjecture, we recall that effect on a symmetric function $f(\boldsymbol{q}) = f(q_1, \ldots, q_k)$ of the operator $\sum_{a=0}^{n-1} u^a e_k^\perp$ may be globally expressed in plethystic notation as

$$\sum_{a=0}^{n-1} u^a e_k^\perp f(\boldsymbol{q}) = f[\boldsymbol{q} - \varepsilon\, u].$$

Thus, assuming that Conjecture 2 holds, we see that (3.17) may be now simply be coined as

$$\mathbb{DBF}_n(q, t; u; \boldsymbol{z}) = \mathcal{E}_n[q + t - \varepsilon\, u; \boldsymbol{z}].$$

[8]With k standing for the numbers of sets of commuting variables, and j for those that are anticommuting.
[9]Obtained by replacing each $s_\mu(\boldsymbol{z})$ by the number, f^μ, of standard tableaux of shape μ.

This immediately[10] suggested that the more general formula of Conjecture 1 should hold. All experiments confirmed this. Known or conjectured formulas (due to various researchers) for the dimensions of $\mathbb{DBF}_n(k;j)$ as functions of n, for small k and j, are displayed in Table 2. Here, $\left\{{n \atop k}\right\}$ stands for the Stirling numbers of the second kind. Currently known or conjectured

$k \setminus j$	0	1	2
0	1	2^{n-1}	$\binom{2n-1}{n}$
1	$n!$	$\sum_{i=1}^{n} i!\left\{{n \atop i}\right\}$	$2^{n-1}\, n!$
2	$(n+1)^{n-1}$	$\sum_{i=0}^{n+1} \binom{n+1}{i} \frac{i^n}{2(n+1)}$	?
3	$2^n(n+1)^{n-2}$	?	?

TABLE 2. Dimensions of $\mathbb{DBF}_n(k,j)$.

formulas for the multiplicities of alternating component in $\mathbb{DBF}_n$ appear in Table 3, where

$k \setminus j$	0	1	2	3
0	0	1	n	n^2-n+1
1	1	2^{n-1}	3^{n-1}	$2^{-1}F_{3n-1}$
2	$\frac{1}{n+1}\binom{2n}{n}$	$s(n)$	$\frac{2^{n-1}}{n+1}\binom{2n}{n}$	?
3	$\frac{2}{n(n+1)}\binom{4n+1}{n-1}$	?	?	?

TABLE 3. Coefficients of $s_{1^n}(\boldsymbol{z})$ in $\mathbb{DBF}_n(k,j)(\boldsymbol{z})$.

$s(n) = \frac{1}{n}\sum_{i=0}^{n-1} \binom{n}{i}\binom{n}{i+1} 2^i$ denotes the n^{th} small Schröder number, and F_n stands for the n^{th} Fibonacci number.

Formulas for low degree components of the corresponding general expressions have also been conjectured to hold. The first of these, see [1], states that

$$\mathbb{DBF}_n(\boldsymbol{q};0;\boldsymbol{z}) =_{(n)} \frac{h_n[\Omega(\boldsymbol{q})\,\boldsymbol{z}]}{h_n[\Omega(\boldsymbol{q})]}, \tag{4.2}$$

where $\Omega(\boldsymbol{q}) = 1 + h_1(\boldsymbol{q}) + h_2(\boldsymbol{q}) + \ldots$; with equality holding for terms of degree at most n in the $\boldsymbol{q}$ variables. It may be worth recalling here that, when $\boldsymbol{q}$ consists of only one variable, the above is well known to be an equality. Indeed, this the symmetric group case of the Chevalley-Shephard-Todd theorem.

A slightly stronger form of a conjecture stated in [6], is that the difference $\mathbb{DBF}_n(q,t;\boldsymbol{u};\boldsymbol{z}) - \mathcal{M}_n(q,t;\boldsymbol{u};\boldsymbol{z})$ is Schur positive in all three sets of variables $\boldsymbol{u} = \{u_1, u_2, \ldots, u_r\}$ (for all r), $\boldsymbol{z}$, and $\{q,t\}$; setting

$$\mathcal{M}_n(q,t;\boldsymbol{u};\boldsymbol{z}) := \sum_{k=0}^{n-1} \sum_{\nu \vdash k} m_\nu(\boldsymbol{u})\, \Theta_{e_\nu} \nabla(e_{n-k})(q,t;\boldsymbol{z}). \tag{4.3}$$

[10] During the January 2019 Banff meeting where Mike Zabrocki presented his conjecture for the first time.

Furthermore, the expression $\mathcal{M}_n(q,t;\boldsymbol{u};\boldsymbol{z})$ is itself Schur positive in all three sets of variables.

Although some of the terms in either of the expressions (right hand-side of, 4.2) and (4.3) may be recovered (via plethysm) from the other, this not the case for all terms. Hence, the two statements are not equivalent.

5. An explicit example

With $n=3$, we have

$$\mathcal{E}_3 = 1\otimes s_3 + (s_1+s_2)\otimes s_{21} + (s_{11}+s_3)\otimes s_{111},$$

from which we deduce that

$$\begin{aligned}\mathbb{DBF}_3 &= 1\otimes 1\otimes s_3 + (s_1\otimes 1 + 1\otimes s_1 + s_2\otimes 1 + s_1\otimes s_1 + 1\otimes s_{11})\otimes s_{21}\\ &\quad + (s_{11}\otimes 1 + s_1\otimes s_1 + 1\otimes s_2 + s_3\otimes 1 + \ s_2\otimes s_1 + s_1\otimes s_{11} + 1\otimes s_{111})\otimes s_{111}.\end{aligned}$$

By specialization, we get

$$\begin{aligned}\mathbb{DBF}_3(q;0;\boldsymbol{z}) &= s_3(\boldsymbol{z}) + (q+q^2)\, s_{21}(\boldsymbol{z}) + q^3\, s_{111}(\boldsymbol{z}),\\
\mathbb{DBF}_3(0;u;\boldsymbol{z}) &= s_3(\boldsymbol{z}) + u\, s_{21}(\boldsymbol{z}) + u^2\, s_{111}(\boldsymbol{z}),\\
\mathbb{DBF}_3(q,t;0;\boldsymbol{z}) &= s_3(\boldsymbol{z}) + (q^2+qt+t^2+q+t)\, s_{21}(\boldsymbol{z})\\
&\qquad + (q^3+q^2t+qt^2+t^3+qt)\, s_{111}(\boldsymbol{z}),\\
\mathbb{DBF}_3(q,t;u;\boldsymbol{z}) &= s_3(\boldsymbol{z}) + (q+t+u+q^2+q\,t+t^2+q\,u+t\,u)\, s_{21}(\boldsymbol{z})\\
&\quad + (q\,t+q\,u+t\,u+u^2+q^3+q^2t+q\,t^2+t^3+q^2u+q\,t\,u+t^2u)\, s_{111}(\boldsymbol{z}),\\
\mathbb{DBF}_3(\boldsymbol{q};0;\boldsymbol{z}) &= s_3(\boldsymbol{z}) + (s_1(\boldsymbol{q})+s_2(\boldsymbol{q}))\, s_{21}(\boldsymbol{z}) + (s_{11}(\boldsymbol{q})+s_3(\boldsymbol{q}))\, s_{111}(\boldsymbol{z}),\\
\mathbb{DBF}_3(0;\boldsymbol{u};\boldsymbol{z}) &= s_3(\boldsymbol{z}) + (s_1(\boldsymbol{u})+s_{11}(\boldsymbol{u}))\, s_{21}(\boldsymbol{z}) + (s_2(\boldsymbol{u})+s_{111}(\boldsymbol{u}))\, s_{111}(\boldsymbol{z}).\end{aligned}$$

The polynomial expressions in k and j of the dimension, and the Frobenius characteristic for $\mathbb{DBF}_3$ are respectively:

$$\begin{aligned}\mathbb{DBF}_3(k;j) &= \frac{1}{6}(k+j+1)\,(k^2+2\,k\,j+j^2+11\,k+5\,j+6), \qquad \text{and}\\
\mathbb{DBF}_3(k;j;\boldsymbol{z}) &= s_3(\boldsymbol{z}) + \frac{1}{2}(k^2+2\,k\,j+t^2+3\,k+j)\, s_{21}(\boldsymbol{z})\\
&\qquad + \frac{1}{6}(k^3+3\,k^2j+3\,k\,j^2+j^3+6\,k^2+6\,k\,j-k+5\,j)\, s_{111}(\boldsymbol{z}).\end{aligned}$$

Explicit values for $n=3,4$, and 5 are the dimensions

$k\setminus j$	0	1	2
0	1	4	10
1	6	13	23
2	16	28	45
3	32	50	74

$k\setminus j$	0	1	2
0	1	8	35
1	24	75	192
2	125	288	597
3	400	785	1440

$k\setminus j$	0	1	2
0	1	16	126
1	120	541	1920
2	1296	3936	10541
3	6912	17072	38912

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