Floating Point

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Based on slides originally by:
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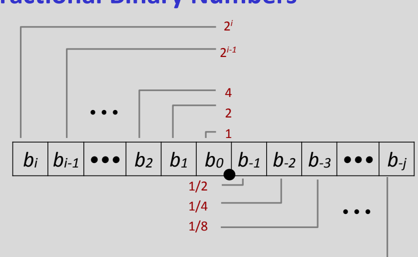
Today: Floating Point

- **Background: Fractional binary numbers**
- **IEEE floating point standard: Definition**
- **Example and properties**
- **Rounding, addition, multiplication**
- **Floating point in C**
- **Summary**

Fractional binary numbers

- **What is 1011.101_2?**

Fractional Binary Numbers



- **Representation** 2^{-j}
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number: $\sum_{k=-j}^{i} b_k \times 2^k$

Fractional Binary Numbers: Examples

- **Value** **Representation**

Value	Representation
5 3/4	101.11_2
2 7/8	10.111_2
1 7/16	1.0111_2

- **Observations**
  - Divide by 2 by shifting right
  - Multiply by 2 by shifting left
  - Numbers of form $0.111111\ldots_2$ are just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - Use notation $1.0 - \varepsilon$

Representable Numbers

- **Limitation**
  - Can only exactly represent numbers of the form $x/2^k$
  - Other rational numbers have repeating bit representations

Value	Representation
1/3	$0.0101010101[01]\ldots_2$
1/5	$0.001100110011[0011]\ldots_2$
1/10	$0.0001100110011[0011]\ldots_2$

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IEEE Floating Point

- **IEEE Standard 754**
  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs
- **Driven by numerical concerns**
  - Nice standards for rounding, overflow, underflow
  - Hard to make fast in hardware
    - Numerical analysts predominated over hardware designers in defining standard

Floating Point Representation

- **Numerical Form:**

$$(-1)^s\ M\ 2^E$$

  - **Sign bit *s*** determines whether number is negative or positive
  - **Significand *M*** normally a fractional value in range [1.0,2.0).
  - **Exponent *E*** weights value by power of two
- **Encoding**
  - MSB s is sign bit *s*
  - exp field encodes *E* (but is not equal to E)
  - frac field encodes *M* (but is not equal to M)

s	exp	frac

Precisions

- **Single precision: 32 bits**

s	exp	frac
1	8-bits	23-bits

- **Double precision: 64 bits**

s	exp	frac
1	11-bits	52-bits

- **Extended precision: 80 bits (Intel only)**

s	exp	frac
1	15-bits	63 or 64-bits

Normalized (Normal) Values

- **Condition: exp ≠ 000…0 and exp ≠ 111…1**
- **Exponent coded as *biased* value: *E* = *Exp* – *Bias***
  - *Exp*: unsigned value exp
  - *Bias* = 2^{k-1} - 1, where *k* is number of exponent bits
    - Single precision: 127 (Exp: 1…254, E: -126…127)
    - Double precision: 1023 (Exp: 1…2046, E: -1022…1023)
- **Significand coded with implied leading 1: *M* = 1.xxx…x$_2$**
  - xxx…x: bits of frac
  - Minimum when 000…0 (*M* = 1.0)
  - Maximum when 111…1 (*M* = 2.0 – ε)
  - Get extra leading bit for "free"

Normalized Encoding Example

- **Value:** `Float F = 15213.0;`
  - $15213_{10} = 11101101101101_2$
    $= 1.1101101101101_2 \times 2^{13}$
- **Significand**
  - *M* = `1.1101101101101`$_2$
  - `frac=` `11011011011010000000000`$_2$
- **Exponent**
  - *E* = 13
  - *Bias* = 127
  - *Exp* = 140 = `10001100`$_2$
- **Result:**

s	exp	frac
0	10001100	11011011011010000000000

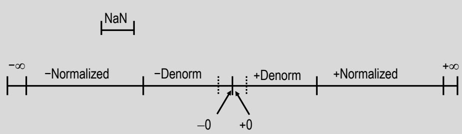
Denormalized Values

- Condition: exp = 000…0
- Exponent value: *E* = –*Bias* + 1 (instead of *E* = 0 – *Bias*)
- Significand coded with implied leading 0: *M* = 0.xxx…x_2
  - `xxx…x`: bits of `frac`
- Cases
  - `exp` = 000…0, `frac` = 000…0
    - Represents zero value
    - Note distinct values: +0 and –0 (why?)
  - `exp` = 000…0, `frac` ≠ 000…0
    - Numbers very close to 0.0
    - Lose precision as get smaller
    - Equispaced

Special Values

- Condition: `exp` = 111…1
- Case: `exp` = 111…1, `frac` = 000…0
  - Represents value ∞ (infinity)
  - Operation that overflows
  - Both positive and negative
  - E.g., 1.0/0.0 = −1.0/−0.0 = +∞, 1.0/−0.0 = −∞
- Case: `exp` = 111…1, `frac` ≠ 000…0
  - Not-a-Number (NaN)
  - Represents case when no numeric value can be determined
  - E.g., sqrt(–1), ∞ − ∞, ∞ × 0

Visualization: Floating Point Encodings



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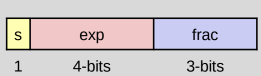
Tiny Floating Point Example



- 8-bit Floating Point Representation
  - the sign bit is in the most significant bit
  - the next four bits are the exponent, with a bias of 7
  - the last three bits are the `frac`
- Same general form as IEEE Format
  - normalized, denormalized
  - representation of 0, NaN, infinity

Dynamic Range (Positive Shown)

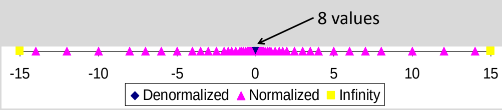
	s	exp	frac	E	Value	
Denormalized numbers	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	closest to zero
	0	0000	010	-6	2/8*1/64 = 2/512	
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0	0001	000	-6	8/8*1/64 = 8/512	smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	largest norm
	0	1111	000	n/a	inf	

Distribution of Values

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 23-1-1 = 3

s	exp	frac
1	3-bits	2-bits

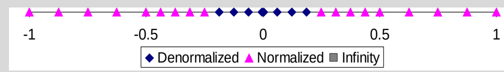
- **Notice how the distribution gets denser toward zero.**



Distribution of Values (close-up view)

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 3

s	exp	frac
1	3-bits	2-bits



Interesting Numbers

`{single,double}`

Description	*exp*	*frac*	*Numeric Value*
Zero	**00…00**	**00…00**	**0.0**
Smallest Pos. Denorm.	**00…00**	**00…01**	$\mathbf{2^{-\{23,52\}} \times 2^{-\{126,1022\}}}$
Single ≈ 1.4×10^{-45}			
Double ≈ 4.9×10^{-324}			
Largest Denormalized	**00…00**	**11…11**	$\mathbf{(1.0 - \varepsilon) \times 2^{-\{126,1022\}}}$
Single ≈ 1.18×10^{-38}			
Double ≈ 2.2×10^{-308}			
Smallest Pos. Normalized	**00…01**	**00…00**	$\mathbf{1.0 \times 2^{-\{126,1022\}}}$
Just larger than largest denormalized			
One	**01…11**	**00…00**	**1.0**
Largest Normalized	**11…10**	**11…11**	$\mathbf{(2.0 - \varepsilon) \times 2^{\{127,1023\}}}$
Single ≈ 3.4×10^{38}			
Double ≈ 1.8×10^{308}			

Special Properties of Encoding

- **FP Zero Same as Integer Zero**
  - All bits = 0
- **Can (Almost) Use Unsigned Integer Comparison**
  - Must first compare sign bits
  - Must consider −0 = 0
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
    - IEEE rule: any comparison with NaN is false!
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity

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Floating Point Operations: Basic Idea

- `x +f y = Round(x + y)`
- `x ×f y = Round(x × y)`
- **Basic idea**
  - First compute exact result
  - Make it fit into desired precision
    - Possibly overflow if exponent too large
    - Possibly round to fit into `frac`

Rounding

- **Rounding Modes (illustrate with $ rounding)**

	$1.40	$1.60	$1.50	$2.50	–$1.50
Towards zero	$1	$1	$1	$2	–$1
Round down (–∞)	$1	$1	$1	$2	–$2
Round up (+∞)	$2	$2	$2	$3	–$1
Nearest Even (default)	$1	$2	$2	$2	–$2

- **What are the different modes good for?**
  - Towards zero: compatible with C integer behavior
  - Round down/up: maintain conservative intervals
  - Nearest even: unbiased, minimal error

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Closer Look at Round-To-Even

- **Default Rounding Mode**
  - All you get in C without doing something special
  - All others are statistically biased
    - Sum of set of positive numbers will consistently be over- or under-estimated
- **Applying to Other Decimal Places / Bit Positions**
  - When exactly halfway between two possible values
    - Round so that least significant remaining digit is even
  - E.g., round to nearest hundredth

1.2349999	1.23	(Less than half way)
1.2350001	1.24	(Greater than half way)
1.2350000	1.24	(Half way—round up)
1.2450000	1.24	(Half way—round down)

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Rounding Binary Numbers

- **Binary Fractional Numbers**
  - "Even" when least significant bit is 0
  - "Half way" when bits to right of rounding position = $100\ldots_2$
- **Examples**
  - Round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
2 3/32	10.00011_2	10.00_2	(<1/2—down)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up)	3
2 5/8	10.10100_2	10.10_2	(1/2—down)	2 1/2

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Exercise break: FP and money?

- **Your sandwich shop uses single-precision floating point for sales amounts**
- **Need to apply a Minneapolis sales tax of 7.75%, rounded up to the nearest cent**
- **On $4.00 purchase, compute:**
  - round_up(4.00 * 0.0775 * 100) = 32 cents
  - Correct tax is 31 cents
- **What went wrong?**
  - Note: 0.0775 = 31/400 exactly

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FP and money: what went wrong?

- **0.0775 = 31/400 cannot be represented exactly in binary**
  - 400 is not a power of 2
- **Actual representation with be like 0.0775 ± ϵ**
  - For single-precision, closest is 0.0775 + ϵ
- **4.00 * (0.775 + ϵ) * 100 = 31 + ϵ**
- **round_up(31 + ϵ) = 32**
- **Similar problems can happen with double precision or other rounding modes**
  - Real Minnesota law is a more complex rule
- **Better choices:**
  - Store cents or smaller fractions as an integer, or
  - Special libraries for decimal arithmetic

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Normalization Example: int to float

- **Steps**
  - Normalize to have leading 1
  - Round to fit within fraction
  - Postnormalize to deal with effects of rounding

s	exp	frac
1	4-bits	3-bits

- **Case Study**
  - Convert 8-bit unsigned numbers to tiny floating point format

Example Numbers

128	10000000
15	00001101
33	00010001
35	00010011
138	10001010
63	00111111

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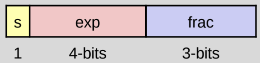
Normalize



- **Requirement**
  - Set binary point so that numbers of form 1.xxxxx
  - Adjust all to have leading one
    - Decrement exponent as shift left

Value	*Binary*	*Fraction*	*Exponent*
128	10000000	1.0000000	7
15	00001101	1.1010000	3
17	00010001	1.0001000	4
19	00010011	1.0011000	4
138	10001010	1.0001010	7
63	00111111	1.1111100	5

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Rounding

1.BBGRXXX

Guard bit: LSB of result

Round bit: 1st bit removed

Sticky bit: OR of remaining bits

- **Round up conditions**
  - Round = 1, Sticky = 1 → > 0.5
  - Guard = 1, Round = 1, Sticky = 0 → Round to even

Value	*Fraction*	*GRS*	*Incr?*	*Rounded*
128	1.0000000	000	N	1.000
15	1.1010000	100	N	1.101
17	1.0001000	010	N	1.000
19	1.0011000	110	Y	1.010
138	1.0001010	011	Y	1.001
63	1.1111100	111	Y	10.000

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Postnormalize

- **Issue**
  - Rounding may have caused overflow
  - Handle by shifting right once & incrementing exponent

Value	*Rounded*	*Exp*	*Adjusted*	*Result*
128	1.000	7		128
15	1.101	3		15
17	1.000	4		16
19	1.010	4		20
138	1.001	7		134
63	10.000	5	1.000/6	64

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FP Multiplication

- $(-1)^{s1}\ M1\ 2^{E1}$ x $(-1)^{s2}\ M2\ 2^{E2}$
- **Exact Result:** $(-1)^{s}\ M\ 2^{E}$
  - Sign *s*: *s1 ^ s2*
  - Significand *M*: *M1* x *M2*
  - Exponent *E*: *E1 + E2*
- **Fixing**
  - If *M* ≥ 2, shift *M* right, increment *E*
  - If *E* out of range, overflow
  - Round *M* to fit `frac` precision
- **Implementation**
  - Most expensive part is multiplying significands

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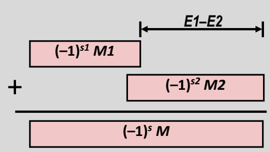
Floating Point Addition

- $(-1)^{s1}\ M1\ 2^{E1}$ + $(-1)^{s2}\ M2\ 2^{E2}$
  - Assume *E1* > *E2*
- **Exact Result:** $(-1)^{s}\ M\ 2^{E}$
  - Sign *s*, significand *M*:
    - Result of signed align & add
  - Exponent *E*: *E1*



- **Fixing**
  - If *M* ≥ 2, shift *M* right, increment *E*
  - if *M* < 1, shift *M* left *k* positions, decrement *E* by *k*
  - Overflow if *E* out of range
  - Round *M* to fit `frac` precision

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Mathematical Properties of FP Add

- **Compare to those of Abelian Group**
  - Closed under addition? ***Yes***
    - But may generate infinity or NaN
  - Commutative? ***Yes***
  - Associative? ***No***
    - Overflow and inexactness of rounding
  - 0 is additive identity? ***Yes***
  - Every element has additive inverse ***Almost***
    - Except for infinities & NaNs
- **Monotonicity**
  - a ≥ b ⇒ a+c ≥ b+c? ***Almost***
    - Except for NaNs (can be produced by infinities)

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Mathematical Properties of FP Mult

- **Compare to Commutative Ring**
  - Closed under multiplication? *Yes*
    - But may generate infinity or NaN
  - Multiplication Commutative? *Yes*
  - Multiplication is Associative? *No*
    - Possibility of overflow, inexactness of rounding
  - 1 is multiplicative identity? *Yes*
  - Multiplication distributes over addition? *No*
    - Possibility of overflow, inexactness of rounding
- **Monotonicity**
  - $a \geq b$ & $c \geq 0 \Rightarrow a * c \geq b * c$? *Almost*
    - Except for infinities & NaNs

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Floating Point in C

- **C Has Two Basic Sizes**
  - `float` single precision
  - `double` double precision (less common: long double)
- **Conversions/Casting**
  - Casting between `int`, `float`, and `double` changes bit representation
  - `double`/`float` → `int`
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN: x86 sets to TMin
  - `int` → `double`
    - Exact conversion, as long as `int` has ≤ 53 bit word size
  - `int` → `float`
    - Will round according to rounding mode

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Floating Point Puzzles

- **For each of the following C expressions, either:**
  - Argue that it is true for all argument values
  - Explain why not true

```
int x = …;
float f = …;
double d = …;
```

Assume neither `d` nor `f` is NaN

- x == (int)(float) x
- x == (int)(double) x
- f == (float)(double) f
- d == (float) d
- f == -(-f);
- 2/3 == 2/3.0
- d < 0.0 ⇒ ((d*2) < 0.0)
- d > f ⇒ -f > -d
- d * d >= 0.0
- (d+f)-d == f

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Summary

- **IEEE Floating Point has clear mathematical properties**
- **Represents numbers of form M x 2^E**
- **One can reason about operations independent of implementation**
  - As if computed with perfect precision and then rounded
- **Not the same as real arithmetic**
  - Violates associativity/distributivity
  - Makes life difficult for compilers & serious numerical applications programmers

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