Logic Design II

CSci 2021: Machine Architecture and Organization
Lecture #37, April 27th, 2015

Your instructor: Stephen McCamant

Truth Tables

- **Combinational circuit = Boolean function**
  - Combinational: no cycles or memory
  - Outputs are determined just by inputs
- **Finite size**
  - A Boolean function has a finite representation
  - If *i* input bits, 2^i possible input combinations
  - Can study by just writing the output for all possible inputs
- **Truth table**
  - Standard way to write a function
  - 2^i rows, input combinations in increasing order
  - One column per intermediate or output

Truth Table Example

a	b	c	(a & b)	(a & b) | c
0	0	0	0	0
0	0	1	0	1
0	1	0	0	0
0	1	1	0	1
1	0	0	0	0
1	0	1	0	1
1	1	0	1	1
1	1	1	1	1

Equivalences with a Truth Table

- **Check whether two Boolean formulas are equal**
  - Write truth table covering both
  - Check two columns have all the same entries
- **Advantages**
  - Straightforward
  - No algebraic insight needed
- **Disadvantages**
  - Effort exponential in number of input bits

Equivalence Example

a	b	c	(b & c)	a | (b & c)	(a | b)	(a | c)	(a | b) & (a | c)
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

Combinational Logic Design

- **Given: description of circuit behavior**
  - Word problem, or truth table
- **Goal: efficient circuit implementation**
  - Usually most important: fewest gates and wires
  - Secondarily: reduce number of levels (propagation delay)
- **Kinds of techniques**
  - Up to 6 inputs: pencil and paper approaches
  - Large but structured: split into repeated pieces
  - Large and unstructured: computer algorithm

DNF / SOP

- An input or its negation is called a *literal*
  - E.g.: a, !b
- An AND of literals is a *product term* or *cube*
  - E.g.: (a & c), (a & !b), (!a & !b & !c), c
- An OR of product terms is a *sum of products* (SOP), or in *disjunctive normal form* (DNF)
  - E.g.: (a & b) | (a & c)
- (Dual: *product of sums* (POS), or *conjunctive normal form* (CNF))

Truth Table → SOP

- Simple but not very efficient
- Create a product term for each 1 entry
- Example with XOR:

a	b	a ^ b
0	0	0
0	1	1
1	0	1
1	1	0

(!a & b)

(a & !b)

Result: (!a & b) | (a & !b)

- (Also possible: dual with 0s and CNF)

Inefficiency of Straight DNF

- Consider another example:

a	b	b
0	0	0
0	1	1
1	0	0
1	1	1

Result: (!a & b) | (a & b)

- By algebra, can simplify back to “b”
  - Factor, (!a | a) = 1, 1 & b = b
- Can we recognize these patterns earlier?

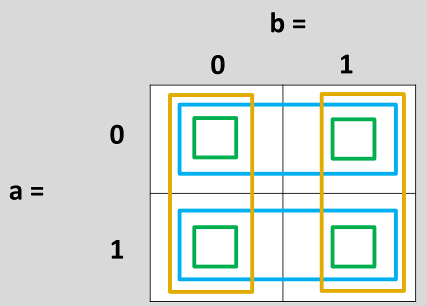
Logistics Intermission

- Sorry, no quiz 2s today
  - Good chance of grades by tomorrow and papers Wednesday
- Cache Lab due tonight
  - Moodle has been having some slowness
  - Suggest you allow a little extra time for final submission
- Assignment V out on Wednesday
  - Mostly logic design

Karnaugh Map Idea

- Write truth table entries in an array
- Product terms represented by certain rectangles
- Visually, find small number of rectangles to cover 1 bits
  - OK to cover more than once, combine with OR
  - Fewer rectangles = smaller circuit

2-variable “Karnaugh Map”



2-variable “Karnaugh Map” example

Result:
!a | b

a \ b =	0	1
0	1	1
1	0	1

Extending to 3 and 4 Variables

- **Put two variables on a side**
  - Weird order: 00 01 11 10
  - “Gray Code”: change only one bit at a time
- **Rectangles can enclose 1, 2, 4, or 8 entries**
  - Bigger is better
- **Rectangles can wrap around the edges**
  - 00 is adjacent to 10

4-variable Karnaugh Map Example

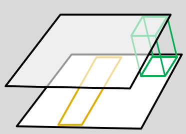
cd = \ ab =	00	10	11	01
00	0	1	0	1
10	0	1	0	0
11	0	1	1	0
01	0	1	1	0

(a & !b)
|
(a & d)
|
(!a & b
& !c & !d)

Extending to 5 and 6 Variables

- **2D is no longer enough**
  - No way to order 3 variables to capture 12 adjacencies
- **Approach: stacking**
  - Make 2 (for 5 inputs) or 4 (for 6 inputs) 4-input Karnaugh maps
  - Corresponding entries are “on top of” each other
  - Rectangles become 3D
  - Usually still drawn as 2D
  - With 6, more possibilities for wrapping too

5-variable Karnaugh Map Example



0	1	0	1
0	1	0	0
0	1	1	0
0	1	1	0

0	0	0	1
0	0	0	0
0	1	1	0
0	1	1	0

Karnaugh Map Tips: Overlap is Good

cd = \ ab =	00	10	11	01
00	0	1	0	0
10	0	1	0	0
11	0	1	1	0
01	0	1	1	0

Karnaugh Map Tips: No 3s

ab =

cd =	00	10	11	01
00	0	0	0	0
10	0	1	0	0
11	0	1	0	0
01	0	1	0	0

19

Karnaugh Map Tips: Wrap Around

ab =

cd =	00	10	11	01
00	1	0	0	1
10	0	0	0	0
11	0	0	0	0
01	1	0	0	1

!a & !c

20

Don't Cares

- Some results don't matter
  - Domain of function is a subset of all n-bit strings
  - Unused bit patterns in encodings
  - Bits sometimes ignored by other circuits
- "Don't care" value could be 0 or 1
  - Usually denoted by X
- Don't-cares allow designs to be simpler
  - Choose the value that allows a simpler circuit
- In early CPUs, led to undocumented instructions
  - Example: x86 ASL vs. SHL
  - On modern CPUs, more error checking

21

Karnaugh Map Tips: Don't Cares

ab =

cd =	00	10	11	01
00	X	0	X	1
10	0	X	X	X
11	X	X	X	X
01	1	X	X	X

22

Dual (POS) Karnaugh Maps

	00	10	11	01
00	1	1	1	1
10	1	1	0	1
11	1	1	0	1
01	1	1	1	1

- Pretend 0s are 1s
  - And vice-versa
- Negate final result

!(a & b & c)

!a | !b | !c

23

Karnaugh Map: Try Yourself

ab =

cd =	00	10	11	01
00	1	0	0	0
10	1	1	0	0
11	1	1	1	1
01	1	1	0	1

24

Automated Methods

- **Karnaugh maps don't scale well beyond 6 inputs**
- **Good job for a computer!**
- **Quine-McCluskey algorithm**
  - Tabular analog to Karnaugh maps
  - Optimal, but suffers from exponential blowup
- **Heuristic methods like "`espresso`"**
  - First, greedily achieve coverage
  - Then, opportunistically improve
  - No optimality guarantee, but good scalability
- **Now a standard part of CAD systems**
  - Like compilers for software