Lecture 12. Options Strategies

Introduction to Options Strategies
Problem 6:23: Assume that a bank can borrow or lend money at the same interest rate in the LIBOR market. The 90-day rate is 10% per annum, and the 180-day rate is 10.2% per annum, both expressed with continuous compounding and actual/actual day count. The Eurodollar futures price for a contract maturing in 91 days is quoted as 89.5. What arbitrage opportunities are open to the bank?

Solution: The Eurodollar futures contract price of 89.5 means that the Eurodollar futures rate is 10.5% per year with quarterly compounding and an actual/360 count. This becomes $10.5 \times 365/360 = 10.646\%$ with actual/actual day count. This is

$$4 \ln(1 + 0.25 \times 0.10646) = 0.1051$$

or 10.51% with contin. compound. The forward rate given by the 91-day rate and the 182-day rate is 10.4% with cont. compounding, since $R_F = \frac{R_{2T2}-R_{1T1}}{T_2-T_1} = \frac{0.102 \times 182 - 0.10 \times 91}{91} = 0.104$. We get the following arbitrage situation:

1. buy eurodollar futures
2. borrow 182-day money
3. invest the borrowed money for 91 days.
Problem 6:25: The futures price for the June 2005 CBOT bond futures contract is 118-23.

- Calculate the conversion factor for a bond maturing on Jan. 1, 2021, paying a coupon of 10%.
- Calculate the conversion factor for a bond maturing on Oct. 1, 2026, paying a coupon of 7%.
- Suppose that the quoted prices of the bonds in the first two parts are $169.00 and $136.00, respectively. Which bond is cheaper to deliver?
- Assuming that the cheapest to deliver bond is actually delivered, what is the cash price received for the bond?

Solution: Recall that the conversion factor equals the quoted price the bond would have per dollar of principal on the first day of the delivery month on the assumption that the interest rate for all maturities equals 6% per annum with semiannual compounding. Bond maturity and times to coupon payment are rounded down to the nearest 3 months.

1. One the first day of the delivery month the bond has 15 years and 7 months to maturity. The value of the bond assuming it lasts 15.5 years and all rates are 6% per annum with semiannual compounding is

\[
\sum_{i=1}^{31} \frac{5}{1.03^i} + \frac{100}{1.03^{31}} = 140.00
\]

The conversion factor is therefore 1.400.
2. On the first day of the delivery month the bond has 21 years and 4 months to maturity. The value of the bond assuming it lasts 21.25 years and all rates are 6% per annum with semiannual compounding is

\[
\frac{1}{\sqrt{1.03}} \left[ 3.5 + \sum_{i=1}^{42} \frac{3.5}{1.03^i} + \frac{100}{1.03^{42}} \right] = 113.66
\]

Subtracting accrued interest of 1.75 yields 111.91. The conversion factor is 1.1191.
To choose cheapest bond, compute

\[ \text{quoted bond price} - \text{settlement price} \times \text{conversion factor} \]

for each bond:

3. For the first bond, the quoted futures price times the conversion factor is

\[ 118.71825 \times 1.400 = 166.2056 \]

This is 2.79444 less than the quoted bond price. For the second bond, the quoted futures price times the conversion factor is

\[ 118.71825 \times 1.1191 = 132.8576 \]

This is 3.1424 less than the quoted bond price. The first bond is therefore the cheapest to deliver.

4. The price received for the bond is 166.2056 plus accrued interest.
Problem 7:20: Company A, a British manufacturer, wishes to borrow US dollars at a fixed rate of interest. Company B, a US multinational, wishes to borrow sterling at a fixed rate of interest. They have been quoted the following rates per annum.

<table>
<thead>
<tr>
<th></th>
<th>Sterling</th>
<th>US dollars</th>
</tr>
</thead>
<tbody>
<tr>
<td>Company A</td>
<td>11.0%</td>
<td>7.0%</td>
</tr>
<tr>
<td>Company B</td>
<td>10.6%</td>
<td>6.2%</td>
</tr>
</tbody>
</table>

Design a swap that will net a bank, acting as intermediary, 10 basis points per annum and that will produce a gain of 15 basis points per annum for each of the two companies.

Solution: The spread between the interest rates offered to A and B is 0.4% (40 basis points) on the sterling loan and 0.8% (80 basis points) on US dollar loans. The total benefit to all parties should be

\[ 80 - 40 = 40 \text{ basis points} \]

Possible to design a swap with 15 basis point benefit to each company and 10 basis points to the intermediary.
Solutions

- Company A borrows 11% sterling from outside lender.
- Company B borrows 6.2% USD from outside lender.
- Intermediary borrows 10.45% sterling from B and lends 6.2% USD to B. Intermediary borrows 6.85% USD from A and lends 11% sterling to A.

Net effect:
- \[ A = -11 + 11 - 6.85 = -6.85 \implies -7.0 \]
- \[ B = -6.2 + 6.2 - 10.45 = -10.45 \implies -11 \]
- \[ \text{intermediary} = -11 + 10.45 - 6.2 + 6.85 = -0.55 + 0.65 = 0.1 \]
Problem 7:21: Under the terms of an interest rate swap, a financial institution has agreed to pay 10% per annum and to receive 3-month LIBOR in return on a notional principal of $100 million with payments being exchanged every 3 months. The swap has a remaining life of 14 months. The average of the bid and offer fixed rates currently being swapped for 3-month LIBOR is 12% per annum for all maturities. The 3-month LIBOR rate 1 month ago was 11.8% per annum. All rates are compounded quarterly. What is the value of the swap?

Solution: The swap can be regarded as a long position in a floating rate bond combined with a short position in a fixed-rate bond. The correct discount rate is 12% per annum with quarterly compounding or 11.82% per annum with continuous compounding. Immediately after the next payment the floating-rate bond will be worth $100 million. The next floating payment is $0.118 \times 100 \times 0.25 = 2.95$ million.

The value of the floating rate bond is

$$102.95e^{-0.1182 \times \frac{2}{4}} = 100.941$$
The value of the fixed-rate bond is

\[2.5e^{-0.1182 \times \frac{2}{12}} + 2.5e^{-0.1182 \times \frac{5}{12}} + 2.5e^{-0.1182 \times \frac{8}{12}} + 2.5e^{-0.1182 \times \frac{11}{12}} + 2.5e^{-0.1182 \times \frac{14}{12}} = 98.678\]

The value of the swap is

\[100.941 - 98.678 = 2.263 \text{ million}\]

A second approach is to consider the swap as a series of forward rate agreements (FRA's). The calculated value is

\[(2.95 - 2.5) e^{-0.1182 \times \frac{2}{12}} + (3.0 - 2.5) e^{-0.1182 \times \frac{5}{12}} + (3.0 - 2.5) e^{-0.1182 \times \frac{8}{12}} + (3.0 - 2.5) e^{-0.1182 \times \frac{11}{12}} + (3.0 - 2.5) e^{-0.1182 \times \frac{14}{12}} = 2.263 \text{ million}\]
Problem 7:22: Suppose that the term structure of interest rates is fat in the US and Australia. The USD interest rate is 7% per annum and the AUD rate is 9% per annum. The current value of the AUD is 0.62 USD. Under the terms of the swap agreement, a financial institution pays 8% per annum in AUD and receives 4% per annum in USD. The principals in the two currencies are $12 million USD and 20 million AUD. Payments are exchanged every year, with one exchange having just taken place. The swap will last 2 more years. What is the value of the swap to the financial institution? Assume all rates are cont. compounded.

Solution: The financial institution is long a dollar bond and short a USD bond. The value of the dollar bond in millions is

\[ 0.48e^{-0.07\times 1} + 12.48e^{-0.07\times 2} = 11.297 \]

The value of the AUD bond in millions is

\[ 1.6e^{-0.09\times 1} + 21.6e^{-0.09\times 2} = 19.504 \]

The value of the swap in millions is therefore,

\[ 11.297 - 19.504 \times 0.62 = -0.795 \]

or -$795,000.
Second approach via forward foreign currency exchange contracts. The one-year forward exchange rate is $0.62 \times e^{-0.02 \times 2} = 0.6077$. The two-year forward exchange rate is $0.62e^{-0.02 \times 2} = 0.5957$. The value of the swap in millions of dollars is

$$(0.48 - 1.6 \times 0.6077) e^{-0.07 \times 1} + (12.48 - 21.6 \times 0.5957) e^{-0.07 \times 2} = -0.795$$
Summary

- Factors that affect the value of a stock:
  - Current price
  - Strike price
  - Expiration date
  - Volatility
  - Risk-free rate
  - Dividends
- Can reach some pricing conclusions without studying volatility via arbitrage arguments
- Have the following bounds
  - \( c \leq S_0 \quad C \leq S_0 \)
  - \( p \leq K \quad P \leq K \)
  - \( c \geq \max\{ S_0 - Ke^{-rT}, 0 \} \)
  - \( p \geq \max\{ Ke^{-rT} - S_0, 0 \} \)
  - \( c \geq \max\{ S_0 - D - Ke^{-rT}, 0 \} \)
  - \( p \geq \max\{ Ke^{-rT} + D - S_0, 0 \} \)
• Put-call parity relates the call price to the put price

\[ c + Ke^{-rT} = p + S_0 \]

• With dividends

\[ c + D + Ke^{-rT} = p + S_0 \]

• Put-call parity doesn't hold for American options, but we get upper and lower bounds for the prices.
We first analyze what happens when we combine stock ownership with a stock option. Consider a long position in a stock plus a long put. Called a **protective put**.

Protects the investor from the loss on the stock if the stock drops sharply. Looks roughly like a **shifted long call**.
Trading Strategies

Next is a short position in a stock plus a short put.

Looks roughly like a *shifted short call*. Opposite of a protective put.
The first is writing a covered-call. This is a long position in a stock plus a short position in a call option. Protects the investor from the payoff on the short call that is necessary if the stock rises sharply. Looks roughly like a shifted short put.
Next is a short position in a stock plus a long call.

Protects the investor from the loss on the stock if the stock rises sharply. Looks roughly like a shifted long put.
Trading Strategies

In each case, owning stock and an option is similar to another type of option. Why?

Put-call parity

\[ p + S_0 = c + D + Ke^{-rT} \]

Thus,

- A long put and a long position in a stock is equivalent to long call plus an amount \( D + Ke^{-rT} \).

- From put-call parity: \( -p - S_0 = -c - (D + Ke^{-rT}) \) or a short put plus a short stock position equals a short call minus an amount \( D + Ke^{-rT} \).

- From put-call parity: \( S_0 - c = Ke^{-rT} + D - p \), so long position in the stock plus a short call is a short put plus an amount \( D + Ke^{-rT} \).

- From put-call parity: \( c - S_0 = p - (Ke^{-rT} + D) \), so long call plus a short position in a stock is a short put minus an amount \( D + Ke^{-rT} \).
Spreads involve taking a position in two or more options. Most common is the bull spread created by two call options at different strike prices. Investor hoping the stock will rise:

Payoff using calls with strike prices of $K_1$ and $K_2$. Call option bought has a lower strike price than the call option sold. Then
### Bull Spread

<table>
<thead>
<tr>
<th>Stock price range</th>
<th>Payoff from long call option</th>
<th>Payoff from short call option</th>
<th>Total payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_T \geq K_2$</td>
<td>$S_T - K_1$</td>
<td>$-(S_T - K_2)$</td>
<td>$K_2 - K_1$</td>
</tr>
<tr>
<td>$K_1 &lt; S_T &lt; K_2$</td>
<td>$S_T - K_1$</td>
<td>0</td>
<td>$S_T - K_1$</td>
</tr>
<tr>
<td>$S_T \leq K_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Strategy limits upside and downside risk.

1. Both call are initially out of the money
2. One call is initially in the money; other call is initially out of the money
3. Both calls are initially in the money

Most aggressive is 1. Usually very inexpensive to set up with low probability of high payoff. Moving down become more conservative.
**Bull Spread**

**Example:** Investor buys for $3 a call with a strike price of $30 and sells for $1 a call with a strike price of $35. The payoff from this bull spread strategy is $5 if the stock price is above $35 and zero if it is below $30. If the stock price is between $30 and $35, the payoff is the amount by which the stock price exceeds $30. The cost of the strategy is $3 - $1 = $2. The profit is

<table>
<thead>
<tr>
<th>Stock range</th>
<th>Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_T \leq 30$</td>
<td>-2</td>
</tr>
<tr>
<td>$30 &lt; S_T &lt; 35$</td>
<td>$S_T - 32$</td>
</tr>
<tr>
<td>$S_T \geq 35$</td>
<td>3</td>
</tr>
</tbody>
</table>
Another common spread is the bear spread created by two put options at different strike prices. Investor hoping the stock will fall:

Payoff using puts with strike prices of $K_1$ and $K_2$. Strike price of the put option bought is higher than the strike price of the put option sold.
## Bear Spreads

<table>
<thead>
<tr>
<th>Stock price range</th>
<th>Payoff from long call option</th>
<th>Payoff from short call option</th>
<th>Total payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_T \geq K_2 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( K_1 &lt; S_T &lt; K_2 )</td>
<td>( K_2 - S_T )</td>
<td>0</td>
<td>( K_2 - S_T )</td>
</tr>
<tr>
<td>( S_T \leq K_1 )</td>
<td>( K_2 - S_T )</td>
<td>(- (K_1 - S_T))</td>
<td>( K_2 - K_1 )</td>
</tr>
</tbody>
</table>
A box spread is a combination of a bull call spread with strike prices $K_1$ and $K_2$ and a bear put spread with the same two strike prices. The payoff is always $K_2 - K_1$. The present value of the payoff is therefore $(K_2 - K_1)e^{-rT}$. Works only on European options.

- If the market price is too low, then buy the box, i.e. buy a call with strike price $K_1$, buy put with strike price $K_2$, selling put with strike price $K_1$.
- If the market price is too high, then sell the box, i.e. buy a call with strike price $K_2$, buy put with strike price $K_1$, selling call with strike price $K_1$, and selling put with strike price $K_2$.

<table>
<thead>
<tr>
<th>Stock price range</th>
<th>Payoff from bull call spread</th>
<th>Payoff from bear put spread</th>
<th>Total payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_T \geq K_2$</td>
<td>$K_2 - K_1$</td>
<td>0</td>
<td>$K_2 - K_1$</td>
</tr>
<tr>
<td>$K_1 &lt; S_T &lt; K_2$</td>
<td>$S_T - K_1$</td>
<td>$K_2 - S_T$</td>
<td>$K_2 - K_1$</td>
</tr>
<tr>
<td>$S_T \leq K_1$</td>
<td>0</td>
<td>$K_2 - K_1$</td>
<td>$K_2 - K_1$</td>
</tr>
</tbody>
</table>
Butterfly Spreads

A butterfly spread entails an option at three different strike prices.

- Buy a call at a low strike price $K_1$
- Buy a call at a high strike price $K_3$
- Sell two calls at a halfway strike price $K_2$.

Butterfly spreads lead to profit if the stock stays close to $K_2$ and a small loss if the stock ventures far from $K_2$ in either direction.

Appropriate if the investor feels the stock will not change much in the future. Strategy requires small investment initially.
Butterfly Spreads

Figure 1: Butterfly spread from call options

<table>
<thead>
<tr>
<th>Stock price range</th>
<th>Payoff from first call</th>
<th>Payoff from second call</th>
<th>Payoff from short calls</th>
<th>Total payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_T \leq K_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$K_1 &lt; S_T &lt; K_2$</td>
<td>$S_T - K_1$</td>
<td>0</td>
<td>0</td>
<td>$K_2 - S_T$</td>
</tr>
<tr>
<td>$K_2 &lt; S_T &lt; K_3$</td>
<td>$S_T - K_1$</td>
<td>0</td>
<td>$-2(S_T - K_2)$</td>
<td>$S_T - K_1$</td>
</tr>
<tr>
<td>$K_3 &lt; S_T$</td>
<td>$S_T - K_1$</td>
<td>$S_T - K_3$</td>
<td>$-2(S_T - K_2)$</td>
<td>0</td>
</tr>
</tbody>
</table>

Example:
Butterfly Spreads

The butterfly spread can also be made from put options.

![Diagram of Butterfly Spread](image)

**Figure 2: Butterfly spread from put options**

<table>
<thead>
<tr>
<th>Stock price range</th>
<th>Payoff from first put</th>
<th>Payoff from second put</th>
<th>Payoff from short puts</th>
<th>Total payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_T \leq K_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$K_1 &lt; S_T &lt; K_2$</td>
<td>$S_T - K_1$</td>
<td>0</td>
<td>0</td>
<td>$K_2 - S_T$</td>
</tr>
<tr>
<td>$K_2 &lt; S_T &lt; K_3$</td>
<td>$S_T - K_1$</td>
<td>0</td>
<td>$-2(S_T - K_2)$</td>
<td>$S_T - K_1$</td>
</tr>
<tr>
<td>$K_3 &lt; S_T$</td>
<td>$S_T - K_1$</td>
<td>$S_T - K_3$</td>
<td>$-2(S_T - K_2)$</td>
<td>0</td>
</tr>
</tbody>
</table>
Possible to create spreads using different maturity dates, called calendar spreads.

- Example is selling a call option with a certain strike price and buying a lower-maturity call option with the same strike price.
- The longer the maturity, the more expensive the call option, hence require an initial investment.
- Investor makes a profit if the stock price at the expiration of the short-maturity option is close to the strike price of the short-maturity option.
- Investor loses money when the stock price is significantly above or below the strike price.
Calendar Spreads

Why?

- Consider if the stock price is very low when the short-maturity option expires.
  - Short-maturity option is worthless
  - Value of the long-maturity option is close to zero.
  - Investor incurs a loss that is close to the cost of setting up the spread
- Consider if the stock price is very high when the short-maturity option expires.
  - The short-maturity option costs the investor $S_T - K$
  - The long-maturity option is worth close to $S_T - K$, where $K$ is the strike price of the options.
  - The investor makes a net loss that is close to the set-up costs.
- If the $S_T$ is close to $K$, then
  - The short-maturity option costs the investor either a small amount or nothing at all
  - The long-maturity option is still very valuable (time value). Then there should be a significant net profit.

A couple of types

- **Neutral calendar spread** - strike price close to the current stock price is chosen
- **Bullish calendar spread** - higher strike price chosen
- **Bearish calendar spread** - lower strike price chosen
Calendar Spreads

Calendar spreads can be made with put options, too:

- Investor buys a short-maturity option
- Sells a long-maturity option
- Small profit arises if the stock prices at the expiration is well above or well below strike price
- Significant loss results if close to strike price.

Also exist reverse calendar spreads with pictures opposite of the two pictures.
Diagonal Spreads

• Possible to have a calendar spread with different strike prices

• **Diagonal spreads** have differences in both the maturity and strike price of call options
Combinations: Straddles

A combination is a trading strategy containing both calls and puts in the same stock.

The first is a straddle. Consists of a call and put with same strike price and expiration date.

Consider first a bottom straddle or straddle purchase, buy a call and put.

Figure 3: Bottom straddle

- If the stock is close to the strike price $K$, then there is a loss due to the setup costs.
- If the stock moves significantly in either direction, then there is a substantial profit.
- Appropriate if investor feels the stock will move in one direction or another (unsure of the direction).
Example: Consider an investor who feels that a stock will move significantly, currently at $69 in the next 3 months.

- Investor creates a straddle by buying both a put and a call with strike price $70 and expiration date 3 months. If the calls costs $4 and the put calls $3.
- Suppose the stock stays at $69, then strategy costs $6. If stock goes to $70 then strategy costs $7.
- If stock jumps to $79, then profit is $9 - $7 = $2. If stock drops to $55, then profit is $15 - $7 = $8.

There is also a top straddle or straddle write which is the reverse of a bottom straddle.

- Investor sells a call and a put at the same strike price and same maturity.
- If the stock stays close to strike price, the investor makes money.
- If the stock moves in either direction, then can be large loss.
Combinations: Strips and Straps

Consider now combinations with more than two options. Investor feels the stock will move and likely move in one direction more than another.

A strip consists of a long position in one call and two puts with the same strike price and maturity. Investor feels the stock will move - more likely down.

A strap consists of a long position in two calls and one put with the same strike price and maturity. Investor feels the stock will move - more likely up.
Combinations: Strangle

A combination of puts and calls with different strike prices is a strangle. Here the investor buys a put and a call with different strike prices and the same maturity. Here the put has a strike price $K_1$ lower than the strike price of the call $K_2$.

Similar to a straddle, but cheaper to buy.
## Combinations: Strangle

Payoff:

<table>
<thead>
<tr>
<th>Range of stock price</th>
<th>Payoff from call</th>
<th>Payoff from put</th>
<th>Total payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_T \leq K_1$</td>
<td>0</td>
<td>$K_1 - S_T$</td>
<td>$K_1 - S_T$</td>
</tr>
<tr>
<td>$K_1 &lt; S_T &lt; K_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>$S_T - K_2$</td>
<td>0</td>
<td>$S_T - K_2$</td>
</tr>
</tbody>
</table>

The reverse of the strangle is a **top vertical combination** - investor feels the stock unlikely to move in either direction, but unlimited liability!
Arbitrary payoffs?

Can approximate any curve from a thoughtful combination of the long/short call/put curves.
Summary

- Many trading strategies involving an option and the stock itself. Equivalent to an option offset by income via the put-call parity.
- Spreads involve taking a position in two or more calls or in two or more puts.
- Bull (bear) spread - buying a call (put) with low strike price and selling a call(put) with high strike price.
- Butterfly spread - buying calls (puts) with low and high strike prices and selling two calls (puts) with intermediate strike prices.
- Calendar spread - selling a call (put) with a short maturity and buying a call (put) with long maturity.
- Combinations involve positions in both calls and puts.
- Straddle - long position in a call and a long position in a put with same strike price and maturity.
- Strip - long position in one call and two long positions in a put.
- Strap - long position in two calls and one long positions in a put.
- Strangle - long position in a call and long position in a put at different strike prices.
**Binomial Trees**

**Binomial trees** are useful tools for pricing options. Construct charts of possible movements of a stock and price according to the movements.

We assume that the stock price is a random walk, i.e. at each time step there is a certain probability of the stock moving in one direction or another (up or down).

Work first from example (artificial):

- Stock is priced currently at $20 and at the end of three months it will be either $22 or $18. Value the European call option to buy the stock for $21 in 3 months.
- If the stock is $22 then the option will be worth $1, and if the stock is worth $18 then the option is worth zero.
- Use arbitrage argument to value this situation.
  - Long position in $\Delta$ shares and a short position in the call option
  - If the stock price moves from $20$ to $22$ then value of shares is $22\Delta$ and the value of the option is $1$. Total value is $22\Delta - 1$.
  - If the stock price moves down from $20$ to $18$ then the value of the shares is $18\Delta$ and the value of the option is zero. Total value is $18\Delta$.
  - Portfolio is risk-less if $\Delta$ chosen so both situations agree. Then

\[ 22\Delta - 1 = 18\Delta \implies \Delta = 0.25 \]
Example, cont.

- Risk-less portfolio is therefore long 0.25 shares and short 1 option.
- In either case the value is $22 \times 0.25 - 1 = 18 \times 0.25 = 4.5$.

- Risk-less portfolios must earn the risk-free interest rate $r$. If it is 12% per year, cont. comp. then the value of the option is
  \[ 4.5 e^{-0.12 \times \frac{3}{12}} = 4.367 \]
at the start time.
- The stock price is $20$ today and the option price is $f$, then the value of the portfolio is
  \[ 20 \times 0.25 - f = 5 - f \]
- But
  \[ 5 - f = 4.367 \implies f = 0.633. \]
Consider now a stock with price $S_0$ and an option with current price $f$.

- Suppose that the option lasts for time $T$ and during the life of the option, the price can go either up from $S_0$ to $S_0u$ (where $u > 1$) or down from $S_0$ to $S_0d$ (where $d < 1$).

- The percentage increase in the stock price in up movement is $u - 1$. The percentage decrease in the stock price in down movement is $1 - d$.

- Let the option payoff for up movement be $f_u$ and the option payoff for down movement be $f_d$. 
Generalization

- Build a portfolio of a long position in $\Delta$ shares and a short position in one option.
- On up movement the value of the portfolio is at the end of the option
  $$S_0 u \Delta - f_u$$
- On down movement the value of the portfolio is at the end of the option
  $$S_0 d \Delta - f_d$$
- Equality if
  $$S_0 u \Delta - f_u = S_0 d \Delta - f_d$$  \hspace{1cm} (1)
  or
  $$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d}$$
  Thus $\Delta$ is the ratio of the change in the options prices to the change in the stock prices.
- Portfolio is now risk-less, and so earns risk-free rate $r$. Present value of the portfolio (due to (??)) is
  $$(S_0 u \Delta - f_u) e^{-rT}$$
Generalization

• Cost of setting up the portfolio is

\[ S_0 \Delta - f \]

• Since cost should equal present value (else arbitrage opportunity)

\[ S_0 \Delta - f = (S_0 u \Delta - f_u) e^{-rT} \]

or

\[ f = S_0 \Delta + (S_0 u \Delta - f_u) e^{-rT} = S_0 \Delta \left(1 - u e^{-rT}\right) + f_u e^{-rT} \]

• Recall \( \Delta = \frac{f_u - f_d}{S_0 u - S_0 d} \) then

\[
\begin{align*}
f &= S_0 \left(1 - u e^{-rT}\right) \frac{f_u - f_d}{S_0 u - S_0 d} + f_u e^{-rT} \\
&= e^{-rT} \left( \frac{(e^{rT} - u) (f_u - f_d)}{u - d} + \frac{f_u (u - d)}{u - d} \right) \\
&= e^{-rT} \left( \frac{(e^{rT} - u) (f_u - f_d)}{u - d} + \frac{f_u - f_d + f_d (u - d)}{u - d} \right) \\
&= e^{-rT} \left( f_u \left( \frac{e^{rT} - d}{u - d} \right) + f_d \left( 1 - \left( \frac{e^{rT} - d}{u - d} \right) \right) \right)
\end{align*}
\]
Generalization

• Or

\[ f = e^{-rT} \left[ pf_u + (1 - p)f_d \right] \]

where

\[ p = \left( \frac{e^{rT} - d}{u - d} \right) \]

**Example:** Recall from our example: \( u = 1.1, \) \( d = 0.9, \) \( r = 0.12, \) \( T = 0.25, \) \( f_u = 1, \) \( f_d = 0.\)

Then

\[ p = \frac{e^{0.12 \times 0.25} - 0.9}{1.1 - 0.9} = 0.6523 \]

and so

\[ f = e^{-0.12 \times 0.25} \left( 0.6523 \times 1 + 0.3477 \times 0 \right) = 0.633 \]

**Remark:** Pricing independent of the probability of the stock moving up or down! Calculating the price in terms of the underlying stock (incorporated in the value of the stock price already).