Lecture 15. Stochastic Processes & Stock Options

Introduction to Stochastic Processes
Wiener Processes and Itô’s Lemma

More sophisticated approach to modeling the behavior of assets underlying derivatives - view motion as a stochastic process.

A stochastic process is a process where future evolution is described by probability distributions.

Two types: discrete-time stochastic process changes values at discrete time steps. A continuous-time stochastic process changes value at any time.

Stochastic process can be continuous variable or discrete variable. A continuous-variable process can take any value within a certain range. (motion of a particle in fluid). A discrete-variable process can take only certain prescribed values. (coin flips)

Markov Process is a stochastic process where only present value of a variable is relevant for predicting the future. Coin flips are Markovian. If we flip the coin 30 times and comes up heads 30 straight times, next flip still 50/50 chance.
Let $\phi(\mu, \sigma)(x)$ denote the normal distribution. Then $\phi$ satisfies

$$
\phi(\mu, \sigma)(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}
$$

Probability of sum of
Consider a random number $x \in (0, 1)$ then most likely within the middle of the curve, if we undo it.

Sums of two normal distributions mean zero is a normal distribution with mean zero and variance that's the sum of the two variances. Proof: next time.
Continuous-Time Stochastic Processes

- Consider a Markov stochastic process. Suppose that the current value is 10 and the change in its value during 1 year is $\phi(0, 1)$.
- After two years? The change in two years is a sum of two one year Markov stochastic process with mean zero and standard deviation 1.
- Therefore, the sum is a normal distribution with mean zero and variance $1 + 1 = 2$. Thus the standard deviation is $\sqrt{2}$.
- Consider now the change in the variable during 6 months. The variance of the change in the value of the variable during 1 year equals the variance of the change during the first 6 months plus the variance of the change during the second 6 months.
- We assume they are the same. Then variance of change during a 6-month period must be 0.5. Thus, the standard deviation of the change is $\sqrt{0.5}$. Thus 6-month distribution is $\phi(0, \sqrt{0.5})$.
- Consider a small time step $\Delta t = \frac{1}{N}$, during which each period is an independent normal distribution. Then sum of the variances are equal to 1, so each variance should be $\Delta t$.
- The standard deviation then is $\sqrt{\Delta t}$.

Uncertainty is proportional to square root of time.
Wiener Processes

We continue letting $\Delta t \to 0$ carefully! This is called the Wiener process or Brownian motion. It is a Markov stochastic process with mean zero and variance 1.0 per year. Therefore, it has

1. Change $\Delta z$ during a small period of time $\Delta t$ is

$$\Delta z = \epsilon \sqrt{\Delta t}$$

where $\epsilon$ has a standardized normal distribution $\phi(0, 1)$. Therefore, $\Delta z$ has a normal distribution with

- mean of $\Delta z = 0$
- standard deviation of $\Delta z = \sqrt{\Delta t}$
- variance of $\Delta z = \Delta t$.

2. Values of $\Delta z$ for any two different short intervals of time $\Delta t$ are independent. Therefore, $z$ follows a Markov process.
Measure the value of \( z(T') - z(0) \) over a long period of time \( T \).

View as a sum of \( N \) small changes over small time changes \( \Delta t \), i.e.

\[
N = \frac{T}{\Delta t} \implies z(T) - z(0) = \sum_{i=1}^{N} \epsilon_i \sqrt{\Delta t}
\]

where \( \epsilon_i \) for \( i \in \{1, \ldots, N\} \) are distributed \( \phi(0, 1) \). The \( \epsilon_i \)'s are independent of each other.

Then \( z(T) - z(0) \) is normally distributed with

- mean of \( [z(T) - z(0)] \) = 0
- variance of \( [z(T) - z(0)] \) = \( N\Delta t = T \)
- standard deviation of \( [z(T) - z(0)] \) = \( \sqrt{\Delta T} \).
Approximating Wiener Process

\[ z \]

Relatively large value of \( \Delta t \)
Approximating Wiener Process

![Graph of Wiener Process]

Smaller value of $\Delta t$
Approximating Wiener Process

1. Expected length of the path followed by \( z \) in any time interval is infinite!

2. Expected number of times \( z \) equals any particular value in any time interval is infinite!
Self-similar structure. Lies in $C^{0,\frac{1}{2}}$. 
Generalized Wiener Process

The mean change per unit time for a stochastic process is known as the drift rate.

The variance per unit time for a stochastic process is known as the variance rate.

A generalized Wiener process for a variable $x$ can be defined in terms of $dz$ as

$$dx = adt + bdz$$

where $adt$ is the expected drift rate of $a$ per unit time.

Holds since $dx = adt \implies \frac{dx}{dt} = a$. Therefore,

$$x = x_0 + at$$

After time $T$ the variable $x$ travels $T$ units.
Generalized Wiener Process

The term $b \, dz$ regarded as noise added to the system, which is $b$ times a Wiener process.

In a small time interval $\Delta t$, the change $\Delta x$ in the variable of $x$ is given by

$$\Delta x = a \Delta t + b \epsilon \sqrt{\Delta t}$$

where $\epsilon$ has a standard normal distribution.

- mean of $\Delta x = a \Delta t$
- standard deviation of $\Delta x = b \sqrt{\Delta t}$
- variance of $\Delta x = b^2 \Delta t$.

The same argument show that the change in the value of $x$ in any time interval $T$ is normally distributed with

- mean of $x = aT$
- standard deviation of $x = b \sqrt{T}$
- variance of $x = b^2 T$. 
Figure 12.2  Generalized Wiener process with $a = 0.3$ and $b = 1.5$. 

\[
dx = a \, dt + b \, dz
\]

Value of variable, $x$
Itô Process

A generalized Wiener process in which the parameters $a$ and $b$ are functions of the value of the underlying variable $x$ and $t$. An Itô process can be written as

$$dx = a(x, t)dt + b(x, t)dz$$

Both the expected drift rate and the variance rate of an Itô process are liable to change over time. In a small time interval between $t$ and $t + \Delta t$, the variable changes from $x$ to $x + \Delta x$, where

$$\Delta x = a(x, t)\Delta t + b(x, t)\epsilon\sqrt{\Delta t}$$

Thus $b^2$ is the variance and $a$ is the mean during the interval between $t$ and $t + \Delta t$. 
Discuss the process that models stock movements for a nondividend paying stock:

- Expected return $\mu = \frac{\text{Expected drift}}{\text{Stock price}}$ is constant
- If $S$ is the stock price a time $t$, then the expected drift rate in $S$ should be assumed to be $\mu S$ for some constant parameter $\mu$.
- So in short period of time $\Delta t$ the expected increase in $S$ should be $\mu S \Delta t$.

If volatility of the stock is zero then model implies

$$\Delta S = \mu S \Delta t$$

In the limit $\Delta t \to 0$, 

$$dS = \mu S dt$$

or

$$\frac{dS}{S} = \mu dt$$

Then

$$S_T = S_0 e^{\mu T}$$
Including volatility then expect: variability of the percentage return in a short period of time $\Delta t$ is the same regardless of the stock price. This suggests that the standard deviation of the change in a short period of time $\Delta t$ should be proportional to the stock price and leads to

$$dS = \mu S \, dt + \sigma S \, dz$$

or

$$\frac{dS}{S} = \mu dt + \sigma dz$$  \hspace{1cm} (1)$$

We use (1) to price stocks. Here $\sigma$ is the volatility and $\mu$ is the expected return rate.

Limiting case of the random walk we saw with binomial trees.

**Example:** Consider a stock that pays no dividends, has a volatility of 30% per annum, and provides an expected return of 15% per annum with continuous compounding. In this case, $\mu = 0.15$ and $\sigma = 0.30$. The process for the stock price is

$$\frac{dS}{S} = 0.15 dt + 0.30 dz$$
Process for a stock price

If $S$ is the stock price at a particular time and $\Delta S$ is the increase in the stock price in the next small interval of time,

$$\frac{\Delta S}{S} = 0.15\Delta t + 0.30\epsilon\sqrt{\Delta t}$$

where $\epsilon$ has a standard normal distribution. Consider a time interval of 1 week or 0.0192 year and suppose that the initial stock price is $100. Then $\Delta t = 0.0192$, $S = 100$, and

$$\Delta S = 100 (0.00288 + 0.0416\epsilon)$$

or

$$\Delta S = 0.288 + 4.16\epsilon$$

showing that the price increase has a normal distribution with mean $0.288$ and standard deviation $2.16$. 
Discrete-Time Model

Discrete-time version of the model is
\[
\frac{\Delta S}{S} = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t}
\]  \hspace{1cm} (2)
so the change in the stock value over a short period of time is
\[
\Delta S = \mu S \Delta t + \sigma S \epsilon \sqrt{\Delta t}
\]

- Variable $\Delta S$ is the change in the stock price $S$ over a small interval of time $\Delta t$ and $\epsilon$ has a standard normal distribution (normal distribution with $\sigma = 1$ and $\mu = 0$).
- $\mu$ is the expected rate of return by the stock in a short period of time $\Delta t$.
- $\sigma$ is the volatility of the stock price.
Discrete-Time Model

Left-hand-side of (2) is the return provided by the stock in a short period of time.

- Term $\mu \Delta t$ is the expected value of the return
- Term $\sigma \epsilon \sqrt{\Delta t}$ is the stochastic component of the return. Variance is $\sigma^2 \Delta t$ (consistent with the definition of volatility defined earlier).

Then $\Delta S / S$ is normally distributed with mean $\mu \Delta t$ and standard deviation $\sigma \sqrt{\Delta t}$, so

$$\frac{\Delta S}{S} \sim \phi(\mu \Delta t, \sigma \sqrt{\Delta t})$$
Monte Carlo simulation of a stochastic process is a numerical procedure for sampling random outcomes for a process.

Example, consider a stock with expected return of 14% per annum with volatility of 20% per annum. Therefore, $\mu = 0.14$ and $\sigma = 0.20$. Suppose $\Delta t = 0.01$ (i.e. 1% of a year). Then

$$\Delta S = 0.14 \times 0.01 S + 0.2\sqrt{0.01}S\epsilon$$

or

$$\Delta S = 0.0014S + 0.02S\epsilon$$

In order to sample - take random number $(0, 1)$ - then use inverse normal distribution:
Monte Carlo

Table 12.1  Simulation of stock price when \( \mu = 0.14 \) and \( \sigma = 0.20 \) during periods of length 0.01 year.

<table>
<thead>
<tr>
<th>Stock price at start of period</th>
<th>Random sample for ( \epsilon )</th>
<th>Change in stock price during period</th>
</tr>
</thead>
<tbody>
<tr>
<td>20.000</td>
<td>0.52</td>
<td>0.236</td>
</tr>
<tr>
<td>20.236</td>
<td>1.44</td>
<td>0.611</td>
</tr>
<tr>
<td>20.847</td>
<td>-0.86</td>
<td>-0.329</td>
</tr>
<tr>
<td>20.518</td>
<td>1.46</td>
<td>0.628</td>
</tr>
<tr>
<td>21.146</td>
<td>-0.69</td>
<td>-0.262</td>
</tr>
<tr>
<td>20.883</td>
<td>-0.74</td>
<td>-0.280</td>
</tr>
<tr>
<td>20.603</td>
<td>0.21</td>
<td>0.115</td>
</tr>
<tr>
<td>20.719</td>
<td>-1.10</td>
<td>-0.427</td>
</tr>
<tr>
<td>20.292</td>
<td>0.73</td>
<td>0.325</td>
</tr>
<tr>
<td>20.617</td>
<td>1.16</td>
<td>0.507</td>
</tr>
<tr>
<td>21.124</td>
<td>2.56</td>
<td>1.111</td>
</tr>
</tbody>
</table>

More on Monte Carlo simulations later...
Parameters

The development of the pricing model depends on $\mu$ and $\sigma$ so far.

For derivatives that depend on the stock, not important to have $\mu$. However, very important to have $\sigma$. We saw this with binomial tree pricing.

The standard deviation of the proportional change in the stock price in a small interval of time $\Delta t$ is $\sigma \sqrt{\Delta t}$. The standard deviation of the proportional change in the stock price over a relatively long period of time $T$ is $\sigma \sqrt{T}$. 
An Itô process is one in which the drift and the volatility depend on both $x$ and $t$. Suppose $x$ is an Itô’s process then

$$dx = a(x, t)dt + b(x, t)dz$$

where $dz$ is a Wiener process and $a, b$ are functions of $x$ and $t$. Then $x$ has a variance $b^2$.

Itô’s Lemma states that a function $G$ of $x$ and $t$ follows the following process:

$$dG = \left( \frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} bdz$$

In particular $G$ is an Itô process with drift rate

$$\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2$$

and variance

$$\left( \frac{\partial G}{\partial x} \right)^2 b^2$$
Itô’s Lemma - formal argument

Assume $G$ is a function of two variables $x$ and $t$ then we can formally take a power series expansion

$$
\Delta G = \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} (\Delta x)^2 + \frac{\partial^2 G}{\partial x \partial t} (\Delta x \Delta t) + \frac{1}{2} \frac{\partial^2 G}{\partial t^2} (\Delta t)^2 + O(\Delta^3)
$$

An Itô process satisfies

$$
dx = a(x, t) \, dt + b(x, t) \, dz
$$

where $dz$ is a Wiener process. Then approximately (at the discrete level) we have

$$
\Delta x = a(x, t) \Delta t + b(x, t) \epsilon \sqrt{\Delta t}.
$$

Then returning to $\Delta G$ we have

$$
\Delta G = \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} (\Delta x)^2 + \frac{\partial^2 G}{\partial x \partial t} (\Delta x \Delta t) + \frac{1}{2} \frac{\partial^2 G}{\partial t^2} (\Delta t)^2 + O(\Delta^3)
$$

$$
= \frac{\partial G}{\partial x} \left[ a \Delta t + b \epsilon \sqrt{\Delta t} \right] + \frac{\partial G}{\partial t} \Delta t
$$

$$
+ \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \left[ a \Delta t + b \epsilon \sqrt{\Delta t} \right]^2 + \frac{\partial^2 G}{\partial x \partial t} \Delta t \left[ a \Delta t + b \epsilon \sqrt{\Delta t} \right] + \frac{1}{2} \frac{\partial^2 G}{\partial t^2} (\Delta t)^2 + O(\Delta^3)
$$

Expand out:
Itô’s Lemma - formal argument

$$\Delta G = \frac{\partial G}{\partial x} [a \Delta t + b \epsilon \sqrt{\Delta t}] + \frac{\partial G}{\partial t} \Delta t$$

$$+ \frac{1}{2} \frac{\partial^2 G}{\partial x^2} [a \Delta t + b \epsilon \sqrt{\Delta t}]^2 + \frac{\partial^2 G}{\partial x \partial t} \Delta t [a \Delta t + b \epsilon \sqrt{\Delta t}] + \frac{1}{2} \frac{\partial^2 G}{\partial t^2} (\Delta t)^2 + O(\Delta^3)$$

$$= \sqrt{\Delta t} b \epsilon \frac{\partial G}{\partial x}$$

$$+ \Delta t \left[ \frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + b^2 \epsilon^2 \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \right]$$

$$+ (\Delta t)^{\frac{3}{2}} \left[ 2ab \epsilon \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \right]$$

$$+ (\Delta t)^2 \left[ a^2 \frac{1}{2} \frac{\partial^2 G}{\partial x^2} + a \frac{\partial^2 G}{\partial x \partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial t^2} \right] + O((\Delta t)^3)$$

$$= \left[ \frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + b^2 \epsilon^2 \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \right] \Delta t + \frac{\partial G}{\partial x} b \epsilon \sqrt{\Delta t} + O((\Delta t)^{\frac{3}{2}})$$
Itô’s Lemma - formal argument

Therefore,

\[ \Delta G = \left[ \frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + b^2 \epsilon^2 \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \right] \Delta t + \frac{\partial G}{\partial x} b \epsilon \sqrt{\Delta t} + O((\Delta t)^{3/2}) \]

Since is a normal distribution, then the variance \( \epsilon^2 \) is 1. Thus \( 1 = E(\epsilon^2) - (E(\epsilon))^2 = E(\epsilon^2) \).

Therefore, the expected value of \( \epsilon^2 \Delta t \) is \( \Delta t \) (small fluctuations cancel out) and hence nonstochastic! Take the limit as \( \Delta t \to 0 \) then get formally

\[ dG = \left[ \frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + b^2 \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \right] dt + \frac{\partial G}{\partial x} b dz \]
Itô's Lemma: Modeling stock movements

We argued that a reasonable model of stock movements should be

\[ dS = \mu S dt + \sigma S dz \]

with \( \mu \) and \( \sigma \) constants.

From Itô’s Lemma we can consider a process \( G \) that depends on \( t \) and \( S \). Then

\[ dG = \left( \frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S dz \]

so both \( S \) and \( G \) are affected by \( dz \) - the noise in the system.
Lognormal Property

Recall that our model requires

\[ dS = \mu S dt + \sigma S dz \]

with \( \mu \) and \( \sigma \) constants.

Define \( G = \ln S \) then

\[
\frac{\partial G}{\partial S} = \frac{1}{S} \quad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2} \quad \frac{\partial G}{\partial t} = 0
\]

by Itô’s Lemma we have

\[
dG = \left( \frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S dz
\]

\[
= \left[ \frac{1}{S} \mu S + 0 + \frac{1}{2} \frac{1}{S^2} \sigma^2 S^2 \right] dt + \frac{1}{S} \sigma S dz
\]

\[
= \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dz
\]

Therefore, \( G \) follows a generalized Wiener process with

- Drift = \( \mu - \frac{\sigma^2}{2} \)
- Variance = \( \sigma^2 \)
Lognormal Property

Therefore, the change in \( \ln S \) between 0 and a future time \( T \) is normally distributed with mean 
\[(\mu - \frac{\sigma^2}{2})T\] and variance \( \sigma^2 T \). Hence:

\[
\ln S_T - \ln S_0 \approx \phi \left[ \left( \mu - \frac{\sigma^2}{2} \right) T, \sigma \sqrt{T} \right]
\]
or

\[
\ln S_T \approx \phi \left[ \ln S_0 + \left( \mu - \frac{\sigma^2}{2} \right) T, \sigma \sqrt{T} \right]
\]

This implies that \( \log \) of the stock price is normally distributed:

A variable has a lognormal distribution if the natural \( \log \) of the variable is normally distributed.

The standard deviation of the logarithm of the stock price is \( \sigma \sqrt{T} \).
Application of Itô’s Lemma to Forward Contracts

A forward contract on a non-dividend-paying stock with interest rate \( r \) then

\[
F_0 = S_0 e^{rT}
\]

Let \( F \) be the forward price at a general time \( t \) and \( S \) the stock price at time \( t < T \). The relationship between \( F \) and \( S \) is

\[
F = S e^{r(T-t)}
\]

If \( S \) is given by

\[
dS = \mu S dt + \sigma S dz
\]

then we compute

\[
\frac{\partial F}{\partial S} = e^{r(T-t)} \quad \frac{\partial^2 F}{\partial S^2} = 0 \quad \frac{\partial F}{\partial t} = -rS e^{r(T-t)}
\]

then by Itô’s Lemma so that

\[
dF = \left[ e^{r(T-t)} \mu S - rS e^{r(T-t)} \right] dt + e^{r(T-t)} \sigma S dz = (\mu - r) F dt + \sigma F dz
\]
Application of Itô’s Lemma to Forward Contracts

Then $F$ is lognormally distributed. Set $G = \ln F$ then

$$dG = \left( \mu - r - \frac{\sigma^2}{2} \right) dt + \sigma dz$$

Thus

$$\ln F - \ln F_0 \approx \phi \left[ \left( \mu - r - \frac{\sigma^2}{2} \right) T, \sigma \sqrt{T} \right]$$
Summary

- Stochastic process describe the probabilistic evolution of the value of a variable through time.
- Markov process is a stochastic process where only the present value of the variable is relevant for predicting the future.
- Wiener process $dz$ is the process describing the evolution of a normally distributed variable. Drift of the process is zero and variance rate is 1.0 per unit time.
- Generalized Wiener process describes the evolution of a normally distributed variable with a drift of $a$ per unit time and a variance rate of $b^2$ per unit time, where $a$ and $b$ are constants. If the variable starts at 0 then it is normally distributed with mean $aT$ and standard deviation of $b\sqrt{T}$ at time $T$.
- Itô process is a process where the drift and variance rate of $x$ can be a function of both $x$ itself and time. The change in $x$ in a very short period of time is normally distributed, but changes over longer periods of time is liable to be nonnormal.
- Itô process is a way of calculating the stochastic process followed by a function of a variable from the stochastic process followed by the variable itself.
Use Itô’s Lemma to help price options. Let $f$ be the price of a call option depending on a $S$ that satisfies our model.

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rS$$

with the boundary condition

$$f = \max\{S - K, 0\}$$

when $t = T$.

Where does this partial differential equation (PDE) come from?

How do we solve it?
Due Oct. 31, 5PM.

• 11.11, 11.15

• Graded: 11.19, 11.20 (Steps (a)&(b)), 11.21 (Steps (a)&(b)), 12.12, 12.15