Midterm Review
Fall 2007
What is a derivative?

A financial instrument whose value derives from the value of underlying variables.
Financial instruments used in derivative transactions

I. Forwards

II. Futures

III. Swaps

IV. Options
Forward Contracts

A **forward contract** is an agreement to buy or sell an asset at a **certain time** for a **certain price**.

Usually traded in the OTC market.

One party assumes a **long position** by agreeing to **buy** the underlying asset on a specified future date for a specified price.

The other party assumes a **short position** by agreeing to **sell** the underlying asset on the same date and at the same price.

A **spot contract** is an agreement to buy or sell an asset **today**.

Spot contracts are for immediate delivery of the asset. Common among currency derivatives.
Payoffs for Forward Contracts, cont.

In general $K$ is the delivery price and $S_T$ is the spot price of the asset at maturity then we define the following payoffs for forward contracts.

The payoff for a long forward contract is

$$S_T - K.$$  

The payoff for a short forward contract is

$$K - S_T.$$
Futures Contracts

• Similar to a forward contract - it is an agreement between two parties to buy or sell an asset at a certain time in the future for a certain price.

• However, futures are usually traded in exchanges.

• Mechanisms by the exchange to guarantee that the contract will be honored.

• Futures on
  – commodities such as pork bellies, orange juice, copper, sugar, etc.
  – financial assets such as stock indices, currencies, Treasury bonds, etc.
Types of derivative traders in the market

- **Hedgers** - use derivatives to reduce risk in the market from potential future market movements.

- **Speculators** - trade derivatives to bet on the future direction of a market variable.

- **Arbitrageurs** - take offsetting positions in two or more instruments to lock in a profit. (usually short-lived)
Hedging

- Forward contracts **neutralize risk** by fixing prices that the hedger will pay or receive for the underlying asset.

- Options contracts offer **insurance** for investors to protect against adverse price movements.
Arbitrageurs

Involve locking in riskless profit by simultaneously entering into transactions in two or more markets.

Mostly possible when future prices become out of line with spot prices.
## Forward vs. Futures Contracts

<table>
<thead>
<tr>
<th>Forward</th>
<th>Futures</th>
</tr>
</thead>
<tbody>
<tr>
<td>Private contract between two parties</td>
<td>Traded on an exchange</td>
</tr>
<tr>
<td>Not standardized</td>
<td>Standardized contract</td>
</tr>
<tr>
<td>Usually one specified delivery date</td>
<td>Range of delivery dates</td>
</tr>
<tr>
<td>Settled at end of contract</td>
<td>Settled daily</td>
</tr>
<tr>
<td>Delivery or final cash settlement usually</td>
<td>Contract is usually closed out</td>
</tr>
<tr>
<td>takes place</td>
<td>prior to maturity</td>
</tr>
<tr>
<td>Some credit risk</td>
<td>Virtually no credit risk</td>
</tr>
</tbody>
</table>

- Under the forward contract, the entire gain or loss is realized at the end of the life of the contract.

- Under the futures contract, the entire gain or loss is realized day-by-day because of **marking to market** or daily settlement.
Minimum Variance Hedge Ratio

We set the following:

\[ \Delta S \equiv \text{Change in spot price over the life of the hedge} \]
\[ \Delta F \equiv \text{Change in spot price over the life of the hedge} \]
\[ \sigma_S \equiv \text{Standard deviation of } S \]
\[ \sigma_F \equiv \text{Standard deviation of } F \]
\[ \rho \equiv \text{Coefficient of correlation between } \Delta S \text{ and } \Delta F \]
\[ h^* \equiv \text{Hedge ratio that minimizes the variance of the hedger’s position} \]

Claim

\[ h^* = \rho \frac{\sigma_S}{\sigma_F} \]

Note that \( \sigma_S, \sigma_F \) and \( \rho \) are estimated from historical data.
Minimum Variance Hedge Ratio

As we saw, choosing

\[ h^* = \rho \frac{\sigma_S}{\sigma_F} \]

minimizes the variance of \( \Delta S - h \Delta F \).

Minimizing the variance ensures that there will be less risk in using futures with an underlying asset that differs from the original asset.
Minimum Variance Hedge Ratio

- If \( \rho = 1 \) (completely correlated) and \( \sigma_F = \sigma_S \) then \( h^* = 1 \). In this case the futures price mimics the spot price.

- If \( \rho = 1 \) and \( \sigma_F = 2\sigma_S \) then \( h^* = 0.5 \). This follows since the futures price always changes by twice the spot price.

- Optimal hedge ratio \( h^* \) is the slope of the best-fit line when \( \Delta S \) is regressed against \( \Delta F \).

- We expect there to be a linear relationship between \( \Delta S \) and \( \Delta F \).
Minimum Variance Hedge Ratio, cont.

Regression of change in spot price against change in futures price

- The **hedge effectiveness** is defined as the proportion of the variance that is eliminated by hedging.

- We define this as $\rho^2$ (correlation squared) or

\[
(h^*)^2 \frac{\sigma_F^2}{\sigma_S^2}
\]
Example of Hedge Ratio calculation

Define

\[ N_A \equiv \text{Size of the position being hedged (units)} \]
\[ Q_F \equiv \text{Size of one futures contract (units)} \]
\[ N^* \equiv \text{Optimal number of futures contracts for hedging} \]

- The number of futures contracts should have a value of \( h^* N_A \).

- The number of futures contracts required is given by

\[
N^* = \frac{\text{value}}{\text{price per contract}} = \frac{h^* N_A}{Q_F}
\]
Rolling the hedge forward

- If the hedge expiration happens after the delivery date of the futures contract, need to roll the hedge

- Accomplished by closing out the futures contract and taking the same futures contract with a later delivery date

Consider a sequence of futures contracts listed $1, 2, 3, \ldots, n$ the company uses:

- $t_1$: Short futures contract 1
- $t_2$: Close contract 1 & Short futures contract 2
- $t_3$: Close contract 2 & Short futures contract 3
- $t_n$: Close contract n-1 & Short futures contract n
An interest rate defines the amount of money a borrower promises to pay a lender. Types of rates:


- **LIBOR - London Interbank Offer Rate**: by a bank is the rate at which the bank is prepared to make a large deposit with other banks. Usually 1-month, 3-month, 6-month, and 12-month LIBOR in major currencies.

- Banks require good (AA) credit ratings to accept a LIBOR quote.

- LIBOR rates are not free of risk, but considered close to risk-free.
Interest rates

Terminal value for an initial deposit of $A$ at a rate of $R$ over $n$ years is

$$A(1 + R)^n$$

Terminal value for an initial deposit of $A$ at a rate of $R$ over $n$ years, compounded $m$ times per year is

$$A \left(1 + \frac{R}{m}\right)^{mn}$$

An investment compounded continuously for $n$ years at a rate of $R$ is

$$Ae^{Rn}$$

Continuous compounding is very close to daily ($m = 365$) compounding.
Continuous Compounding

- **Compounding** a sum of money at a continuously compounded rate $R$ for $n$ years involves multiplying by $e^{Rn}$.

- **Discounting** a sum of money at a continuously compounded rate $R$ for $n$ years involves multiplying by $e^{-Rn}$.

- Derivatives use continuously compounded interest rates more frequently than other types.

Converting continuously compounded rate $R_C$ to $m$-times compounding per year rate $R_m$:

$$Ae^{R_C n} = A \left(1 + \frac{R_m}{m}\right)^{mn}$$

$$R_C = m \ln \left(1 + \frac{R_m}{m}\right)$$

$$R_m = m \left(e^{\frac{R_C}{m}} - 1\right)$$
Zero Rates

- An $n$-year zero-coupon interest rate is the rate of interest on an investment that starts today and last $n$ years.

- All interest and principle is realized at the end of the $n$th year (no intermediate payments).

- Also called $n$-spot rate or $n$-year zero rate.
Bond Pricing

• Bonds usually provide **coupons** or interest payments, periodically.

• Principal (called **face or par value**) is payed at the end of the life.

• Price of a bond can be calculated as the present value of all cash that will be received by the owner of the bond.
• The yield on a bond is the discount rate that, when applied to all cash flows, gives a bond price equal to the market price.

• Consider the previous case with a theoretical value of 98.39. The yield, $y$, is calculated via

$$3e^{-y\times0.5} + 3e^{-y\times1.0} + 3e^{-y\times1.5} + 103e^{-y\times2.0} = 98.39.$$ 

• This is a fourth order polynomial in $e^{-y/2}$. Easy to calculate with a computer:

$$y = 6.76\%$$
The par yield for a certain maturity is the coupon rate that causes the bond prices to equal its par value.

Consider our 2 year bond with $c$-annual coupon rate (so $\frac{c}{2}$ semi-annually). Then

$$
\frac{c}{2}e^{-0.05\times0.5} + \frac{c}{2}e^{-0.058\times1.0} + \frac{c}{2}e^{-0.064\times1.5} + \left(100 + \frac{c}{2}\right)e^{-0.068\times2.0} = 100
$$

This is a linear equation. $c = 6.87\%$. Thus the 2-year par yield is 6.87\% per year with semiannual compounding.
Determining Treasury Zero Rates

To calculate the Treasury Zero Rates, bootstrap using data. Calculation via an example:

<table>
<thead>
<tr>
<th>Bond Principle</th>
<th>Time to Maturity</th>
<th>Annual Coupon</th>
<th>Bond Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.25</td>
<td>0</td>
<td>97.5</td>
</tr>
<tr>
<td>100</td>
<td>0.50</td>
<td>0</td>
<td>94.9</td>
</tr>
<tr>
<td>100</td>
<td>1.00</td>
<td>0</td>
<td>90.0</td>
</tr>
<tr>
<td>100</td>
<td>1.50</td>
<td>8</td>
<td>96.0</td>
</tr>
<tr>
<td>100</td>
<td>2.00</td>
<td>12</td>
<td>101.6</td>
</tr>
</tbody>
</table>

With half coupons paid semiannually. Use $R_C = m \ln \left(1 + \frac{R_m}{m}\right)$ to determine $R_C$.

- 3-month bond: return of $2.5 on an investment of $97.5. Then

$$\frac{4 \times 2.5}{97.5} = 0.10256$$

per annum. Then

$$R_C = 4 \ln \left(1 + \frac{0.10256}{4}\right) = 0.10127 \text{ per annum}$$
Determining Treasury Zero Rates, cont.

• 6-month bond: return of $5.1 on an investment of $94.9. Then

\[
\frac{2 \times 5.1}{94.9} = 0.10748
\]

per annum. Then

\[
R_C = 2 \ln \left( 1 + \frac{0.10748}{2} \right) = 0.10469 \text{ per annum}
\]

• 1-year bond: return of $10 on an investment of $90. Then

\[
\frac{1 \times 10}{90} = 0.11111
\]

per annum. Then

\[
R_C = \ln (1 + 0.11111) = 0.10536 \text{ per annum}
\]
• 18-month bond: three coupon payments of $4 per 6 months.
  – At 6 months, $4 coupon. Discount rate from above is 10.469%.
  – At 1 year, $4 coupon. Discount rate from above is 10.536%.
  – For the payment of $104 at 1.5 years, denote the discount rate $R$. The present value is $96, so:

\[
4e^{-0.10469 \times 0.5} + 4e^{-0.10536 \times 1.0} + 104e^{-R \times 1.5} = 96
\]

or

\[
e^{-R \times 1.5} = 0.85196 \quad R = -\frac{\ln 0.85196}{1.5} = 0.10681
\]

• 2-year bond: four coupon payments of $6 per 6 months. Follow the same procedure:

\[
6e^{-0.10469 \times 0.5} + 6e^{-0.10536 \times 1.0} + 6e^{-0.10681 \times 1.0} + 106e^{-R \times 2.0} = 101.6
\]

or $R = 0.10808$ or 10.808%.
Determining Treasury Zero Rates, cont.

Bond data.

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Resulting Zero Coupon Rates

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<tr>
<th>Maturity</th>
<th>Zero Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>10.127</td>
</tr>
<tr>
<td>0.50</td>
<td>10.469</td>
</tr>
<tr>
<td>1.00</td>
<td>10.536</td>
</tr>
<tr>
<td>1.50</td>
<td>10.681</td>
</tr>
<tr>
<td>2.00</td>
<td>10.808</td>
</tr>
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Forward Rates

The Forward interest rates are the interest rates implied by current zero rates for periods of times in the future. Rates are assume to compound continuously.

Example:

- Suppose an investment returns 3% per year for a one year investment. Then $100 \rightarrow 100e^{0.03 \times 1} = 103.05$ after one year.

- Suppose the investment returns 4% per year for a two year investment. Then $100 \rightarrow 100e^{0.04 \times 2} = 108.33$ after two years.

- The forward rate is implied by the zero rates for the period of time between the end of the first year and end of the second year

\[ 100e^{0.04 \times 2} = 108.33 = 100e^{0.03 \times 1} e^{R \times 1} \]

so $R = 0.05$. This is the averaged rate.
Forward Rates, cont.

- If the 3 year rate is 4.6%, then the 3-year forward rate comes from

\[ 100e^{0.046 \times 3} = 100e^{0.03 \times 1}e^{0.05 \times 1}e^{R \times 1} \]

so \( R = 0.138 - 0.08 = 0.058 \).

- If \( R_1 \) and \( R_2 \) are the zero rates for maturities \( T_1 \) and \( T_2 \) then the forward interest rate \( R_F \) yields

\[ R_F = \frac{R_2T_2 - R_1T_1}{T_2 - T_1} \]
The duration of a bond is a measure of how long the holder of the bond has to wait to receive cash payments (averaged).

- An $n$-year zero coupon bond has a duration of $n$ years (since payment at the end)
- A coupon-bearing bond has a duration $< n$.

Suppose a bond gives cash flows $c_i$ at time $t_i$ for times $t_1, \ldots, t_n$. The price $B$ and yield $y$, continuously compounded, satisfies

$$B = \sum_{i=1}^{n} c_i e^{-y t_i}$$

The duration $D$ is defined as

$$D = \frac{1}{B} \sum_{i=1}^{n} t_i c_i e^{-y t_i} = \sum_{i=1}^{n} t_i \left[ \frac{c_i e^{-y t_i}}{B} \right]$$
Duration

Note this is a filter: sum of weights is one. Consider a change in the yield $\Delta y$ then the bond value changes via:

$$
\Delta B = \sum_{i=1}^{n} c_i e^{-y_2 t_i} - \sum_{i=1}^{n} c_i e^{-y_1 t_i}
$$

$$
\Delta B = - (\Delta y) BD
$$

This gives us the duration relationship with yield changes

$$
\frac{\Delta B}{B} = - D \Delta y
$$
Pricing Nonincome producing Forward Contracts

• Consider a forward contract on investment asset with price $S_0$ with no income

• Let $F_0$ be the future position then

$$F_0 = S_0 e^{rT}$$

Why?

• If $F_0 > S_0 e^{rT}$ then arbitrageur follows the strategy:
  – Borrow $S_0$ dollars at an interest rate $r$ for $T$ years
  – Enter into a short forward contract on the asset

• If $F_0 < S_0 e^{rT}$ then arbitrageur follows the strategy:
  – Short the asset for $S_0$ dollars
  – Enter into a long forward contract on the asset

The formula follows from buying one unit of the asset and enter a short forward contract to sell it for $F_0$ at time $T$. This costs $S_0$ and leads to a cash influx of $F_0$ at time $T$. Therefore, $S_0$ must equal the present value of $F_0$, or

$$S_0 = F_0 e^{-rT} \equiv \text{ if and only if } F_0 = S_0 e^{rT}$$
Consider a forward contract on investment asset with price $S_0$ with income with present value $I$ over life of contract.

Let $F_0$ be the future position then

$$F_0 = (S_0 - I) e^{rT}$$
Pricing Forward Contracts with Known Yield

We now consider pricing forward contracts with fixed known yield instead of a known income. In particular the income is expressed as a percentage of the asset’s value at the time the income is paid.

Let \( q \) be the average yield per annum on an asset during the life of a forward contract with continuous compounding, then

\[
F_0 = S_0 e^{(r-q)T}.
\]
We consider how to determine the price of stock indices.

- Stock index can usually be regarded as the price of an investment asset that pays dividends.
- The investment asset is the portfolio of stocks underlying the index.
- Dividends paid by the investment asset are the dividends that would be received by the holder of the portfolio.
- Usually assumed that dividends provide a known yield rather than a known cash income.
- If $q$ is the dividend yield rate gives the future price $F_0$ as

$$F_0 = S_0 e^{(r-q)T}$$
• Consider the price of contracts on currency exchanges
• Let $S_0$ define the current spot price, in dollars, of one unit of foreign currency.
• Let $F_0$ define the forward or futures price in dollars of one unit of foreign currency.
• This is not necessarily the way in which such contracts are quoted.
• Major exchange rates (outside of the pound, euro, Australian dollar, and New Zealand dollar) quote number of units of currency equivalent to one dollar.
• Foreign currency earns interest at the risk-free rate in the foreign country.

Let $r_f$ be the value of foreign risk-free interest rate when money is invested for time $T$. Let $r$ be the US dollar risk-free rate when money is invested for time $T$.

$$F_0 = S_0 e^{(r-r_f)T}$$
Valuing Forward Contracts

Forward contracts initially have a value of zero, but the value changes daily with the marking to market.

\[ K \equiv \text{delivery price for a contract that was negotiated in the past} \]
\[ T \equiv \text{time to delivery date, in years} \]
\[ r \equiv \text{the } T\text{-year risk-free interest rate} \]
\[ F_0 \equiv \text{forward price that would be applicable if negotiated the contract today} \]
\[ f \equiv \text{value of the forward contract today} \]

\[ f_0 = (F_0 - K) e^{-rT} \]
Valuing Forward Contracts

- Value of forward contracts with no income

\[ f = S_0 - Ke^{-rT} \]

- Value of forward contracts with known income

\[ f = S_0 - I - Ke^{-rT} \]

- Value of forward contracts with known yield

\[ f = S_0e^{-rT} - Ke^{-rT} \]
We consider pricing futures contracts on commodities that are primarily investment assets, such as precious metals.

- Gold owners can earn income from leasing the gold.
- The interest charged from leasing gold is the gold lease rate.
- Also holds for silver.

In the absence of storage and income, the forward price of the commodity that is an investment asset is

$$F_0 = S_0 e^{rT}$$

We treat storage costs as negative income, then set $U$ to be the present value of all storage costs, net of income, during the life of a forward contract, then

$$F_0 = (S_0 + U) e^{rT}$$
Cost of Carry

Cost of carry is storage costs and interest that is paid to finance the asset less the income earned on the asset.

- Nondividend-paying stocks, the cost of carry is $r$ (no storage & no income)
- Stock index, the cost of carry is $r - q$ (income earned at rate $q$)
- Currency futures, the cost of carry is $r - r_f$ (difference in the foreign risk-free rate)
- Commodity with income $q$ and storage costs at rate $u$, the cost of carry is $r - q + u$

Cost of carry is

$$F_0 = S_0 e^{cT}$$

for investment assets and

$$F_0 = S_0 e^{(c-y)T}$$

for consumption assets with $y$ the convenience yield.
Swaps

- A swap is an agreement between two companies to exchange cash flows in the future.
- An agreement includes the dates when the cash flows are paid and the way in which they are calculated.
- Most common swap is a plain vanilla swap: One company agrees to pay cash flows equal to interest at a predetermined fixed rate on a notional principal for a number of years.
- In return, the other company pays interest at a floating rate on the same notional principal for the same period of time.
- Most common floating rate is the LIBOR. Typically 1-month, 3-month, 6-month, and 12-month rates.

Example: Consider 5-year bond with a rate of interest specified as 6-month LIBOR plus 0.5% per annum.

- Life the bond is divided into 10 6-month periods.
- Each period has a rate of interest set at 0.5% per annum above the 6-month LIBOR rate at the beginning of the period.
- Interest is paid at the end of the period.
Swaps, cont.

- Notional Principal is not exchanged.

- Since principal is not exchanged, then the swap can be viewed as an exchange of:

  fixed-rate bond $\iff$ floating-rate bond

Role of Financial Intermediary

- Non-financial companies usually do not get in touch with each other to arrange swaps.
- Rather companies deal through a financial intermediary.
- Plain-vanilla fixed-point swaps are structured so that financial institutions earn about 3 or 4 basis points on a pair of offsetting transactions per year.
Valuation

Principal is not exchange in swap agreements.

In valuing swaps helpful to think of the principal as being exchanged. Then from the perspective of the floating-rate payer, a swap can be regarded as a long position in fixed rate bond and short position in floating rate bond. Yields a swap value of

\[ V_{swap} = B_{fix} - B_{fl} \]

where \( B_{fl} \) is the value of the floating-rate bond and \( B_{fix} \) is the value of the fixed-rate bond.

From the perspective of the fixed-rate payer, a swap can be regarded as a short position in fixed rate bond and long position in floating rate bond. Yields a swap value of

\[ V_{swap} = B_{fl} - B_{fix} \]

Note that a bond is worth the notional interest immediately after interest payment, since LIBOR has just been rolled-over and fair market value has been issued.
Valuation of swaps in terms of FRA

A swap can be characterized as a portfolio of FRA’s.

In the 3-year Microsoft-Intel swap, the first exchange of payments is known at the swap negotiations.

The next 5 exchanges can be regarded as forward rate agreements, i.e. the exchange on Sept. 5, 2005 is an FRA of 5% for interest at the 6-month rate observed March 5, 2005, etc.

Since FRA’s can be valued by assuming that forward interest rates are realized, then our plain vanilla swap can be valued via FRA valuing:

1. Use the LIBOR/swap zero curve to calculate forward rates for each of the LIBOR rates that will determine swap cash flows
2. Calculate swap cash flows on the assumption that the LIBOR rates will equal the forward rates
3. Discount these swap cash flows using the LIBOR/swap zero curve to obtain the swap value
Currency Swaps

A currency swap is an agreement to exchange principal and interest payments in one currency for principal and interest payments in another.

- Requires the principal be specified in each of the two currencies.
- Principal amounts (usually approximately equivalent) are usually exchanged at the beginning and at the end of the swap.
- At the end of the swap, values may be very different.
Valuation of Currency Swaps

Fixed-for-fixed currency swaps can be valued by considering it as the difference between bonds or a portfolio of forward foreign exchange contracts.

Valuation in Terms of Bond Prices:

Let $V_{swap}$ denote the value in US dollars of an outstanding swap where dollars are received and foreign currency is paid.

$$V_{\text{swap}} = B_D - S_0 B_F$$

- $B_F$ is the value, measured in foreign currency, of the bond defined by the foreign cash flows on the swap.
- $B_D$ is the value of the bond defined by the domestic cash flows on the swap
- $S_0$ is the spot exchange rate (expressed as number of dollars per unit of foreign currency)
- Value of swap is determined from
  - LIBOR rates in the two currencies.
  - Term structure of interest rates in the domestic currency
  - Spot exchange rate

Other hand have reverse value for foreign currency received and USD are paid is

$$V_{\text{swap}} = S_0 B_F - B_D$$
Two most common types of swaps are plain vanilla interest rate swaps and currency swaps. In an interest rate swap, one party agrees to pay the other party interest at a fixed rate on a notional principal for a number of years. In return it receives interest at a floating rate on the same notional principal amount for the same period of time.

Principal amounts usually are not exchanged in an interest rate swaps. In a currency swap, principal amounts are usually exchanged at both the beginning and the end of the life of the swap.

In a currency swap one party agrees to pay interest on a principal amount in one currency. In return it receives interest on a principal amount in a different currency.

For the party paying interest in the foreign currency, the foreign principal is received, and the domestic principal is paid at the beginning of the life of the swap.

At the end of the swap, the foreign principal is paid and the domestic principal is received.

Interest rate swap can be used to transform floating-rate loan/asset into a fixed-rate loan/asset, or vice-versa.

Currency swap can be used to transform a loan/investment in one currency into a loan/investment in another currency.

Two ways of valuing interest rate an currency swaps. First method: swap is a long position in one bond and a short position in another bond. Second method: swap is a portfolio of forward contracts.
Options are different from Forward and Futures contracts in that an option gives the right to do something, but does not need to exercise this right.

Forward and Future contracts ensure that the parties commit to an agreement.

**Types of Options:**

- **Call option** - gives the owner the right to buy an asset by a certain date for a certain price
- **Put option** - gives the owner the right to sell an asset by a certain date for a certain price
- **Expiration date or maturity date** - is the date in which the call/put option expires.
- **Exercise price or strike price** - is the price specified in the option contract.
- **American option** - is an option that can be exercised any time up to expiration date.
- **European option** - is an option that can be exercised only at expiration date.
So far we have discussed owning the call option and the put option. This is the long position in a call or put option.

On the other end is the seller of the call or put option. This is the short position.

The short position earns money up-front; however, there are liabilities to selling the position.

We characterize the European option position in terms of the terminal value. Let $S_T$ be the spot price and $K$ the strike price.
Option Positions

There are four positions:

• **long position** in a call option - then the payoff is:

\[ \max\{S_T - K, 0\} \]

• **short position** in a call option - this is the opposite payoff as the long call position. Therefore, the payoff is:

\[ -\max\{S_T - K, 0\} = \min\{K - S_T, 0\} \]

• **long position** in a put option - then the payoff is:

\[ \max\{K - S_T, 0\} \]

• **short position** in a put option - this is the opposite payoff as the long call position. Therefore, the payoff is:

\[ -\max\{K - S_T, 0\} = \min\{S_T - K, 0\} \]
Figure 8.5  Payoffs from positions in European options: (a) long call; (b) short call; (c) long put; (d) short put. Strike price = $K$; price of asset at maturity = $S_T$. 
Properties of Stock Options

Factors affecting options prices

1. Current Stock Price $S_0$
2. Strike Price $K$
3. Time to Expiration $T$
4. Volatility of the Stock Price $\sigma$
5. Risk-free Interest Rate $r$
6. Dividends expected during the Life of the Option

Effect on Options Price:

<table>
<thead>
<tr>
<th>Variable</th>
<th>European call</th>
<th>European put</th>
<th>American call</th>
<th>American put</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current stock price</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>Strike price</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>Time to expiration</td>
<td>?</td>
<td>?</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Volatility</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>Risk-free rate</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>Amount of future dividends</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>
Upper Bounds

• A call option gives the owner the right to buy one share for a certain price. If the option was worth more than the stock, then the arbitrageur can buy the stock at $S_0$ and sell the call option at $C$ for a risk-free profit. Hence

$$c \leq S_0 \quad \quad C \leq S_0$$

• A put option gives the owner the right to sell one share for a certain price $K$. It follows that

$$p \leq K \quad \quad P \leq K$$

Otherwise, sell the put option at $p$. Then no matter what happens you earn at least $p - K > 0$.

• For European options, at maturity the option is worth less than $K$. It follows that the option cannot be worth more than the present value of $K$ today:

$$p \leq K e^{-rT}$$
Lower Bound for Calls on Non-Dividend-Paying Stocks

A lower bound for the price of a European call option on a non-dividend-paying stock is

\[ c \geq S_0 - K e^{-rT} \]

Why? Consider two portfolios,

1. one European call option plus an amount of cash equal to \( Ke^{-rT} \)
2. one share

In Portfolio 1, if the cash is invested in risk-free interest rate, will grow to \( K \) in time \( T \). If \( S_T > K \), the call option is exercised at maturity and portfolio 1 is worth \( S_T \). If \( S_T < K \), the call option expires worthless and the portfolio is worth \( K \). Hence at time \( T \) Portfolio 1 is worth

\[ \max\{S_T, K\} \]
Portfolio 2 is worth $S_T$ at time $T$. Therefore, Portfolio 1 is always worth as much as (an perhaps more than) Portfolio 2 at the option’s maturity. If follows that in the absence of arbitrage, it must be true today. Hence,

$$c + Ke^{-rT} \geq S_0 \implies c \geq S_0 - Ke^{-rT}$$

Because the worst case is that the call option is not exercised (worthless) then $c \geq 0$ so

$$c \geq \max\{S_0 - Ke^{-rT}, 0\}$$
Lower Bound for European Puts on Non-Dividend-Paying Stocks

A lower bound for the price of a European call option on a non-dividend-paying stock is

\[ p \geq Ke^{-rT} - S_0 \]

Why? Consider two portfolios,

1. one European put option plus one share
2. an amount of cash equal to \( Ke^{-rT} \)

If \( S_T < K \) then the option in Portfolio 1 is exercised at option maturity, and the portfolio becomes worth \( K \). If \( S_T > K \) then the put option expires worthless and the portfolio is worth \( S_T \) at this time. Hence Portfolio is worth

\[ \max\{S_T, K\} \]

in time \( T \).
Lower Bound for European Puts on Non-Dividend-Paying Stocks

Assuming the cash is invested at the risk-free interest rate portfolio 2 is worth $K$ at time $T$. Hence portfolio 1 is always worth as much as (maybe more than) portfolio 2 in time $T$. It follows that portfolio 1 must be worth at least as much as portfolio 2 today. Hence,

$$p + S_0 \geq Ke^{-rT} \implies p \geq Ke^{-rT} - S_0$$

Since $p \geq 0$ then

$$p \geq \max\{Ke^{-rT} - S_0, 0\}$$
Put-Call Parity

We can derive a relationship between $p$ and $c$.

1. one European call option plus an amount of cash equal to $Ke^{-rT}$
2. one European put option plus one share

Claim: both portfolios are worth $\max\{S_T, K\}$ at expiration.

If $S_T > K$ then for portfolio 1 the call is exercised and the stock is purchased with a profit of $K + (S_T - K) = S_T$. For portfolio 2 the put expires and the value of the portfolio is the share value $S_T$.

If $S_T < K$ then for portfolio 1, the call expires and the portfolio is worth $K$. For portfolio 2 the put is exercised and the stock is sold with value $S_T + (K - S_T) = K$.

Since the options are European, the options cannot be exercised until maturity. Therefore, the portfolios must have identical values today.

$$c + Ke^{-rT} = p + S_0$$

This relationship is called the put-call parity.
Put-call parity on American Options

What about a Put-Call parity for American Options? Start with a similar portfolio as before,

1. one European call plus cash equal to \( Ke^{-rT} \)
2. one American put plus one share

From Put-Call parity for European options, we would have

\[
p + S_0 = c + Ke^{-rT}
\]

if there was a European put instead of an American put. But \( P \geq p \) so

\[
P \geq p = c + Ke^{-rT} - S_0.
\]

We also know that an American call is equal to a European call if there is no dividend. Thus

\[
C = c, \text{ so}
\]

\[
C - P \leq Ke^{-rT} - S_0.
\]
Put-call parity on American Options

Next, consider the following portfolio,

1. one European call plus cash equal to $K$
2. one American put plus one share

Portfolio 1 has the cash invested in risk-free rate, so worth $K e^{rT}$. Therefore, Portfolio 1 is worth

$$\max\{S_T - K, 0\} + Ke^{rT} = \max\{S_T, K\} - K + Ke^{rT}$$

If the put is not exercised early then Portfolio 2 is worth $\max\{S_T, K\}$ at time $T$. This implies 

Port. 1 $\geq$ Port. 2.

Now suppose Port. 2 is exercised early at time $t < T$. Then Port. 2 is worth

$S_T + (K - S_T) = K$. However, Port.1 is worth at least $Ke^{rT}$, hence more than .

Since Port. 1 is worth more than Port. 2 for all times, we have

$$c + K \geq P + S_0$$

and since $C = c$ then

$$C - P \geq S_0 - K$$
Put-call parity on American Options

We have

$$S_0 - K \leq C - P \leq S_0 - K^{-rT}$$
Lower Bound for Calls

1. one European call option plus an amount of cash equal to $D + Ke^{-rT}$
2. one share

After time $T$ we have two scenarios:

If $S_T > K$ then Port. 1 is worth $(S_T - K) + De^{rT} + K = S_T + De^{rT}$. Port. 2 is worth $S_T + De^{rT}$. Therefore, Port.1 $\geq$ Port. 2.

If $S_T < K$ then Port. 1 is worth $De^{rT} + K$. Port. 2 is worth $S_T + De^{rT}$. Since $K > S_T$ then Port 1. $\geq$ Port. 2.

In total since Port.1 is always bigger than Port.2 then

$$c + D + Ke^{-rT} \geq S_0$$

or

$$c \geq S_0 - D - Ke^{-rT}$$
Lower Bound for Puts

After time $T$ we have two scenarios:

1. one European put option plus one share
2. an amount of cash equal to $D + Ke^{-rT}$

If $S_T > K$ then Port. 1 is worth $De^{rT} + S_T$. Port. 2 is worth $De^{rT} + K$. Since Therefore, Port.1 $\geq$ Port. 2.

If $S_T < K$ then the put is exercised and Port. 1 is worth $(K - S_T) + S_T + De^{rT} = K + De^{rT}$. Port. 2 is worth $De^{rT} + K$. Then Port 1. $=$ Port. 2.

In total since Port.1 is always bigger than or equal to Port.2 then

$$p + S_0 \geq D + Ke^{-rT}$$

or

$$p \geq D + Ke^{-rT} - S_0$$
Put-call parity with dividends

1. one European put option plus one share
2. one European call option plus an amount of cash equal to \( D + Ke^{-rT} \)

We get the following scenario at time \( T \)

If \( S_T \geq K \) then put expires, so Port. 1 is worth \( S_T + De^{rT} \). Port. 2 exercises the call and is worth \( (S_T - K) + De^{rT} + K = S_T + De^{rT} \).

If \( S_T \leq K \) then put is exercised, so Port.1 is worth \( (K - S_T) + S_T + De^{rT} = K + De^{rT} \). In Port2. the call expires and so \( DrT + K \).

In both cases we get \( De^{rT} + \max\{S_T, K\} \). Thus portfolios are equivalent, and we get

\[
c + D + Ke^{-rT} = p + S_0
\]

Following the same procedure as before yields

\[
S_0 - D - K \leq C - P \leq S_0 - Ke^{-rT}
\]
• Factors that affect the value of a stock:
  – Current price
  – Strike price
  –Expiration date
  – Volatility
  – Risk-free rate
  – Dividends
• Can reach some pricing conclusions without studying volatility via arbitrage arguments
• Have the following bounds
  – \( c \leq S_0 \quad C \leq S_0 \)
  – \( p \leq K \quad P \leq K \)
  – \( c \geq \max\{S_0 - Ke^{-rT}, 0\} \)
  – \( p \geq \max\{Ke^{-rT} - S_0, 0\} \)
  – \( c \geq \max\{S_0 - D - Ke^{-rT}, 0\} \)
  – \( p \geq \max\{Ke^{-rT} + D - S_0, 0\} \)
Summary

• Put-call parity relates the call price to the put price

\[ c + K e^{-rT} = p + S_0 \]

• With dividends

\[ c + D + K e^{-rT} = p + S_0 \]

• Put-call parity doesn’t hold for American options, but we get upper and lower bounds for the prices.
Summary

- Many trading strategies involving an option and the stock itself. Equivalent to an option offset by income via the put-call parity.
- Spreads involve taking a position in two or more calls or in two or more puts.
- Bull (bear) spread - buying a call (put) with low strike price and selling a call(put) with high strike price.
- Butterfly spread - buying calls (puts) with low and high strike prices and selling two calls (puts) with intermediate strike prices.
- Calendar spread - selling a call (put) with a short maturity and buying a call (put) with long maturity.
- Combinations involve positions in both calls and puts.
- Straddle - long position in a call and a long position in a put with same strike price and maturity
- Strip - long position in one call and two long positions in a put.
- Strap - long position in two calls and one long positions in a put.
- Strangle - long position in a call and long position in a put at different strike prices.
Consider now a stock with price $S_0$ and an option with current price $f$.

- Suppose that the option lasts for time $T$ and during the life of the option, the price can go either up from $S_0$ to $S_0u$ (where $u > 1$) or down from $S_0$ to $S_0d$ (where $d < 1$).

- The percentage increase in the stock price in up movement is $u - 1$. The percentage decrease in the stock price in down movement is $1 - d$.

- Let the option payoff for up movement be $f_u$ and the option payoff for down movement be $f_d$. 
Generalization

- Build a portfolio of a long position in $\Delta$ shares and a short position in one option.
- On up movement the value of the portfolio is at the end of the option
  \[ S_0 u \Delta - f_u \]
- On down movement the value of the portfolio is at the end of the option
  \[ S_0 d \Delta - f_d \]
- Equality if
  \[ S_0 u \Delta - f_u = S_0 d \Delta - f_d \] (1)
  or
  \[ \Delta = \frac{f_u - f_d}{S_0 u - S_0 d} \]
  Thus $\Delta$ is the ratio of the change in the options prices to the change in the stock prices.
- Portfolio is now risk-less, and so earns risk-free rate $r$. Present value of the portfolio (due to (1)) is
  \[ (S_0 u \Delta - f_u) e^{-rT} \]
Generalization

- Cost of setting up the portfolio is
  \[ S_0 \Delta - f \]
- Since cost should equal present value (else arbitrage opportunity)
  \[ S_0 \Delta - f = (S_0 u \Delta - f_u) e^{-rT} \]
  or
  \[ f = S_0 \Delta + (S_0 u \Delta - f_u) e^{-rT} = S_0 \Delta \left(1 - u e^{-rT}\right) + f_u e^{-rT} \]
- Recall \( \Delta = \frac{f_u - f_d}{S_0 u - S_0 d} \) then
  \[ f = e^{-rT} \left[p f_u + (1 - p) f_d\right] \]
  where
  \[ p = \left(\frac{e^{rT} - d}{u - d}\right) \]

Remark: Pricing independent of the probability of the stock moving up or down! Calculating the price in terms of the underlying stock (incorporated in the value of the stock price already).
To price American options follow the following procedure:

1. Final nodes are priced as for European options

2. Previous nodes we price via taking the greater of
   (a) Value given by
      
      \[ f = e^{-r\Delta t} [pf_u + (1 - p)f_d] \]
      
      \[ p = \frac{a - d}{u - d} \]
      
      with \( a = e^{rT} \) the growth rate.
   
   (b) Value given by early exercise
Binomial trees on Options paying continuous dividend yield

Suppose a stock pays a dividend with yield at a rate $q$.

- Total return from the dividends and capital gains in risk neutral world is $r$.
- Dividends provide a return rate of $q$.
- Capital gains is $r - q$.
- Stock starts at $S_0$ then expected value after one time step of $\Delta t$ must be $S_0 e^{(r-q)\Delta t}$. Thus

$$pS_0 u + (1 - p)S_0 d = S_0 e^{(r-q)\Delta t}$$

hence

$$p = \frac{e^{(r-q)\Delta t} - d}{u - d}$$

- Match volatility by setting $u = e^{\sigma \sqrt{\Delta t}}$ and $d = \frac{1}{u}$.
- We have

$$f = e^{-r\Delta t} [pf_u + (1 - p)f_d]$$

where

$$a = e^{(r-q)\Delta t}$$

is the growth factor.
It does not cost anything to enter into a futures contract on either side (unlike stock). Therefore, if $F_0$ is the initial futures price, the expected futures price at the end of one time step of length $\Delta t$ should also be $F_0$. This implies

$$pF_0u + (1 - p)F_0d = F_0$$

or

$$p = \frac{a - d}{u - d}$$

with growth rate $a = 1$.

Repeat the pricing argument.
Define *Delta*, $\Delta$, to be the ratio of the change in the price of the stock option to the change in the price of the underlying stock.

Number of units of stock that we should hold for each option shorted in order to create a risk-less hedge.

- Tells investor how to adjust the amount of stock in portfolio so that the portfolio remains risk-less.
- A risk-less hedge is referred to as *delta hedging*.
Matching Volatility with $u$ and $d$

When constructing binomial trees, choose $u$ and $d$ to match the volatility of the stock price.

We present the following method:

- Let $\mu$ be the expected return on the stock (in the real world).

- Let $\sigma$ be the volatility. The volatility $\sigma$ of a stock price is defined so that $\sigma \sqrt{\Delta t}$ is the standard deviation of the return on the stock price in a short period of time of length $\Delta t$. More later.

- Let $\Delta t$ be the interval of time.

- Suppose the stock price starts at $S_0$ and stock moves to either $S_0u$ or $S_0d$ at the end of $\Delta t$. 
Matching Volatility with $u$ and $d$

- Probability of up movement is $p^*$. Then the expected return at the first time step is
  \[ S_0 e^{\mu \Delta t} \]
  and the expected stock price at this time is
  \[ p^* S_0 u + (1 - p^*) S_0 d \]
  which should match the return. Then
  \[ p^* S_0 u + (1 - p^*) S_0 d = S_0 e^{\mu \Delta t} \]
- Therefore,
  \[ p^* = \frac{e^{\mu \Delta t} - d}{u - d} \]
- Now we use the volatility into the picture. Since the volatility $\sigma \sqrt{\Delta t}$ is the standard deviation of the return on the stock price over short periods of time $\Delta t$, then $\sigma^2 \Delta t$ is the variance on the return on the stock price over short periods of time.
Matching Volatility with $u$ and $d$

- But the variance on the return is $E(X^2) - (E(X))^2$ for $X = u, d$ then

$$\sigma^2 t = p^* u^2 + (1 - p^*) d^2 - (p^* u + (1 - p^*) d)^2$$

Using

$$p^* = \frac{e^{\mu \Delta t} - d}{u - d}$$

then

$$\sigma^2 t = e^{\mu \Delta t} (u + d) - ud - e^{2\mu \Delta t}$$

Can solve for $u$ and $d$ when $\Delta t \ll 1$ and get

$$u = e^{\sigma \sqrt{\Delta t}}$$

$$d = e^{-\sigma \sqrt{\Delta t}}$$