
Solution to Black-Sholes Equation
Process for a stock price

Including volatility then expect: variability of the percentage return in a short period of time $\Delta t$ is the same regardless of the stock price.

Standard deviation of the change in a short period of time $\Delta t$ should be proportional to the stock price and leads to

$$dS = \mu S dt + \sigma S dz$$

or

$$\frac{dS}{S} = \mu dt + \sigma dz \quad (1)$$

We use (1) to price stocks. Here $\sigma$ is the volatility and $\mu$ is the expected return rate.
Itô’s Lemma: Modeling stock movements

We argued that a reasonable model of stock movements should be

\[ dS = \mu S dt + \sigma S dz \]

with \( \mu \) and \( \sigma \) constants.

From Itô’s Lemma we can consider a process \( G \) that depends on \( t \) and \( S \). Then

\[ dG = \left( \frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S dz \]

so both \( S \) and \( G \) are affected by \( dz \) - the noise in the system.
Derivation of Black-Scholes-Merton Differential Equation

1. The stock price follows the process defined earlier for $\mu$ and $\sigma$:

\[
\frac{dS}{S} = \mu dt + \sigma dz
\]

2. Short selling of securities with full use of proceeds is permitted
3. There are no transactions costs or taxes. All securities are perfectly divisible
4. There are no dividends during the life of the derivative
5. There are no riskless arbitrage opportunities
6. Security trading is continuous
7. The risk-free rate of interest, $r$, is constant and the same for all maturities
Derivation of Black-Scholes-Merton Differential Equation

We now build a portfolio that will eliminate the stochasticity of the process. The appropriate portfolio (as we will see) is

- -1 option

- $\frac{\partial f}{\partial S}$ shares (recall $\Delta = \frac{f_u - f_d}{S_0 u - S_0 d}$ in the binomial tree)

which changes continuously over time. Let $\Pi$ be the value of the portfolio then

$$\Pi = -f + \frac{\partial f}{\partial S}$$

and $\Delta \Pi$ be the value of the portfolio in the time interval $\Delta t$ then

$$\Delta \Pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S$$
Then

\[ \Delta \Pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S \]

\[ = - \left[ \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right] \Delta t \]

Note that \( \Delta \Pi \) does not depend on \( dz \), therefore there is no risk during time \( \Delta t \)!

Thus the portfolio must instantaneously earn the same rate of return as other short-term risk-free securities.

Thus:

\[ \Delta \Pi = r \Pi \Delta t \]
Derivation of Black-Scholes-Merton Differential Equation

where \( r \) is the risk-free rate. Then

\[
\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf
\]  

(2)

Equation (2) is the Black-Scholes partial differential equation. Any solution corresponds to the price of a derivative overlying a particular stock.

Boundary conditions for European call options:

\[
f = \max\{S - K, 0\}
\]

when \( t = T \). Boundary conditions for European put options:

\[
f = \max\{K - S, 0\}
\]

when \( t = T \). The portfolio created is riskless only for infinitesimally short periods.
The Black-Scholes formulas for the price at time 0 of a European call option on a non-dividend-paying stock and for a European put option on a non-dividend-paying stock are

\[ c = S_0 N(d_1) - Ke^{-rT} N(d_2) \]

and

\[ p = KN(-d_2) - S_0 e^{-rT} N(-d_1) \]

where

\[ d_1 = \frac{\ln \frac{S_0}{K} + \left( r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \]

\[ d_2 = \frac{\ln \frac{S_0}{K} + \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T} \]

and \( N(x) \) is the cumulative probability distribution function.
The variables $c$ and $p$ are the European call and put prices, $S_0$ is the current stock price at time 0, $K$ is the strike price, $r$ is the continuously compounded risk-free rate, $\sigma$ is the stock price volatility, and $T$ is the time to maturity of the option. Why?
We do a change of variables on (??). Set

\[ x = \ln \frac{S}{K} + \left( r - \frac{\sigma^2}{2} \right) (T - t) \]

\[ \tau = T - t \]

\[ u = f e^{r(T-t)} \]

Blackboard Calculation
Volatility

In order to measure $\sigma$ we use historical data. Stocks typically have a $\sigma$ between 15% to 60%.

Recall that the stock process implies

$$\frac{\Delta S}{S} \sim \phi \left[ \mu T, \sigma \sqrt{T} \right];$$

i.e. $\sigma \sqrt{T}$ is approximately equal to the standard deviation of the percentage change in the stock price in time $T$.

Example: Suppose a stock has a volatility $\sigma = 20\%$ per annum with a current stock price of $30$. What are the range of stock prices after two weeks?

The volatility of 20% per annum becomes a volatility of

$$20 \times \sqrt{\frac{14}{365}} = 3.92\%$$

A one-standard-deviation move in the stock price in two weeks is $30 \times 0.03912 = 1.175$ or $1.76$. 
Volatility

Estimating Volatility from Historical Data

Define

- $n + 1$: Number of observations
- $S$: Stock price at end of the $i$th interval, with $i = 0, 1, \ldots, n$.
- $\tau$: Length of time interval in years

and let

$$u_i = \ln \left( \frac{S_i}{S_{i-1}} \right)$$

for $i = 0, 1, \ldots, n$.

Then the sample standard deviation of $u_i$ is given by

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (u_i - \bar{u})^2}$$
Volatility

or

\[ s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} u_i^2 - \frac{1}{n(n-1)} \left( \sum_{i=1}^{n} u_i \right)^2} \]

From our stock process argument, the standard deviation of \( u_i \) is \( \sigma \sqrt{\tau} \). The variable \( s \) is an estimate of \( \sigma \sqrt{\tau} \), then \( \sigma \) can be estimate as \( \hat{\sigma} \), where

\[ \hat{\sigma} = \frac{s}{\sqrt{\tau}}. \]

The standard error of this calculation can be shown to be approximately \( \frac{\hat{\sigma}}{\sqrt{2n}} \).
Example: Consider a sequence of stock prices during 21 consecutive trading days. Then

\[ \sum u_i = 0.09531 \quad \sum u_i^2 = 0.00326 \]

The standard deviation of the daily return is

\[ \sqrt{\frac{0.0326}{10} - \frac{0.09531^2}{380}} = 0.01216 \]

or 1.216%. Assume that there are 252 trading days per year, \( \tau = 1/252 \) and the data give an estimate for the volatility per annum of \( 0.01216 \sqrt{252} = 0.193 \), or 19.3%. The standard error of this estimate is

\[ \frac{0.193}{\sqrt{2 \times 20}} = 0.031 \]

or 3.1% per annum.
Dividends

Used the assumption that no dividends are payed to establish Black-Scholes equation.

Assume the amount and timing of the dividends during the life of an option can be predicted with certainty. For short-life options this is not an unreasonable assumption. For long-life options it is usual to assume that the dividend yield rather than the cash dividend payments are known.

By the time the option matures, the dividends will have been paid and the riskless component will no longer exist. The Black-Scholes formula is therefore correct if $S_0$ is equal to the risky component of the stock price and $\sigma$ is the volatility of the process followed by the risky component.

Operationally, this means that the Black-Scholes formula can be used provided that the stock price is reduced by the present value of all dividends during the life of the option.

The discounting being done from the ex-dividend dates at the risk-free rate. A dividend is counted as being during the life of the option if its ex-dividend date occurs during the life of the option.

**Example:** Consider a European call option on a stock when there are ex-dividend dates in two months and five months. The dividend on each ex-dividend date is expected to be $0.50. The current share price is $40, the exercise price is $40, the stock price volatility is 30% per annum, the risk-free rate of interest is 9% per annum, and the time to maturity is six months. The present value of the dividends is
Dividends

\[ 0.5 e^{-0.1667 \times 0.09} + 0.5 e^{-0.4167 \times 0.09} = 0.9741 \]

The option price can therefore be calculated from the Black-Scholes formula, with

\[ S_0 = 40 - 0.9741 = 39.0259, \quad K = 40, \quad r = 0.09, \quad \sigma = 0.3, \quad \text{and} \quad T = 0.5: \]

\[ d_1 = \frac{\ln(39.0259/40) + (0.09 + 0.3^2/2) \times 0.5}{0.3\sqrt{0.5}} = 0.2017 \]

\[ d_2 = \frac{\ln(39.0259/40) + (0.09 - 0.3^2/2) \times 0.5}{0.3\sqrt{0.5}} = -0.0104 \]

Then \( N(d_1) = 0.5800 \) and \( N(d_2) = 0.4959. \)

Then the call price is

\[ c = S_0 N(d_1) - K e^{-rT} N(d_2) = 39.0259 \times 0.5800 - 40e^{-0.09 \times 0.5} \times 0.4959 = 3.67 \]

or $3.67.
In the absence of dividend payments, there should be no early exercise of American Options (seen from our binomial tree calculations too).

When dividends are paid, it is optimal to exercise only at a time immediately before the stock goes ex-dividend. Assume that \( n \) ex-dividend dates are anticipated and that they are at times

\[ t_1, t_2, \ldots, t_n \text{ with } t_1 < t_2 < \cdots < t_n \]

The dividend payments corresponding to these dates are denoted

\[ D_1, D_2, \ldots, D_n \]

- Consider first an early exercise just prior to the final ex-dividend date. If the option is exercised at \( t_n \) the investor receives \( S(t_n) - K \), where \( S(t) \) denotes the stock price at time \( t \).
- If the option is not exercised, the stock price drops to \( S(t_n) - D_n \). The value of the option is greater than

\[
S(t_n) - D_n - Ke^{-r(T-t_n)}
\]

Then if

\[
S(t_n) - D_n - Ke^{-r(T-t_n)} \geq S(t_n) - K
\]
American Options

i.e.

\[ D_n \leq K \left[ 1 - e^{-r(T-t_n)} \right] \]  \hspace{1cm} (3)

then it cannot be optimal to exercise at time \( t_n \).

- On the other hand if

\[ D_n > K \left[ 1 - e^{-r(T-t_n)} \right] \]

for any reasonable assumption about the stochastic process followed by the stock price, it can be shown that it is always optimal to exercise at time \( t_n \) for a sufficiently high value of \( S(t_n) \).

This inequality will tend to be satisfied when the final ex-dividend date is fairly close to the maturity of the option \((T - t)\) small) and the dividend is large.

- Consider the next time \( t_{n-1} \). If the option is exercised immediately prior to time \( t_{n-1} \), the investor receives \( S(t_{n-1}) - K \). If the option is not exercised at time \( t_{n-1} \), the stock drops to \( S(t_{n-1}) - D_{n-1} \) and the earliest subsequent time at which exercise could take place is \( t_n \).

Hence a lower bound to the option price if it is not exercised at time \( t_{n-1} \) is

\[ S(t_{n-1}) - D_{n-1} - Ke^{-r(t_n-t_{n-1})} \]
American Options

- It follows that if

\[ S(t_{n-1}) - D_{n-1} - Ke^{-r(t_n-t_{n-1})} \geq S(t_{n-1}) - K \]

or

\[ D_{n-1} \leq K \left[ 1 - e^{-r(t_n-t_{n-1})} \right] \]

it is not optimal to exercise immediately prior to time \( t_{n-1} \). Similarly, for any \( i < n \) if

\[ D_i \leq K \left[ 1 - e^{-r(t_{i+1}-t_i)} \right] \quad (4) \]

or approximately

\[ D_i \leq Kr(t_{i+1} - t_i) \]

- Assuming that \( K \) is fairly close to the current stock price, the dividend yield on the stock has to be either close to above the risk-free rate of interest for this inequality not to be satisfied.

- Conclude that the most likely time for the early exercise of an American call is immediately before the final ex-dividend date \( t_n \). Furthermore, if inequality (4) holds for \( i = 1, 2, \ldots, n - 1 \) and inequality (3) holds, we can be certain that early exercise is never optimal.