Lecture 20. Black-Sholes Equation

More on Black-Sholes Equation
The Black-Scholes formulas for the price at time 0 of a European call option on a non-dividend-paying stock and for a European put option on a non-dividend paying stock are

\[
c = S_0 N(d_1) - Ke^{-rT} N(d_2)
\]

and

\[
p = Ke^{-rT} N(-d_2) - S_0 N(-d_1)
\]

where

\[
d_1 = \frac{\ln \frac{S_0}{K} + \left( r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}
\]

\[
d_2 = \frac{\ln \frac{S_0}{K} + \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T}
\]

and \( N(x) \) is the cumulative probability distribution function.
Black-Scholes Pricing Formulas

The variables $c$ and $p$ are the European call and put prices, $S_0$ is the current stock price at time 0, $K$ is the strike price, $r$ is the continuously compounded risk-free rate, $\sigma$ is the stock price volatility, and $T$ is the time to maturity of the option. Why?

$N(x) = \int_{-\infty}^{x} \phi(t) dt$

$1 - N(x) = N(-x)$
Another Black-Scholes Calculation

What about the Power-Option?

Power call option pays off \((\max\{S - K, 0\})^2\) at time \(T\).

Change of variables

\[
x = \ln \frac{S}{K} + \left( r - \frac{\sigma^2}{2} \right) (T - t)
\]

\[
\tau = T - t
\]

\[
u = f e^{r(T-t)}
\]

then \(u\) satisfies

\[
u(\tau, x)_{\tau} = \frac{\sigma^2}{2} u(\tau, x)_{xx}
\]

on \(-\infty < x < \infty\). Boundary condition?

\[
u(x, 0) = (\max\{S - K, 0\})^2 = (\max\{Ke^x - K, 0\})^2
\]
Then, we have

\[
\begin{align*}
  u(x, \tau) &= \frac{1}{\sqrt{4\sigma^2 \pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\sigma^2 t}} u_0(y) \, dy \\
  &= \frac{1}{\sigma \sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2\sigma^2 t}} (\max\{ K e^y - K, 0\})^2 \, dy \\
  &= \frac{K^2}{\sigma \sqrt{2\pi t}} \int_{0}^{\infty} e^{-\frac{(x-y)^2}{2\sigma^2 t}} (e^y - 1)^2 \, dy
\end{align*}
\]

Well-behaved for \( t < \frac{1}{2\sigma^2} \).
In order to measure $\sigma$ we use historical data. Stocks typically have a $\sigma$ between 15% to 60%.

Recall that the stock process implies

$$\frac{\Delta S}{S} \sim \phi \left[ \mu T, \sigma \sqrt{T} \right];$$

i.e. $\sigma \sqrt{T}$ is approximately equal to the standard deviation of the percentage change in the stock price in time $T$.

**Example:** Suppose a stock has a volatility $\sigma = 20\%$ per annum with a current stock price of $30$. What are the range of stock prices after two weeks?

The volatility of 20% per annum becomes a volatility of

$$20 \times \sqrt{\frac{14}{365}} = 3.92\%$$

A one-standard-deviation move in the stock price in two weeks is $30 \times 0.03912 = 1.175$ or $1.76$. 
Volatility

Estimating Volatility from Historical Data

Define

- \( n + 1 \): Number of observations
- \( S \): Stock price at end of the \( i \)th interval, with \( i = 0, 1, \ldots, n \).
- \( \tau \): Length of time interval in years

and let

\[
    u_i = \ln \left( \frac{S_i}{S_{i-1}} \right)
\]

for \( i = 0, 1, \ldots, n \).

Then the sample standard deviation of \( u_i \) is given by

\[
    s = \sqrt{ \frac{1}{n - 1} \sum_{i=1}^{n} (u_i - \bar{u})^2 }
\]
Volatility

or

\[ s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} u_i^2 - \frac{1}{n(n-1)} \left( \sum_{i=1}^{n} u_i \right)^2} \]

From our stock process argument, the standard deviation of \( u_i \) is \( \sigma \sqrt{\tau} \). The variable \( s \) is an estimate of \( \sigma \sqrt{\tau} \), then \( \sigma \) can be estimate as \( \hat{\sigma} \), where

\[ \hat{\sigma} = \frac{s}{\sqrt{\tau}}. \]

The standard error of this calculation can be shown to be approximately \( \frac{\hat{\sigma}}{\sqrt{2n}} \).
Example: Consider a sequence of stock prices during 21 consecutive trading days. Then

$$\sum u_i = 0.09531 \quad \sum u_i^2 = 0.00326$$

The standard deviation of the daily return is

$$\sqrt{\frac{0.0326}{10} - \frac{0.09531^2}{380}} = 0.01216$$

or 1.216%. Assume that there are 252 trading days per year, $\tau = 1/252$ and the data give an estimate for the volatility per annum of $0.01216\sqrt{252} = 0.193$, or 19.3%. The standard error of this estimate is

$$\frac{0.193}{\sqrt{2 \times 20}} = 0.031$$

or 3.1% per annum.
Dividends

Used the assumption that no dividends are payed to establish Black-Scholes equation.

Assume the amount and timing of the dividends during the life of an option can be predicted with certainty. For short-life options this is not an unreasonable assumption. For long-life options it is usual to assume that the dividend yield rather than the cash dividend payments are known.

By the time the option matures, the dividends will have been paid and the riskless component will no longer exist. The Black-Scholes formula is therefore correct if $S_0$ is equal to the risky component of the stock price and $\sigma$ is the volatility of the process followed by the risky component.

Operationally, this means that the Black-Scholes formula can be used provided that the stock price is reduced by the present value of all dividends during the life of the option.

The discounting being done from the ex-dividend dates at the risk-free rate. A dividend is counted as being during the life of the option if its ex-dividend date occurs during the life of the option.

**Example:** Consider a European call option on a stock when there are ex-dividend dates in two months and five months. The dividend on each ex-dividend date is expected to be $0.50. The current share price is $40, the exercise price is $40, the stock price volatility is 30% per annum, the risk-free rate of interest is 9% per annum, and the time to maturity is six months. The present value of the dividends is
\[ 0.5e^{-0.1667\times 0.09} + 0.5e^{-0.4167\times 0.09} = 0.9741 \]

The option price can therefore be calculated from the Black-Scholes formula, with 
\[ S_0 = 40 - 0.9741 = 39.0259, \quad K = 40, \quad r = 0.09, \quad \sigma = 0.3, \text{ and } T = 0.5: \]

\[
d_1 = \frac{\ln(39.0259/40) + (0.09 + 0.3^2/2) \times 0.5}{0.3\sqrt{0.5}} = 0.2017
\]

\[
d_2 = \frac{\ln(39.0259/40) + (0.09 - 0.3^2/2) \times 0.5}{0.3\sqrt{0.5}} = -0.0104
\]

Then \( N(d_1) = 0.5800 \) and \( N(d_2) = 0.4959 \).

Then the call price is

\[
c = S_0N(d_1) - Ke^{-rT}N(d_2) = 39.0259 \times 0.5800 - 40e^{-0.09 \times 0.5} \times 0.4959 = 3.67
\]

or $3.67.
American Options

In the absence of dividend payments, there should be no early exercise of American Options (seen from our binomial tree calculations too).

When dividends are paid, it is optimal to exercise only at a time immediately before the stock goes ex-dividend. Assume that $n$ ex-dividend dates are anticipated and that they are at times

$$t_1, t_2, \ldots, t_n \text{ with } t_1 < t_2 < \cdots < t_n$$

The dividend payments corresponding to these dates are denoted

$$D_1, D_2, \ldots, D_n$$

- Consider first an early exercise just prior to the final ex-dividend date. If the option is exercised at $t_n$ the investor receives $S(t_n) - K$, where $S(t)$ denotes the stock price at time $t$.
- If the option is not exercised, the stock price drops to $S(t_n) - D_n$. The value of the option is greater than

$$S(t_n) - D_n - Ke^{-r(T-t_n)}$$

Then if

$$S(t_n) - D_n - Ke^{-r(T-t_n)} \geq S(t_n) - K$$
American Options

\[ D_n \leq K \left[ 1 - e^{-r(T-t_n)} \right] \]  \hspace{1cm} (1)

then it cannot be optimal to exercise at time \( t_n \).

- On the other hand if

\[ D_n > K \left[ 1 - e^{-r(T-t_n)} \right] \]

for any reasonable assumption about the stochastic process followed by the stock price, it can be shown that it is always optimal to exercise at time \( t_n \) for a sufficiently high value of \( S(t_n) \). This inequality will tend to be satisfied when the final ex-dividend date is fairly close to the maturity of the option \((T - t \text{ small})\) and the dividend is large.

- Consider the next time \( t_{n-1} \). If the option is exercised immediately prior to time \( t_{n-1} \), the investor receives \( S(t_{n-1}) - K \). If the option is not exercised at time \( t_{n-1} \), the stock drops to \( S(t_{n-1}) - D_{n-1} \) and the earliest subsequent time at which exercise could take place is \( t_n \). Hence a lower bound to the option price if it is not exercised at time \( t_{n-1} \) is

\[ S(t_{n-1}) - D_{n-1} - Ke^{-r(t_n-t_{n-1})} \]
American Options

• It follows that if

\[ S(t_{n-1}) - D_{n-1} - Ke^{-r(t_n-t_{n-1})} \geq S(t_{n-1}) - K \]

or

\[ D_{n-1} \leq K \left[ 1 - e^{-r(t_n-t_{n-1})} \right] \]

it is not optimal to exercise immediately prior to time \( t_{n-1} \). Similarly, for any \( i < n \) if

\[ D_i \leq K \left[ 1 - e^{-r(t_{i+1}-t_i)} \right] \quad (2) \]

or approximately

\[ D_i \leq Kr(t_{i+1} - t_i) \]

• Assuming that \( K \) is fairly close to the current stock price, the dividend yield on the stock has to be either close to above the risk-free rate of interest for this inequality not to be satisfied.

• Conclude that the most likely time for the early exercise of an American call is immediately before the final ex-dividend date \( t_n \). Furthermore, if inequality (2) holds for \( i = 1, 2, \ldots, n - 1 \) and inequality (1) holds, we can be certain that early exercise is never optimal.
Options on Stocks paying a dividend

Dividends cause stock prices to reduce on the ex-dividend date by the amount of the dividend payment.

The payment of a dividend yield at rate $q$, therefore, causes the growth rate in the stock price to be less than it would otherwise be by an amount $q$.

If the stock, with yield $q$, grows from $S_0$ today to $S_T$ at time $T$, then the absence of dividends it would grow from $S_0$ today to $S_T e^{qT}$ at time $T$.

Alternatively in the absence of dividends it would grow from $S_0 e^{-qT}$ today to $S_T$ at time $T$.

Argument shows that we get the same probability distribution for the stock price at time $T$ in each of the following two cases:

- The stock starts at price $S_0$ and provides a dividend yield at rate $q$.
- The stock starts at price $S_0 e^{-qT}$ and pays no dividends.

This leads to a simple rule:

When valuing a European option lasting for time $T$ on a stock paying a known dividend yield at rate $q$, we reduce the current stock price from $S_0$ to $S_0 e^{-qT}$ and then value the option as though the stock pays no dividends.
Pricing results

Lower bounds for Option prices:

Substitute $S_0 e^{-qT}$ for $S_0$ in our call lower bound, then

$$c \geq \max\{S_0 e^{-qT} - Ke^{-rT}, 0\}$$

For the European put option we get similar results:

$$p \geq \max\{Ke^{-rT} - S_0 e^{-qT}, 0\}$$

Can be proved by arbitrage argument.
Replacing $S_0$ by $S_0e^{-qT}$ in the Black-Scholes formulas, then the price $c$ of a European call and price $p$ of a European put on a stock providing a dividend yield at rate $q$ as

$$c = S_0e^{-qT}N(d_1) - Ke^{-rT}N(d_2)$$

and

$$p = Ke^{-rT}N(-d_2) - S_0e^{-qT}N(-d_1).$$

Since $\ln \frac{S_0e^{-qT}}{K} = \ln \frac{S_0}{K} - qT$ then $d_1$ and $d_2$ are

$$d_1 = \frac{\ln \frac{S_0}{K} + \left( r - q + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}$$

$$d_2 = \frac{\ln \frac{S_0}{K} + \left( r - q - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T}$$

and $N(x)$ is the cumulative probability distribution function.
Options Pricing result

To prove the result we start with the stock process:

\[ dS = \mu S dt + \sigma S dz \]

where \( dz \) is the Wiener process. The \( \mu \) and \( \sigma \) are the expected growth rate and volatility of the underlying stock. By Itô’s Lemma for the stock process yields:

\[
df = \left( \frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S} \sigma S dz
\]

Set up a portfolio with

- -1 derivative
- \(+\frac{\partial f}{\partial S}\) stock

then value \( \Pi \) of the portfolio is

\[
\Pi = -f + \frac{\partial f}{\partial S} S \quad (3)
\]
Then the change in value of the portfolio (as before) is

\[ \Delta \Pi = \left( -\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t, \] (4)

In time \( \Delta t \) the holder of the portfolio earns capital gains equal to \( \Delta \Pi \) and dividends on the stock position equal to

\[ qS \frac{\partial f}{\partial S} \Delta t. \]

Define \( \Delta W \) as the change in the wealth of the portfolio holder in time \( \Delta t \). It follows that

\[ \Delta W = \left( -\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 + qS \frac{\partial f}{\partial S} \right) \Delta t \] (5)

Because this is independent of the Wiener process, the portfolio is instantaneously riskless. Hence

\[ \Delta W = r \Pi \Delta t \]
Substituting (3) and (4) into (5) yields

\[
\left( -\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 + qS \frac{\partial f}{\partial S} \right) \Delta t = r \left( -f + \frac{\partial f}{\partial S} S \right) \Delta t
\]

so that

\[
\frac{\partial f}{\partial t} + (r - q) S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf
\]

This can be solved in a similar fashion as we solved the Black-Scholes equation last time...
Binomial tree process

We can value the stock index by binomial trees

\[ u = e^{\sigma \sqrt{\Delta t}} \quad d = e^{-\sigma \sqrt{\Delta t}} \]

where \( \Delta t \) is the length of the time step. The risk-neutral probability \( p \) of an up movement is chosen so that the expected return is \( r - q \). This implies

\[ pSu + (1 - p)Sd = Se^{(r-q)\Delta t} \]

or

\[ p = \frac{a - d}{u - d} \]

where

\[ a = e^{(r-q)\Delta t} \]

as we discussed pre-midterm.
1. Which of the following is true? (circle one)
   (a) Both forward and futures contracts are traded on exchanges.
   (b) Forward contracts are traded on exchanges, but futures contracts are not.
   (c) Futures contracts are traded on exchanges, but forward contracts are not.
   (d) Neither futures contracts nor forward contracts are traded on exchanges.

2. What is the payoff for a short position in a call option where $S_T$ is the stock price and $K$ is the strike price, assuming no transaction cost? (circle one)
   (a) $\max\{K - S_T, 0\}$
   (b) $\max\{S_T - K, 0\}$
   (c) $\min\{K - S_T, 0\}$
   (d) $\min\{S_T - K, 0\}$
3. Which of the following decreases the cost of an American put option?
   (a) Increase in strike price.
   (b) Increase in current stock price.
   (c) Increase in time to expiration.
   (d) Increase in volatility.

4. The short term risk-free rate usually used by derivatives traders is (circle one)
   (a) The Treasury rate
   (b) The LIBOR rate
   (c) The repo rate
   (d) The commercial paper rate

5. Which of the following is a consumption asset (circle one)
   (a) Copper.
   (b) The S&P 500 index.
   (c) The Canadian dollar.
   (d) IBM shares.
6. Which of the following is true (circle one)
   (a) Principals are not usually exchanged in a currency swap
   (b) The principal amounts usually flow in the opposite direction to interest payments at the beginning of a currency swap and the same direction as interest payments at the end of the swap.
   (c) The principal amounts usually flow in the same direction to interest payments at the beginning of a currency swap and the opposite direction as interest payments at the end of the swap.
   (d) Principals are not usually specified in a currency swap.

7. Which type of derivatives trader aims to reduce risk (circle one)
   (a) Arbitrageur.
   (b) Hedger.
   (c) Speculator.
8. An interest rate is 15% per annum when expressed with annual compounding.
   (a) What is the equivalent rate with continuous compounding?

   \[
   \left(1 + \frac{0.15}{1}\right) = e^{-r \times 1} \implies -r = \ln (1.15)
   \]

   or \( r = 13.98\% \).

   (b) What is the equivalent rate with quarterly compounding?

   \[
   \left(1 + \frac{0.15}{1}\right)^{\frac{1}{4}} = \left(1 + \frac{r}{4}\right)^4 \implies \frac{r}{4} = 1.15^{\frac{1}{4}} - 1
   \]

   implies \( r = 4 \left(1.15^{\frac{1}{4}} - 1\right) = 14.22\% \).
9. The three-year zero rate is 7% and the four-year zero rate is 7.5% (both continuously compounded). What is the forward rate for the fourth year?

\[ r_f = \frac{r_2 t_2 - r_1 t_1}{t_2 - t_1} = \frac{0.075 \times 4 - 0.07 \times 3}{4 - 3} = 0.09 \]

or 9%.
10. It is May 1. A 6-month Treasury bond in the US with principle $100 costs $96. A 12-month Treasury bond in the US with principle $100 and annual coupon of $6 (paid semiannually) costs $95.

(a) Compute the zero rates for maturities of 6-months and 12-months.

For the 6-month zero rate we have

\[ 96 = 100e^{-\frac{r}{2}} \implies -\frac{r}{2} = \ln 0.96 \implies r = 0.0816 \]

or \( r = 8.16\% \). For the 12-month zero rate we need to take into account the coupon payments. So

\[ 95 = 3e^{-0.0816 \times \frac{1}{2}} + 103e^{-r \times 1} \implies e^{-r} = \frac{95 - 3e^{-0.0816 \times \frac{1}{2}}}{103} \]

Solving for \( r \) yields \( r = 0.1116 \). Thus \( r = 11.16\% \).
(b) Compute the yield of the 12-month bond. The yield rate $y$ satisfies

$$95 = 3e^{-\frac{y}{2}} + 103e^{-y}$$

This can be solved exactly by the quadratic equation. Set $x = e^{-\frac{y}{2}}$, then

$$x = \frac{-3 \pm \sqrt{3^2 + 4 \times 103 \times 95}}{2 \times 103} = 0.94593$$

hence

$$y = -2 \ln 0.94593 = 0.111179 \text{ or } y = 11.12\%.$$
11. Prove by a simple arbitrage argument that the price of a European put option is less than the strike price, i.e. \( p \leq K \).

If \( p > K \) then we should short the put option since it is overpriced. We gain $ \( p \). At the expiration we have two cases.

If \( S_T \geq K \), then the put option is not exercised, hence we make \( p \) dollars.

If \( S_T < K \), then the put is exercised. The value is \( - (K - S_T) = S_T - K \). Since \( p > K \) then \( p + (S_T - K) = p - K + S_T > 0 \).

We purchase a share of stock at the price \( S_T \) and we lose \( K - S_T \).
12. Three-month European call options with strike prices of $50, $55, and $60 cost $2, $4, and $7, respectively.

(a) What is the maximum gain when a butterfly spread is created from the call options?
(b) What is the maximum loss when a butterfly spread is created from the call options?
(c) For what two values of $S_T$ does the holder of the butterfly spread breakeven with a profit of zero, where $S_T$ is the stock price in three months?
(d) Draw the profit curve.

The butterfly consists of a long 50, 2 short 55's, and a long 60. Then the value of the portfolio is

$$
\text{max}\{S - 50, 0\} - 2 - 2 \times \text{max}\{S - 55, 0\} - 4 + \text{max}\{S - 60, 0\} - 7
$$

$$
= \text{max}\{S - 50, 0\} - 2 \times \text{max}\{S - 55, 0\} + \text{max}\{S - 60, 0\} - 1
$$

<table>
<thead>
<tr>
<th>Share size</th>
<th>Profit</th>
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</thead>
<tbody>
<tr>
<td>$S &lt; 50$</td>
<td>$0 + 0 + 0 - 1 = 1$</td>
</tr>
<tr>
<td>$50 &lt; S &lt; 55$</td>
<td>$(S - 50) + 0 + 0 - 1 = S - 51$</td>
</tr>
<tr>
<td>$55 &lt; S &lt; 60$</td>
<td>$(S - 50) + 2(55 - S) + 0 - 1 = 59 - S$</td>
</tr>
<tr>
<td>$60 &lt; S$</td>
<td>$(S - 50) + 2(55 - S) + (S - 60) - 1 = -1$</td>
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Maximum profit at the peak, $S = 55$, of $4$.
Maximum loss $\$ - 1$.
Break-even point when $0 = S - 51$ or $59 - S = 0$.  

13. Consider a 6-month put option on a futures contract when the futures price is 40, the strike price is 38, risk–free rate is 6%, and volatility is 35%. Find the value of the option using a two–step binomial tree if the option is
(a) European
(b) American. Is there ever early exercise? When?
14. Consider a portfolio of a long position in a stock and a long position in a put option. Draw the profit curve. Explain the shape via the put-call parity.

\[ c + Ke^{-rT} = p + S_0 \]

**Credit** What is the put-call parity for European options? Prove the put-call parity via an arbitrage argument. (Hint: One of the portfolios should contain a call option and cash of value \( Ke^{-rT} \)).

(a) one European call option plus an amount of cash equal to \( Ke^{-rT} \)
(b) one European put option plus one share

Both portfolios are worth \( \max\{S_T, K\} \) at expiration.

If \( S_T > K \) then for portfolio 1 the call is exercised and the stock is purchased with a profit of \( K + (S_T - K) = S_T \). For portfolio 2 the put expires and the value of the portfolio is the share value \( S_T \).

If \( S_T < K \) then for portfolio 1, the call expires and the portfolio is worth \( K \). For portfolio 2 the put is exercised and the stock is sold with value \( S_T + (K - S_T) = K \).

Since the options are European, the options cannot be exercised until maturity. Therefore, the portfolios must have identical values today.

\[ c + Ke^{-rT} = p + S_0 \]
Due Nov. 21, 5PM.

- 13.18
- Graded: 13.29, 13.30