Lecture 22. Greek Letters

Fine tuning portfolios with $\Delta, \Theta, \Gamma$, etc...
Greek Letters

The Greek Letters measure different dimensions of risk that correspond to a options position.

Try to approach Hedging in a more sophisticated way.

Example: Consider a large financial institution that has sold a European call option on 100,000 shares of a non-dividend yielding stock for $300,000.

Assume the stock price is $49, the strike price is $50, the risk-free interest rate is 5% per annum, the stock price volatility is 20% per annum, the time to maturity is 20 weeks, and the expected return from the stock is 13% per year.
Then

\[ S_0 = 49 \quad K = 20 \quad r = 0.05 \quad \sigma = 0.20 \quad T = \frac{20}{52} = 0.3846 \quad \mu = 0.13 \]

Black-Scholes formula implies

\[
d_1 = \frac{\ln \frac{S_0}{K} + \left( r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}
\]

\[
= \frac{\ln \frac{49}{20} + \left( 0.05 + \frac{0.2^2}{2} \right) 0.3846}{0.2 \sqrt{0.3846}}
\]

\[
d_2 = d_1 - \sigma \sqrt{T} = -0.2 \sqrt{0.3846}
\]

so

\[
c = S_0 N(d_1) - Ke^{-rT} N(d_2) = 49 - 20e^{-0.05 \times 0.3846} = 2.4
\]

Thus Black-Scholes says the price should be $240,000. The financial institution sells the options
for $60,000 more than the pricing model says they’re worth.
Naked & Covered Positions

- In a **naked position** the investor does nothing to hedge against losses. In our example, this approach does well so long as the stock remains below $50. Then

- Alternatively the investor house can take a **covered position**. This involves buying 100,000 shares as soon as the option has been sold. If the option is exercised, the strategy works well. If the stock drops then there is a large loss. By the put-call parity this is similar to

\[
c + Ke^{-rT} = p + S_0 \implies -c + S_0 = -p + Ke^{-rT}
\]

so it is the same as writing a put option. Therefore, the covered position is bad if the stock price goes down.
Stop-Loss Strategy

The stop-loss strategy involves the following:

• Consider a bank that has written a call option with strike price $K$.
• The bank buys one unit of stock as soon as the price rises above $K$ and selling it as soon as its price is less than $K$.
• Point is to hold a naked position whenever the stock is less than $K$ and a covered position whenever the stock price is greater than $K$.
• The scheme is designed to ensure that at time $T$ the institution owns the stock if the option closes in the money and does not own it if the option closes out of the money.
• Strategy seems to produce payoffs that are the same as the payoffs on the option.
• In this example, we buy the stock at time $t_1$, sell it at time $t_2$, buy it at time $t_3$, selling at time $t_4$, buying at time $t_5$, and deliver at time $T$.

• Denote the initial stock price $S_0$. The initial cost of the setting up the hedge is $S_0$ if $S_0 > K$ and zero otherwise.

• Is the total cost of the hedge $Q = \max\{S_0 - K, 0\}$, since the purchase/sale of stocks occur always at $S = K$?

• True if **no transaction costs**! The hedger would make a riskless profit by writing the option and hedging.
Two problems:

- Cash flows to the hedger occur at different times and must be discounted
- Purchases and sales cannot be made at exactly the same price $K$. Crucial point...If the stock purchases are made at $K + \epsilon$ and sold at $K - \epsilon$ then every purchase and sale incurs a loss of $2\epsilon$.

If the stock prices change continuously (as it is modeled on a Brownian motion) then we expect the curve $S$ to cross our line $S = K$ an infinite number of times! Our profit will go away due to excessive number of transactions.
Delta Hedging

Instead of designing a portfolio with a stop-loss strategy, a different strategy is to design a delta hedge.

- Recall that $\Delta$ of an option is the rate of change of the option price with respect to the price of the underlying asset.
- If the $\Delta$ of a call option on a stock is 0.4, then when the stock changes by a small amount, the option value changes by 40% of that amount.
- We have that $\Delta = \frac{\partial c}{\partial S}$ where $c$ is the price of the call option and $S$ is the stock price.

**Figure 15.2** Calculation of delta.
• Suppose that the stock price is $100 and the option price is $10. An investor has sold 20 call option contracts = option to buy 2000 shares
• The investor position can be hedged by buying

\[ 0.6 \times 2000 = 1200 \]

shares
• The gain (loss) on the option position would tend to be offset by the loss (gain) on the stock position
• For example, if the stock goes up by $1, there is a gain in the stock of $1200, the option price will tend to go up by \(0.6 \times 1 \times 2000\) producing a loss of $1200.
• If the stock goes down by $1, there is a loss in the stock of $1200, the option price will tend to go down by \(0.6 \times 1 \times 2000\) producing a gain of $1200.
• The investor loses \(1200\Delta S\) on the short option position when the stock price increases by \(\Delta S\). The delta of the stock is 1.0, so the long position in 1200 shares has a delta of \(+1200\). The delta of the investor’s overall position is zero.
• A position with \(\Delta = 0\) is referred to as being delta neutral.
Since delta changes over time, the investor’s position remains delta hedged (delta neutral) for relatively short periods of time.

A hedge is rebalanced, or adjusted periodically, to remain delta neutral.

We will describe a dynamic-hedging scheme that rebalances the portfolio periodically to ensure a delta-neutral portfolio.

This is in contrast to static hedging schemes where the hedge is set up and left alone. Such schemes are called hedge and forget schemes.

We will use Black-Scholes analysis to help devise a good delta hedge scheme. Recall that the Black-Scholes portfolio that is riskless is

\[
-1 : \text{option} \\
+\Delta : \text{shares of stock}
\]
Delta of European Stock Options

A European call option on a non-dividend-paying stock is

\[ \Delta(\text{call}) = N(d_1) \]

and a European put option on a non-dividend-paying stock is

\[ \Delta(\text{put}) = N(d_1) - 1 \]

Figure 15.3  Variation of delta with stock price for (a) a call option and (b) a put option on a non-dividend-paying stock.

Keeping a delta hedge for a long position in a European call option involves maintaining a short position of \( N(d_1) \) shares at any given time.
Note that the $\Delta$ for a European put option is negative, so that the a long position in a put option should be hedged with a long position in the underlying stock, and a short position in a put option should be hedged with a short position in the underlying stock.

Why $N(d_1)$? Recall that

$$c = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

and we wish to calculate $\frac{\partial c}{\partial S}$.

We first compute a formula for $N'(x)$. In particular

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$
and since $d_2 = d_1 - \sigma \sqrt{T - t}$ then

$$N'(d_1) = N'(d_1 + \sigma \sqrt{T - t})$$

$$= \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{d_2^2}{2} - \sigma d_2 \sqrt{T - t} - \frac{1}{2} \sigma^2 (T - t) \right]$$

$$= N'(d_2) \exp \left[ -\sigma d_2 \sqrt{T - t} - \frac{1}{2} \sigma^2 (T - t) \right]$$

But

$$d_2 = \frac{\ln \frac{S}{K} + (r - \frac{\sigma^2}{2})(T - t)}{\sigma \sqrt{T - t}}$$

so

$$\exp \left[ -\sigma d_2 \sqrt{T - t} - \frac{1}{2} \sigma^2 (T - t) \right] = \exp \left[ -\ln \frac{S}{K} - (r - \frac{\sigma^2}{2})(T - t) - \frac{1}{2} \sigma^2 (T - t) \right]$$

$$= \frac{K}{S} e^{-r(T-t)}$$

So,

$$SN'(d_1) = K e^{-r(T-t)} N'(d_2) \quad (1)$$
Next we compute the derivatives of $d_1$ and $d_2$. In particular

\[
\frac{\partial d_1}{\partial S} = \partial_S \left[ \frac{\ln \frac{S}{K} + (r - \frac{\sigma^2}{2})(T - t)}{\sigma \sqrt{T - t}} \right] = \frac{1}{S \sigma \sqrt{T - t}}
\]

\[
\frac{\partial d_2}{\partial S} = \frac{1}{S \sigma \sqrt{T - t}}
\]

since $d_2 = d_1 - \sigma \sqrt{T - t}$.

We now differentiate the Black-Scholes formula:

\[
\frac{\partial c}{\partial S} = \partial_S \left[ SN(d_1) - Ke^{-r(T-t)}N(d_2) \right] = N(d_1) + SN'(d_1)\partial_Sd_1 - Ke^{-r(T-t)}N'(d_2)\partial_Sd_2
\]

\[
= N(d_1) + SN'(d_1)\frac{1}{S \sigma \sqrt{T - t}}N'(d_1) - Ke^{-r(T-t)}\frac{1}{S \sigma \sqrt{T - t}}N'(d_2)
\]

\[
= N(d_1) + \frac{1}{\sigma \sqrt{T - t}}N'(d_1) - \frac{1}{\sigma \sqrt{T - t}}N'(d_1)
\]

\[
= N(d_1)
\]
So,
\[
\frac{\partial c}{\partial S} = N(d_1)
\] (2)

For the put option we note that
\[
p = Ke^{-rT}N(-d_2) - S_0N(-d_1)
\]

and following the same approach yields
\[
\Delta(put) = N(d_1) - 1
\]

Variation of delta with time to maturity for in-the-money, at-the-money, and out-of-the-money call options.
Delta of Other European Options

For European call options on an asset paying a yield $q$, 

$$ \Delta(\text{call}) = e^{-qT} N(d_1) $$

where $d_1$ is defined by 

$$ d_1 = \frac{\ln \frac{S_0}{K} + (r - q + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}} $$

and for European puts 

$$ \Delta(\text{put}) = e^{-qT} [N(d_1) - 1] $$

for the same $d_1$.

If the asset is a currency, we replace $q$ with $r_f$, the foreign risk-free interest rate. If the asset is a futures contract, they are correct with $q$ equal to the risk-free interest rate $r$ and $S_0 = F_0$ in the definition of $d_1$. 
Example: A US bank has sold 6-month put options on £1 million with a strike price of 1.6000 and wishes to make its portfolio delta neutral. Suppose that the current exchange rate is 1.6200, the risk-free interest rate in the UK is 13% per annum, the risk-free interest rate in the US is 10% per annum, and the volatility of pounds is 15%. In this case

\[ S_0 = 1.6200 \quad K = 1.6000 \quad r = 0.10 \quad r_f = 0.13 \quad \sigma = 0.15 \quad T = 0.5 \]

The delta of a put option on a currency is

\[ [N(d_1) - 1] e^{-r_f T} \]

where

\[ d_1 = \frac{\ln \frac{1.6200}{1.6000} + (0.10 - 0.13 + \frac{0.15^2}{2})0.5}{0.15\sqrt{0.5}} = 0.0287 \]

and so \( N(d_1) = 0.5115 \). Thus \( \Delta = [N(d_1) - 1] e^{-r_f T} = -0.458 \).

Therefore, the delta of the put option is -0.458. This is the delta of a long position in one put option. In particular when the exchange rate increases by \( \Delta S \), the price of the put option goes down by 45.8% of \( \Delta S \).

The delta of the bank’s total sort option is +458,000. To make the position delta neutral, the bank must add a short sterling position of £458,000 to the option position. This short position has a delta of -458,000 and neutralizes the delta of the option position.
Delta of a Futures Contract

The futures price for a contract on a non-dividend-paying stock is $S_0 e^{rT}$, where $T$ is the time to maturity of the futures contract.

This shows that when the price of the stock changes by $\Delta S$, with all else remaining the same, the futures price changes by $\Delta e^{rT}$. Since the futures contracts are marked to market daily, the holder of a long futures contract makes an almost immediate gain of this amount.

The delta of a futures contract is therefore $e^{rT}$. For a futures contract on an asset providing a dividend yield at rate $q$, and we see that the delta is $e^{(r-q)T}$.

Sometimes futures contract is used to achieve a delta-neutral position. Define:

- $T$: maturity of futures contract
- $H_A$: required position in asset for delta hedging
- $H_F$: alternative required position in futures contracts for delta hedging

If the underlying asset is a non-dividend-paying stock, the analysis we have just give shows

$$H_F = e^{-rT} H_A$$
When the underlying asset pays a dividend yield \( q \),

\[
H_F = e^{-(r-q)T} H_A
\]

For a stock index, we set \( q \) equal to the dividend yield on the index; for currency we set it equal to the foreign risk-free rate \( r_f \) so that

\[
H_F = e^{-(r-r_f)T} H_A
\]

**Example:** Consider the option before where hedging using the currency requires a short position of £458,000. From above hedging using 9-month currency futures requires a short futures position of

\[
e^{-0.10-0.13 \times \frac{9}{12}} \times 458,000 = £468,442
\]

Because each futures contract is for the purchase or sale of £62,500, \( \frac{468,442}{62,500} = 7.495 \approx 7 \) contracts should be shorted.
Consider the operation of a delta hedging for our first example.

\[ S_0 = 49 \quad K = 50 \quad r = 0.05 \quad \sigma = 0.20 \quad T = 0.3846 \quad \mu = 0.13 \]
on 100,000 shares of stocks. The European call option has been written for $300,000. The initial \( \Delta \) is calculated:

\[
d_1 = \frac{\ln \frac{S_0}{K} + \left( r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}
= \frac{\ln \frac{49}{50} + \left( 0.05 + \frac{0.02^2}{2} \right) 0.3846}{0.2 \sqrt{0.3846}}
\]

and

\[ N(d_1) = 0.522 \]
Once option has been written, the investor has to buy

\[ 100,000 \times 0.522 = 52,200 \text{ shares} \]

for a cost of \( 52,200 \times 49 = 2,557,800 \). The interest rate is 5%, so after one week the interest costs

\[ 2,557,800e^{0.05 \times \frac{1}{52}} = 2,500 \]

Suppose now that the stock drops to $48.12. The delta declines to 0.458. The hedge needs to be

\[ 100,000 \times 0.458 = 45,800 \text{ shares} \]

Therefore, the bank needs to sell 6400 shares. The strategy realizes $308,000 in cash, and the borrowings become \( 2,557,800 + 2,500 - 308,000 = 2,252,300 \). The interest over this period is

\[ 2,252,300 \times e^{0.05 \times \frac{1}{52}} = 2,200 \]

Suppose now that the stock drops to 47.37. The delta declines to 0.400. The hedge needs to be

\[ 100,000 \times 0.400 = 40,000 \text{ shares} \]

Therefore, the bank needs to sell 5800 shares. The strategy realizes \( 5800 \times 47.37 = \$274,746 \). The borrowings become \( 2,252,300 + 2,200 - 274,746 = 1,979,754 \). The interest over the period is

\[ 1,979,754 \times e^{0.05 \times \frac{1}{52}} = 1,900 \]

And on...
Table 15.2  Simulation of delta hedging. Option closes in the money and cost of hedging is $263,300.

<table>
<thead>
<tr>
<th>Week</th>
<th>Stock price</th>
<th>Delta</th>
<th>Shares purchased</th>
<th>Cost of shares purchased ($000)</th>
<th>Cumulative cost including interest ($000)</th>
<th>Interest cost ($000)</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>49.00</td>
<td>0.522</td>
<td>52.200</td>
<td>2,557.8</td>
<td>2,557.8</td>
<td>2.5</td>
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<tr>
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<td>48.12</td>
<td>0.458</td>
<td>(6.400)</td>
<td>(308.0)</td>
<td>2,252.3</td>
<td>2.2</td>
</tr>
<tr>
<td>2</td>
<td>47.37</td>
<td>0.400</td>
<td>(5.800)</td>
<td>(274.7)</td>
<td>1,979.8</td>
<td>1.9</td>
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<td>0.596</td>
<td>19.600</td>
<td>984.9</td>
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<td>1.000</td>
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<td>1.000</td>
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<td>0.0</td>
<td>5,263.3</td>
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In this simulation, the stock price climbs.

As it becomes evident that the option will be exercised at the maturity date, and delta approaches 1.0.

By week 20, the hedger has a fully covered position. The hedger receives $5 million for the stock held, so that the total cost of writing the option and hedging it is $263,300.

On the other hand consider a sequence of events such that the option closes out of the money. As it becomes clear that the option will not be exercised, delta approaches zero.
Table 15.3  Simulation of delta hedging. Option closes out of the money and cost of hedging is $256,600.

<table>
<thead>
<tr>
<th>Week</th>
<th>Stock price</th>
<th>Delta</th>
<th>Shares purchased</th>
<th>Cost of shares purchased ($000)</th>
<th>Cumulative cost including interest ($000)</th>
<th>Interest cost ($000)</th>
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<td>52.200</td>
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<td>(12,600)</td>
<td>(630.0)</td>
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<td>(12,000)</td>
<td>(580.6)</td>
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<td>(77.2)</td>
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<td>(672.7)</td>
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<td>2.0</td>
</tr>
<tr>
<td>15</td>
<td>47.50</td>
<td>0.236</td>
<td>(16,400)</td>
<td>(779.0)</td>
<td>1,324.4</td>
<td>1.3</td>
</tr>
<tr>
<td>16</td>
<td>48.00</td>
<td>0.261</td>
<td>2,500</td>
<td>120.0</td>
<td>1,445.7</td>
<td>1.4</td>
</tr>
<tr>
<td>17</td>
<td>46.25</td>
<td>0.062</td>
<td>(19,900)</td>
<td>(920.4)</td>
<td>526.7</td>
<td>0.5</td>
</tr>
<tr>
<td>18</td>
<td>48.13</td>
<td>0.183</td>
<td>12,100</td>
<td>582.4</td>
<td>1,109.6</td>
<td>1.1</td>
</tr>
<tr>
<td>19</td>
<td>46.63</td>
<td>0.007</td>
<td>(17,600)</td>
<td>(820.7)</td>
<td>290.0</td>
<td>0.3</td>
</tr>
<tr>
<td>20</td>
<td>48.12</td>
<td>0.000</td>
<td>(700)</td>
<td>(33.7)</td>
<td>256.6</td>
<td></td>
</tr>
</tbody>
</table>
By week 20, the hedger has naked position and has incurred costs totalling $256,600.

Note that the costs of hedging the option, when discounted to the beginning of the period, are close but not the same as the Black-Scholes price of $240,000.

If the hedging worked perfectly, the cost of hedging would, after discounting, be exactly equal to the Black-Scholes price for every simulated stock price path.

The reason for the variation in the cost of delta hedging is that the hedge is rebalanced only once a week.

As rebalancing takes place more frequently, the variation in the cost of hedging is reduced.

Note we assume that no transaction costs are incurred.
Delta of a Portfolio

The delta of a portfolio of options or other derivatives dependent on a single asset whose price is $S$ is

$$\frac{\partial \Pi}{\partial S}$$

where $\Pi$ is the value of the portfolio.

- The delta of the portfolio can be calculated from the deltas of the individual options in the portfolio. If a portfolio consists of a quantity $w_i$ of option, the delta of the portfolio is given by

$$\Delta = \sum_{i=1}^{n} w_i \Delta_i$$

where $\Delta_i$ is the delta of the $i$th option.

- The formula can be used to calculate the position in the underlying asset necessary to make the delta of the portfolio zero. When this position has been taken, the portfolio is referred to as being **delta neutral**
Example: Suppose a bank has the following three positions in options in Australian dollars

- A long position in 100,000 call options with strike price 0.55 and an expiration date in 3 months. The delta of each option is 0.533.
- A short position in 200,000 call options with strike price 0.56 and an expiration date in 5 months. The delta of each option is 0.468.
- A short position in 50,000 put options with strike price 0.56 and an expiration date in 2 months. The delta of each option is -0.508.

The delta of the whole portfolio is

\[
100,000 \times 0.533 - 200,000 \times 0.468 - 50,000 \times (-0.508) = -14,900
\]

Therefore, the entire portfolio can be made delta neutral with a long position of 14,900 Australian dollars.

Delta neutrality can be achieved also by a 6-month forward contract. Suppose the risk-free rate is 8% per annum in Australia and 5% in the US. The delta of a forward contract maturing at time \( T \) on one Australian dollar is \( e^{-r_f T} = e^{-0.08 \times 0.05} = 0.9608 \).

The long position in Australian dollar forward contracts for delta neutrality is therefore

\[
\frac{14,900}{0.9608} = 15,508.
\]
Maintaining a delta-neutral position in a single option and underlying asset, in the way we used, is prohibitively expensive because of costs due to transactions on each trade.

For a large portfolio of options, delta neutrality is more feasible. One trade in the underlying asset is necessary to zero out delta for the whole portfolio.

The hedging transactions costs are absorbed by the profits on many different trades.
Theta

The theta $\Theta$ of a portfolio of options is the rate of change of the value of the portfolio with respect to the passage of time with all else remaining the same.

Theta is sometimes referred as the **time decay** of the portfolio For a European call option on a non-dividend-paying stock, then

$$\Theta(\text{call}) = -\frac{S_0 N'(d_1) \sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2)$$

where $d_1$ and $d_2$ are defined as before and

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

For a European put option on the stock

$$\Theta(\text{put}) = -\frac{S_0 N'(d_1) \sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2)$$
For a European call on a dividend paying stock with rate $q$ is

$$\Theta(\text{call}) = -\frac{S_0 N'(d_1) e^{-qT} \sigma}{2\sqrt{T}} - rKe^{-rT} N(d_2)$$

where

$$d_1 =$$

for a European put on a dividend paying stock with rate $q$ is

$$\Theta(\text{call}) = -\frac{S_0 N'(d_1) e^{-qT} \sigma}{2\sqrt{T}} + rKe^{-rT} N(-d_2)$$

Why?

$$\frac{\partial c}{\partial t} = \partial_t \left[ SN(d_1) - Ke^{-(T-t)} N(d_2) \right]$$

$$= SN'(d_1) \partial_t d_1 - rKe^{-(T-t)} N(d_2) - Ke^{-r(T-t)} N'(d_2) \partial_t d_2$$

$$= -rKe^{-(T-t)} N(d_2) + SN'(d_1) \partial_t d_1 - SN'(d_1) \partial_t d_2$$

$$= -rKe^{-(T-t)} N(d_2) + SN'(d_1) \left[ \partial_t d_1 - \partial_t d_2 \right]$$

and $d_1 - d_2 = \sigma \sqrt{T - t}$, so
\[
[\partial_t d_1 - \partial_t d_2] = \partial_t \left[ \sigma \sqrt{T - t} \right] = -\frac{\sigma}{2\sqrt{T - t}}
\]

Thus,
\[
\frac{\partial c}{\partial t} = -r Ke^{-(T-t)} N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T - t}}
\]

(3)

Here, time is measured in years.

Figure 15.5 Variation of theta of a European call option with stock price.

Theta is usually negative for an option (cases for put options when it is positive and call options with high interest rates).

- As time to maturity decreases with all else remaining the same, the option tends to become less valuable.
- When the stock price is low, theta is close to zero. For an at-the-money call option, the theta is large and negative.
- As the stock price becomes larger, theta tends to \(-r Ke^{-rT}\).
• Theta is different type of hedge parameter as delta, but is useful in delta-neutral portfolios.
• More later.
**Example:** Consider a 4-month put option on a stock index. The current value of the index is 305, the strike price is 300, the dividend yield is 3% per annum, the risk-free interest rate is 8% per annum, and the volatility of the index is 25% per annum. Thus

\[ S_0 = 305 \quad K = 300 \quad q = 0.03 \quad r = 0.08 \quad \sigma = 0.25 \quad T = 0.3333 \]

The theta is

\[
- S_0 N'(d_1) e^{-qT} \sigma - r K e^{-rT} N(d_2) = -18.15
\]

The theta is \(-\frac{18.15}{365} = -0.0467\) per calendar day or \(-\frac{18.15}{252} = -0.0720\) per trading year.
The gamma, $\Gamma$, of a portfolio of options on an underlying asset is the rate of change of the portfolio’s delta with respect to the price of the underlying asset.

It is the second partial derivative of the portfolio with respect to asset price:

$$\Gamma = \frac{\partial^2 \Pi}{\partial S^2}$$

If gamma is small, delta changes slowly, and adjustments to keep a portfolio delta neutral need to be made only relatively infrequently.

If gamma is large in absolute terms, delta is highly sensitive to the price of the underlying asset.

It is then quite risky to leave a delta-neutral portfolio unchanged for any length of time.

When the stock price moves from $S$ to $S'$, delta hedging assumes that the option price moves from $C$ to $C'$, when in fact it moves from $C$ to $C''$. The difference between $C'$ and $C''$ leads to a hedging error.

This error depends on the curvature of the relationship between the option price and the stock price. Gamma measures this curvature.
We see this here:

Figure 15.7  Hedging error introduced by nonlinearity.

Suppose that $\Delta S$ is the price change of an underlying asset during a small interval of time $\Delta t$ and $\Delta \Pi$ is the corresponding price change in the portfolio. We will see that:

$$\Delta \Pi = \Theta \Delta t + \frac{1}{2} \Gamma (\Delta S)^2$$

for a delta-neutral portfolio, where $\Theta$ is the theta of the portfolio.
For European call options on non-dividend-paying stocks, we have

$$\Gamma = \frac{N'(d_1)}{S\sigma\sqrt{T-t}}$$

Why?

Recall that:

$$\frac{\partial c}{\partial S} = N(d_1)$$

so

$$\Gamma = \frac{\partial^2 c}{\partial S^2} = \partial_S [N(d_1)] N'(d_1) \partial_S d_1 = N'(d_1) \frac{1}{S\sigma\sqrt{T-t}}$$

At the initial time we have

$$\Gamma = \frac{N'(d_1)}{S_0\sigma\sqrt{T}}$$

**Example**: Suppose that the gamma of a delta-neutral portfolio of options on an asset is - 10,000. Then if the change of +2 or -2 in the price of the asset occurs over a short period of time, there is an unexpected decrease in the value of the portfolio of approximately

$$0.5 \times 10,000 \times 2^2 = \$20,000$$
Aside

Recall that

\[ \frac{\partial c}{\partial S} = N(d_1) \]

\[ \frac{\partial^2 c}{\partial S^2} = N'(d_1) \frac{1}{S \sigma \sqrt{T-t}} \]

\[ \frac{\partial c}{\partial t} = -r Ke^{-(T-t)} N(d_2) - SN'(d_1) \frac{\sigma}{2 \sqrt{T-t}} \]

Then

\[ \partial_t c + r S \partial_S c + \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} = -r Ke^{-(T-t)} N(d_2) - SN'(d_1) \frac{\sigma}{2 \sqrt{T-t}} \]

\[ + r SN(d_1) + \frac{1}{2} \sigma^2 S^2 N'(d_1) \frac{1}{S \sigma \sqrt{T-t}} \]

\[ = r \left[ SN(d_1) - Ke^{-r(T-t)} N(d_2) \right] \]

\[ = rc \]

which implies our formula satisfies the Black-Scholes PDE...
Making a Portfolio Gamma Neutral

- A position in the underlying asset itself or a forward contract on the underlying asset both have zero gamma and cannot be used to change the gamma of a portfolio.
- What is required is a position in an instrument (such as an option) that is not linearly dependent on the underlying asset.
- Suppose that a delta-neutral portfolio has a gamma equal to $\Gamma$, and a traded option has a gamma equal to $\Gamma_T$.
- If the number of traded options added to the portfolio is $w_T$, the gamma of the portfolio is
  \[ w_T \Gamma_T + \Gamma \]
- Thus the position in the traded option necessary to make the portfolio gamma neutral is $-\frac{\Gamma}{\Gamma_T}$.
- Including the traded option is likely to change the delta of the portfolio, so the position in the underlying asset cannot be changed continuously when delta hedging is used.
- Delta neutrality provides protection against larger movements in this stock price between hedge rebalancing.
Example: Suppose a portfolio is delta neutral and has a gamma of -3000. The delta and gamma of a particular traded call option are 0.62 and 1.50, respectively.

The portfolio can be made gamma neutral by including in the portfolio a long position of

\[ \frac{3000}{1.5} = 2000 \]

in the call option.

However, the delta of the portfolio will then change from zero to \(2000 \times 0.62 = 1240\). So a quantity 1240 of the underlying asset must be sold from the portfolio to keep it delta neutral.

**Figure 15.9** Variation of gamma with stock price for an option.
Other Greek Letters?

We will also study Vega $= \frac{\partial \Pi}{\partial \sigma}$, $\rho = \frac{\partial \Pi}{\partial r}$.

Other Greek letters?

Vega Gamma or Volga $= \frac{\partial^2 \Pi}{\partial \sigma^2}$

Vanna $= \frac{\partial^2 V}{\partial S \partial \sigma}$

Delta Decay or Charm $= \frac{\partial^2 \Pi}{\partial t \partial S}$

Color $= \frac{\partial^3 \Pi}{\partial S^2 \partial t}$