Lecture 23. Greek Letters

Fine tuning portfolios with $\Delta, \Theta, \Gamma$, etc...
Greek Letters

The Greek Letters measure different dimensions of risk that correspond to an options position.

Try to approach Hedging in a more sophisticated way.
Naked & Covered Positions

- In a **naked position** the investor does nothing to hedge against losses. In our example, this approach does well so long as the stock remains below $50. Then
- Alternatively, the investor house can take a **covered position**. This involves buying 100,000 shares as soon as the option has been sold. If the option is exercised, the strategy works well. If the stock drops then there is a large loss. By the put-call parity this is similar to

\[ c + Ke^{-rT} = p + S_0 \implies -c + S_0 = -p + Ke^{-rT} \]

so it is the same as writing a put option. Therefore, the covered position is bad if the stock price goes down.
The stop-loss strategy involves the following:

- Consider a bank that has written a call option with strike price $K$.
- The bank buys one unit of stock as soon as the price rises above $K$ and selling it as soon as its price is less than $K$.
- Point is to hold a naked position whenever the stock is less than $K$ and a covered position whenever the stock price is greater than $K$.
- The scheme is designed to ensure that at time $T$ the institution owns the stock if the option closes in the money and does not own it if the option closes out of the money.
- Strategy seems to produce payoffs that are the same as the payoffs on the option.
• In this example, we buy the stock at time $t_1$, sell it at time $t_2$, buy it at time $t_3$, selling at time $t_4$, buying at time $t_5$, and deliver at time $T$.

• Denote the initial stock price $S_0$. The initial cost of the setting up the hedge is $S_0$ if $S_0 > K$ and zero otherwise.

• Is the total cost of the hedge $Q = \max\{S_0 - K, 0\}$, since the purchase/sale of stocks occur always at $S = K$?

• True if no transaction costs! The hedger would make a riskless profit by writing the option and hedging.
Two problems:

- Cash flows to the hedger occur at different times and must be discounted.
- Purchases and sales cannot be made at exactly the same price $K$. Crucial point...If the stock purchases are made at $K + \epsilon$ and sold at $K - \epsilon$ then every purchase and sale incurs a loss of $2\epsilon$.

If the stock prices change continuously (as it is modeled on a Brownian motion) then we expect the curve $S$ to cross our line $S = K$ an infinite number of times! Our profit will go away due to excessive number of transactions.
Delta Hedging

Instead of designing a portfolio with a stop-loss strategy, a different strategy is to design a **delta hedge**.

- Recall that $\Delta$ of an option is the rate of change of the option price with respect to the price of the underlying asset.
- If the $\Delta$ of a call option on a stock is 0.4, then when the stock changes by a small amount, the option value changes by 40% of that amount.
- We have that $\Delta = \frac{\partial c}{\partial S}$ where $c$ is the price of the call option and $S$ is the stock price.

![Figure 15.2 Calculation of delta.](image)

- A position with $\Delta = 0$ is referred to as being **delta neutral**.
Since delta changes over time, the investor’s position remains delta hedged (delta neutral) for relatively short periods of time.

A hedge is rebalanced, or adjusted periodically, to remain delta neutral.

We will describe a dynamic-hedging scheme that rebalances the portfolio periodically to ensure a delta-neutral portfolio.

This is in contrast to static hedging schemes where the hedge is set up and left alone. Such schemes are called hedge and forget schemes.

We will use Black-Scholes analysis to help devise a good delta hedge scheme. Recall that the Black-Scholes portfolio that is riskless is

\[-1 : \text{ option} \]
\[+\Delta : \text{ shares of stock} \]
Delta of European Stock Options

A European call option on a non-dividend-paying stock is

$$\Delta_{(\text{call})} = N(d_1)$$

and a European put option on a non-dividend-paying stock is

$$\Delta_{(\text{put})} = N(d_1) - 1$$

Figure 15.3  Variation of delta with stock price for (a) a call option and (b) a put option on a non-dividend-paying stock.

Keeping a delta hedge for a long position in a European call option involves maintaining a short position of $N(d_1)$ shares at any given time.
Note that the $\Delta$ for a European put option is negative, so that the a long position in a put option should be hedged with a long position in the underlying stock, and a short position in a put option should be hedged with a short position in the underlying stock.

For the put option we note that

$$p = K e^{-rT} N(-d_2) - S_0 N(-d_1)$$

and following the same approach yields

$$\Delta(\text{put}) = N(d_1) - 1$$
Delta of Other European Options

For European call options on an asset paying a yield $q$,

$$\Delta(\text{call}) = e^{-qT} N(d_1)$$

where $d_1$ is defined by

$$d_1 = \frac{\ln \frac{S_0}{K} + (r - q + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}}$$

and for European puts

$$\Delta(\text{put}) = e^{-qT} [N(d_1) - 1]$$

for the same $d_1$.

If the asset is a currency, we replace $q$ with $r_f$, the foreign risk-free interest rate. If the asset is a futures contract, they are correct with $q$ equal to the risk-free interest rate $r$ and $S_0 = F_0$ in the definition of $d_1$. 
Consider the operation of a delta hedging for our first example.

\[
S_0 = 49 \quad K = 50 \quad r = 0.05 \quad \sigma = 0.20 \quad T = 0.3846 \quad \mu = 0.13
\]
on 100,000 shares of stocks. The European call option has been written for $300,000. The initial \( \Delta \) is calculated:

\[
d_1 = \frac{\ln \frac{S_0}{K} + \left( r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}
\]

\[
= \frac{\ln \frac{49}{50} + \left( 0.05 + \frac{0.02^2}{2} \right) 0.3846}{0.2 \sqrt{0.3846}}
\]

and

\[
N(d_1) = 0.522
\]
• Once option has been written, the investor has to buy

\[ 100,000 \times 0.522 = 52,200 \text{ shares} \]

for a cost of \( 52,200 \times 49 = 2,557,800 \). The interest rate is 5%, so after one week the interest costs

\[ 2,557,800e^{0.05 \times \frac{1}{52}} = 2,500 \]

• Suppose now that the stock drops to $48.12. The delta declines to 0.458. The hedge needs to be

\[ 100,000 \times 0.458 = 45,800 \text{ shares} \]

Therefore, the bank needs to sell 6400 shares. The strategy realizes $308,000 in cash, and the borrowings become \( 2,557,800 + 2,500 - 308,000 = 2,252,300 \). The interest over this period is

\[ 2,252,300 \times e^{0.05 \times \frac{1}{52}} = 2,200 \]

• Suppose now that the stock drops to 47.37. The delta declines to 0.400. The hedge needs to be

\[ 100,000 \times 0.400 = 40,000 \text{ shares} \]

Therefore, the bank needs to sell 5800 shares. The strategy realizes \( 5800 \times 47.37 = $274,746 \). The borrowings become

\[ 2,252,300 + 2,200 - 274,746 = 1,979,754 \]. The interest over the period is

\[ 1,979,754 \times e^{0.05 \times \frac{1}{52}} = 1,900 \]

• And on...
Table 15.2  Simulation of delta hedging. Option closes in the money and cost of hedging is $263,300.

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<tr>
<th>Week</th>
<th>Stock price</th>
<th>Delta</th>
<th>Shares purchased</th>
<th>Cost of shares purchased ($000)</th>
<th>Cumulative cost including interest ($000)</th>
<th>Interest cost ($000)</th>
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In this simulation, the stock price climbs.

As it becomes evident that the option will be exercised at the maturity date, and delta approaches 1.0, since

\[ d_1 = \frac{\ln \frac{S_0}{K} + (r - q + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}} \rightarrow +\infty \]

as \( T \to 0 \). Thus \( N(d_1) = 1 \).

By week 20, the hedger has a fully covered position. The hedger receives $5 million for the stock held, so that the total cost of writing the option and hedging it is $263,300.

On the other hand consider a sequence of events such that the option closes out of the money. As it becomes clear that the option will not be exercised, delta approaches zero.
Table 15.3 Simulation of delta hedging. Option closes out of the money and cost of hedging is $256,600.

<table>
<thead>
<tr>
<th>Week</th>
<th>Stock price</th>
<th>Delta</th>
<th>Shares purchased</th>
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By week 20, the hedger has naked position and has incurred costs totalling $256,600, since

\[ d_1 = \ln \frac{S_0}{K} + \left( r - q + \frac{\sigma^2}{2} \right) T \] \[ \sigma \sqrt{T} \rightarrow -\infty \]

as \( T \rightarrow 0 \). Thus \( N(d_1) = 0 \).

Note that the costs of hedging the option, when discounted to the beginning of the period, are close but not the same as the Black-Scholes price of $240,000.

If the hedging worked perfectly, the cost of hedging would, after discounting, be exactly equal to the Black-Scholes price for every simulated stock price path.

The reason for the variation in the cost of delta hedging is that the hedge is rebalanced only once a week.

As rebalancing takes place more frequently, the variation in the cost of hedging is reduced.

Note we assume that no transaction costs are incurred.
Delta of a Portfolio

The delta of a portfolio of options or other derivatives dependent on a single asset whose price is $S$ is

$$\frac{\partial \Pi}{\partial S}$$

where $\Pi$ is the value of the portfolio.

- The delta of the portfolio can be calculated from the deltas of the individual options in the portfolio. If a portfolio consists of a quantity $w_i$ of option, the delta of the portfolio is given by

$$\Delta = \sum_{i=1}^{n} w_i \Delta_i$$

where $\Delta_i$ is the delta of the $i$th option.

- The formula can be used to calculate the position in the underlying asset necessary to make the delta of the portfolio zero. When this position has been taken, the portfolio is referred to as being **delta neutral**
Transaction Costs

Maintaining a delta-neutral position in a single option and underlying asset, in the way we used, is prohibitively expensive because of costs due to transactions on each trade.

For a large portfolio of options, delta neutrality is more feasible. One trade in the underlying asset is necessary to zero out delta for the whole portfolio.

The hedging transactions costs are absorbed by the profits on many different trades.
The theta $\Theta$ of a portfolio of options is the rate of change of the value of the portfolio with respect to the passage of time with all else remaining the same.

Theta is sometimes referred as the **time decay** of the portfolio. For a European call option on a non-dividend-paying stock, then

$$\Theta(\text{call}) = -\frac{S_0 N'(d_1) \sigma}{2\sqrt{T}} - rK e^{-rT} N(d_2)$$

where $d_1$ and $d_2$ are defined as before and

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

For a European put option on the stock

$$\Theta(\text{put}) = -\frac{S_0 N'(d_1) \sigma}{2\sqrt{T}} + rK e^{-rT} N(-d_2)$$
For a European call on a dividend paying stock with rate $q$ is

$$\Theta(\text{call}) = -\frac{S_0N'(d_1)e^{-qT}\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2)$$

where

$$d_1 =$$

for a European put on a dividend paying stock with rate $q$ is

$$\Theta(\text{call}) = -\frac{S_0N'(d_1)e^{-qT}\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2)$$
The **gamma**, $\Gamma$ of a portfolio of options on an underlying asset is the rate of change of the portfolio’s delta with respect to the price of the underlying asset.

It is the second partial derivative of the portfolio with respect to asset price:

$$\Gamma = \frac{\partial^2 \Pi}{\partial S^2}$$

If gamma is small, delta changes slowly, and adjustments to keep a portfolio delta neutral need to be made only relatively infrequently.

If gamma is large in absolute terms, delta is highly sensitive to the price of the underlying asset. It is then quite risky to leave a delta-neutral portfolio unchanged for any length of time.

When the stock price moves from $S$ to $S'$, delta hedging assumees that the option price moves from $C$ to $C'$, when in fact it moves from $C$ to $C''$. The difference between $C'$ and $C''$ leads to a hedging error.

This error depends on the curvature of the relationship between the option price and the stock price. Gamma measures this curvature.
We see this here:

**Figure 15.7** Hedging error introduced by nonlinearity.

Suppose that $\Delta S$ is the price change of an underlying asset during a small interval of time $\Delta t$ and $\Delta \Pi$ is the corresponding price change in the portfolio. We will see that:

$$\Delta \Pi = \Theta \Delta t + \frac{1}{2} \Gamma (\Delta S)^2$$

for a delta-neutral portfolio, where $\Theta$ is the theta of the portfolio.
We can see this from a Taylor series expansion of the value $\Pi$ in terms of $S$ and $t$. In particular:

$$\Delta \Pi = \frac{\partial \Pi}{\partial S} \Delta S + \frac{\partial \Pi}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 \Pi}{\partial S^2} (\Delta S)^2 + \frac{1}{2} \frac{\partial^2 \Pi}{\partial t^2} (\Delta t)^2 + \frac{\partial^2 \Pi}{\partial t \partial S} (\Delta t)(\Delta S) + \cdots$$

Delta neutral portfolio, the first term on RHS is zero, so

$$\Delta \Pi = \Theta \Delta t + \frac{1}{2} \Gamma (\Delta S)^2$$

For European call options on non-dividend-paying stocks, we have

$$\Gamma = \frac{N'(d_1)}{S \sigma \sqrt{T-t}}$$

At the initial time we have

$$\Gamma = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}}$$
Making a Portfolio Gamma Neutral

- A position in the underlying asset itself or a forward contract on the underlying asset both have zero gamma and cannot be used to change the gamma of a portfolio.
- What is required is a position in an instrument (such as an option) that is not linearly dependent on the underlying asset.
- Suppose that a delta-neutral portfolio has a gamma equal to $\Gamma$, and a traded option has a gamma equal to $\Gamma_T$.
- If the number of traded options added to the portfolio is $w_T$, the gamma of the portfolio is
  \[ w_T \Gamma_T + \Gamma \]
- Thus the position in the traded option necessary to make the portfolio gamma neutral is $-\frac{\Gamma}{\Gamma_T}$.
- Including the traded option is likely to change the delta of the portfolio, so the position in the underlying asset cannot be changed continuously when delta hedging is used.
- Delta neutrality provides protection against larger movements in this stock price between hedge rebalancing.
Relationship between Delta, Theta and Gamma

As we defined last time,

\[
\Delta = \frac{\partial \Pi}{\partial S} \quad \Theta = \frac{\partial \Pi}{\partial t} \quad \Gamma = \frac{\partial^2 \Pi}{\partial S^2}
\]

and these are the components of the Black-Scholes equation:

\[
\frac{\partial \Pi}{\partial t} + rS\frac{\partial \Pi}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial \Pi}{\partial t} = r\Pi
\]

Therefore,

\[
\Theta + rS\Delta + \frac{\sigma^2}{2}S^2\Gamma = r\Pi
\]

When a portfolio is delta-neutral, \( \Delta = 0 \), so

\[
\Theta + \frac{\sigma^2}{2}S^2\Gamma = r\Pi
\]

Thus, when \( \Theta \) is large and positive, \( \Gamma \) tends be large and negative. Thus, \( \Theta \) can be viewed as a replacement for \( \Gamma \) in \( \Delta \) neutral portfolios.
So far have assumed that the volatility of the asset underlying a derivative is constant.

Volatilities change over time. In particular the value of a derivative is liable to change because of movements in volatility as well as because of changes in the asset price and the passage of time.

We set vega of a portfolio of derivatives, \( \mathcal{V} \), is the rate of change of the value of the portfolio with respect to the volatility of the underlying asset:

\[
\mathcal{V} = \frac{\partial \Pi}{\partial \sigma}
\]

- If vega is high in absolute terms, then the portfolio’s value is very sensitive to small changes in volatility.
- If vega is low in absolute terms, then volatility changes have little impact on the value of the portfolio.
Note that a position in the underlying asset has zero vega, but the vega of a portfolio can be changed by adding a position in a traded option.

- If $\mathcal{V}$ is the vega of the portfolio and $\mathcal{V}_T$ is the vega of a traded option, a position of $-\frac{\mathcal{V}}{\mathcal{V}_T}$ in the traded option makes the portfolio instantaneously vega neutral.
- However, a portfolio that is gamma neutral will not generally be vega neutral and vice-versa.
- If a hedger requires a portfolio to be both gamma and vega neutral, at least two traded derivatives dependent on the underlying asset must usually be used.

**Example:** Consider a portfolio that is delta neutral, with a gamma of -5000 and a vega of -8000. A traded option has a gamma of 0.5, a vega of 2.0, and delta of 0.6.

The portfolio can be made vega neutral by including a long position in 4000 traded options. This increases the delta to 2400 and require that 2400 units of the asset be sold to maintain delta neutrality. The gamma of the portfolio would change from -5000 to -3000.

To make the portfolio gamma and vega neutral, we suppose that there is a second traded option with gamma of 0.8, vega of 1.2, and delta of 0.5.

If $w_1$ and $w_2$ are the quantities of the two traded options included in the portfolio, we require that

\[ 0 = -5000 + 0.5w_1 + 0.8w_2 \]
\[ 0 = -8000 + 2.0w_1 + 1.2w_2 \]
Solving these yield $w_1 = 400$ and $w_2 = 6000$. The portfolio can be made gamma and vega neutral by including 400 of the first traded option and 6000 of the second traded option.

The delta of the portfolio after the addition of the positions in the two traded options is $400 \times 0.6 + 6000 \times 0.5 = 3240$. Thus we need to sell 3240 units of the asset to maintain delta neutrality.
Calculating vega

For a European call or put on a non-dividend-paying stock, vega is given by

\[
\mathcal{V} = S_0 \sqrt{T} N'(d_1)
\]

where

\[
d_1 = \frac{\ln \left( \frac{S}{K} \right) + \left( r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}.
\]

For a European call or put on a dividend-paying stock with yield \( q \), the vega is

\[
\mathcal{V} = S_0 \sqrt{T} N'(d_1) e^{-qT}
\]

where

\[
d_1 = \frac{\ln \left( \frac{S}{K} \right) + \left( r - q + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}.
\]

When the asset is a stock index, \( q \) is the dividend yield. When it is a currency contract, then set \( q \) to be the risk-free foreign rate \( r_f \). When it is a futures contract, \( S_0 = F_0 \) and \( q = r \).
**Example:** A financial institution has the following portfolio of OTC options on Sterling:

<table>
<thead>
<tr>
<th>Type</th>
<th>Position</th>
<th>Delta of option</th>
<th>Gamma of option</th>
<th>Vega of option</th>
</tr>
</thead>
<tbody>
<tr>
<td>Call</td>
<td>-1000</td>
<td>0.50</td>
<td>2.2</td>
<td>1.8</td>
</tr>
<tr>
<td>Call</td>
<td>-500</td>
<td>0.80</td>
<td>0.6</td>
<td>0.2</td>
</tr>
<tr>
<td>Put</td>
<td>-2000</td>
<td>-0.40</td>
<td>1.3</td>
<td>0.7</td>
</tr>
<tr>
<td>Call</td>
<td>-500</td>
<td>0.70</td>
<td>1.8</td>
<td>1.4</td>
</tr>
</tbody>
</table>

A traded option is available with delta of 0.6, gamma of 1.5, vega of 0.8.

- What position in the traded option and in sterling would make the portfolio both gamma neutral and delta neutral?

  The delta of the portfolio is

  \[ -1000 \times 0.5 - 500 \times 0.8 - 2000 \times (-0.4) - 500 \times 0.7 = -450 \]

  The gamma of the portfolio is

  \[ -1000 \times 2.2 - 500 \times 0.6 - 2000 \times 1.3 - 500 \times 1.8 = -6000 \]

  The vega of the portfolio is

  \[ -1000 \times 1.8 - 500 \times 0.2 - 2000 \times 0.7 - 500 \times 1.4 = -4000 \]
Thus a long position in 4000 traded options will give a gamma-neutral portfolio since a long position has a gamma of $4000 \times 1.5 = 6000$. The delta of the portfolio is

$$4000 \times 0.6 - 450 = 1950$$

Hence, in addition to the 4000 traded options, a short position in £1950 is necessary to have both gamma and delta neutrality.

**What position in the traded option and in sterling would make the portfolio both vega neutral and delta neutral?**

In this case we need 5000 traded options to get a vega-neutral portfolio since the long position yields a vega of $5000 \times 0.8 = 4000$. The delta of the whole portfolio is then

$$5000 \times 0.6 - 450 = 2550$$

Hence we need a short position in £2550 in addition to the 5000 traded options to maintain vega and delta neutrality.
Vega

- Confusing to compute the vega of a portfolio since we derived Black-Scholes by assuming constant volatility
- $\nu$ calculated from stochastic volatility models with constant volatility works well.
- $\nu$ neutrality protects from large changes in the volatility $\sigma$, whereas $\Gamma$ neutrality protects against large swings in the price of the underlying asset between hedge rebalancing.
The **rho** of a portfolio of options is the rate of change of the value of the portfolio with respect to the interest rate

\[ \text{rho} = \frac{\partial \Pi}{\partial r} \]

which measures the sensitivity of the value of the portfolio to interest rate changes.

For a European call option on non-dividend-paying stock:

\[ \text{rho} = K T e^{-rT} N(d_2) \]

where \( d_2 \) is defined as before. For a European put option

\[ \text{rho} = -K T e^{-rT} N(-d_2) \]

The formulas also hold for European calls and puts for dividend paying stocks with \( d_2 \) adjusted.
Example: Consider a 4-month put option on a stock index. The current value of the index is 305, strike price is 300, dividend yield is 3\% per annum, the risk-free rate is 8\% per annum, and the volatility is 25\% per annum.

So

\[ S_0 = 305 \quad K = 300 \quad q = 0.03 \quad r = 0.08 \quad \sigma = 0.25 \quad T = \frac{4}{12} \]

The rho is

\[ -KTe^{-rT}N(-d_2) = -42.6 \]

So a change of 1\% in the risk-free interest rate, from say 8\% to 7\%, corresponds to an increase in the value of the option by \( 0.426 = 0.01 \times 42.6 \).

- When we consider currency options there are two rhos corresponding to the two different interest rates. The rho corresponding to the domestic interest rate is given by the formulas with \( d_2 \) as before

\[ d_2 = \frac{\ln \frac{S}{K} + (r - r_f + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}} \]

- The rho corresponding to the foreign interest rate for a European call on a currency is

\[ -Te^{-r_fT}S_0N(d_1) \]

and a European put is

\[ Te^{-r_fT}S_0N(-d_1) \]

- The rho for a European call futures option is \( -cT \) and for European put futures option rho is \( -pT \).
So it is possible to design a portfolio with zero delta, zero gamma, and zero vega.

Is this done in the ”real world”?

Large portfolios usually work to maintain zero delta, but maintaining zero gamma and zero vega is much more difficult, since we need efficient access to large positions of nonlinear derivatives.

If the positions are too large, then even maintain zero delta is difficult. But moderately sized portfolios it is reasonable, since profits of trading cover the cost of daily rebalancing.
Portfolio Insurance

In general a portfolio manager wishes to acquire a put option to protect against large declines while achieving gains if the market appreciates.

One approach is to buy put options on a market index. Another approach is to create the put synthetically.

To create a synthetic put option, one maintains a position in the underlying asset so that the delta of the position is equal to the delta of the required option.

This can be more attractive than buying the put from the market:

- Options markets do not always have the liquidity to absorb trades that managers of large funds would like to have access to.
- Fund managers often require strike prices and exercise dates that are different from those available from the exchange-traded markets.
How to synthetically create the put?

The option can be created by trading the portfolio or by trading in index futures contracts.

- Consider the first approach - creating a put option by trading the portfolio. Recall the delta of a European put on the portfolio is

\[ \Delta = e^{-qt} [N(d_1) - 1] \]  

(1)

where

\[ d_1 = \frac{\ln \frac{S_0}{K} + (r - q + \sigma^2/2)T}{\sigma \sqrt{T}} \]

with \( S_0 \) the value of the portfolio, \( K \) the strike price, \( r \) the risk-free rate, \( q \) the dividend yield on the portfolio, \( \sigma \) the volatility of the portfolio, and \( T \) the life of the option.

Assume that the volatility of the portfolio is beta times the volatility of a well-diversified market index.

Therefore, to create the put option synthetically, the fund manager should ensure that at any given time a proportion

\[ e^{-qt} [N(d_1) - 1] \]

of the stocks in the original portfolio has been sold and the proceeds invested in riskless assets.
As the value of the original portfolio declines, the delta of the put given by (1) becomes more negative and the proportion of the original portfolio sold must be increased.

As the value of the original portfolio increases, the delta of the put becomes less negative and the proportion of the original portfolio sold must be decreased (and shares purchased)

- This strategy to create portfolio insurance entails dividing funds between the stock portfolio on which the insurance is required and riskless assets.
- As the value of the stock portfolio increases, riskless assets are sold and the position in the stock portfolio is increased.
- As the value of the stock portfolio decreases, the position in the stock portfolio is decreased and riskless assets are purchased.
- Insurance costs arise as the fact that selling occurs after a decline in the market and buying occurs after a rise in the market.

**Example:** Consider a portfolio worth $90 million. To protect against downturns in the market, the manager requires a 6-month European put option on the portfolio with a strike price of $87 million. The risk free rate is 9% per annum, the dividend yield is 3% per annum, and the volatility of the portfolio is 25% per annum.
The S&P 500 index is at 900. As the portfolio is considered to mimic the S&P 500 closely, one approach is to buy 1000 put option contracts on the S&P 500 with a strike price of 870.

Another approach is to create the put synthetically. Here $S_0 = 90$ milion, $K = 87$ million

$$r = 0.09 \quad \sigma = 0.25 \quad T = 0.5$$

so

$$d_1 = \ln \frac{90}{87} + (0.09 - 0.03 + \frac{0.25^2}{2})0.5 \over 0.25\sqrt{0.5} = 0.4499$$

the delta of the required option is

$$e^{-qT} [N(d_1) - 1] = -0.3215$$

Thus 32.15% of the portfolio should be sold initially to match the delta of the required option. The amount of the portfolio sold must be monitored frequently.
Stock Market Volatility

Note that the portfolio insurance created by synthetic means (selling portfolio at downswings and purchasing on upswings), can accentuate trends:

If a market declines, portfolio managers sell stock or sell index futures contracts. Both actions accentuate the decline. Sale of stock is liable to drive down market indices further.

This creates selling pressure on stocks - cause of 1987 stock market crash?

Whether portfolio insurance schemes affect volatility depends on how easily a market can absorb the trades that are generated by portfolio insurance.

If portfolio insurance trades are a small fraction of all trades, there is like no effect. As portfolio insurance becomes more popular, it is liable to have a destabilizing effect.
Summary

- The Delta $\Delta$ of an option is the rate of change of its price with respect to the price of the underlying asset. Delta hedging involves creating a position with zero delta or a delta-neutral position.
- Because the delta of the underlying asset is 1.0, one way is to take a position of $-\Delta$ in the underlying asset for each long option being hedged. Position needs to be adjusted frequently.
- Once an option position has been made, one can examine the $\Gamma$ of the portfolio - the rate of change of its delta with respect to the price of the underlying asset.
- Gamma measures the curvature of the relationship between the option price and the asset price. The impact of this curvature on the performance of the hedge can be reduced by making the portfolio gamma neutral.
- If $\Gamma$ is the gamma of the position being hedged, this reduction is usually achieved by a position in a traded option with a total gamma of $-\Gamma$.
- Both $\Delta$ and $\Gamma$ are calculated assuming that volatility is constant. However, in practice volatility does change.
- The $\mathcal{V}$ of a portfolio measures the rate of change of its value with respect to volatility.
• Trader can make hedge position vega neutral by taking an offsetting position. To make the portfolio both Gamma and Vega neutral, the trader needs two different derivatives on the same asset.

• Other variables that measure the risk in a portfolio are the rho and theta which measure sensitivity of the portfolio to rate changes and time changes, respectively.

• In practice traders usually maintain delta neutrality daily and monitor Gamma and Vega for large swings.

• Traders can sometimes build put options synthetically for large equity portfolios. They do so by either trading the portfolio or by trading index futures on the portfolio. Trading the portfolio entails splitting the portfolio between equities and risk-free securities. As the market declines more is invested in risk-free securities. As the market increases, more is invested in equities.
Volatility Smiles

How close are market prices to those predicted by Black-Scholes? Are Black-Scholes formulas used to price options?

Not entirely. Traders typically allow for volatility to depend on *price strike price and time to maturity*.

Plot of implied volatility of an option as a function of strike price is known as a *volatility smile*. 
Due Dec. 5, 5PM.
Final Exam: December 10th in class.

- Graded: 14.42, 15.26, 15.28