Lecture 23. Volatility Smiles

Adjusting volatility to the market
Volatility Smiles

How close are market prices to those predicted by Black-Scholes? Are Black-Scholes formulas used to price options?

Not entirely. Traders typically allow for volatility to depend on price strike price and time to maturity.

Plot of implied volatility of an option as a function of strike price is known as a volatility smile.
Implied Volatility

There are two ways to think about volatility:

- From price changes, we can compute the volatility via standard deviation.
- Another method is to consider data used in Black-Scholes:

\[ c = S_0 N(d_1) - Ke^{-rT} N(d_2) \]

with

\[ d_1 = \frac{\ln \frac{S_0}{K} + \left( r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \]

\[ d_2 = \frac{\ln \frac{S_0}{K} + \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T} \]

If we know \( S_0, K, r, \) and \( c, \) then we can solve, implicitly, for \( \sigma. \) The resulting \( \sigma \) is called the **implied volatility**.
Recall the put-call parity
\[ c + Ke^{-rT} = p + S_0e^{-qT} \]
which is based on a simple arbitrage argument.

- It does not require any information on probabilities of underlying stock movements or volatility!
- True if the stock process is lognormal or not.

Consider a situation where for a particular volatility we compute \( p_{BS} \) and \( c_{BS} \), the value of the European put and call using the Black-Scholes formulae.
\[ c_{BS} + Ke^{-rT} = p_{BS} + S_0e^{-qT} \]

On the other hand consider \( p_{mkt} \) and \( c_{mkt} \) to be the actual market values of the options.
\[ c_{mkt} + Ke^{-rT} = p_{mkt} + S_0e^{-qT} \]
otherwise our simple arbitrage argument works...

So the put-call parity holds for both situations.
Therefore, we get by subtraction:

\[ p_{BS} - p_{mkt} = c_{BS} - c_{mkt} \]  \hspace{1cm} (1)

So the pricing error when BS is used to price a European put option should be the same as the pricing error when it is used to price a European call option.

**Example** Suppose the implied volatility of the put is 22%. Thus \( p_{BS} = p_{mkt} \) when a volatility of 22% is used in BS model. From (1) we see that \( c_{BS} = c_{mkt} \) when same volatility used.

Therefore, implied volatility of a European call is always the same as the implied volatility of a European put.

As a consequence the correct volatility to use for the European call should be the same as for the European put. Approximately true for the American options too...
Example: Value of Australian dollar is $0.60. The risk free interest rate is 5% per annum in the US and 10% per annum in Australia. The market price of a European call option on the Australian dollar with maturity of 1 year and a strike price of $0.59 is 0.0236.

The implied volatility computed from this information on the call is 14.5%. For there to be no arbitrage, the put-call parity relationship must apply with \( q \) equal to the risk-free rate.

The price \( p \) of a European put option with strike price $0.59 and maturity of 1 year satisfies

\[
p + 0.60e^{-0.10 \times 1} = 0.0236 + 0.59e^{-0.05 \times 1}
\]

or \( p = 0.0419 \). Note when we compute the implied volatility with this price, we get 14.5%.
We now consider our first **volatility smile**. This is a graph of volatility as a function of strike price. We assumed for Black-Scholes that this is a constant function...

On the other hand traders use the following volatility smile

![Volatility Smile Diagram](image)

**Figure 16.1** Volatility smile for foreign currency options.

- Volatility is relatively low for at-the-money options.
- Volatility is relatively high for the more in-the-money or out-of-the-money the strike price is.
The associated probability distribution should no longer be lognormal, since the crucial ingredient to \( S \) being lognormal was

\[
\frac{dS}{S} = \mu dt + \sigma \epsilon dz
\]

Now \( \sigma \) is a function....

The **implied distribution** turns out to be
The distribution with the same mean and same standard deviation has

- fat tails
- steeper

Why does this distribution hold? Consider deep out-of-the-money call option with high strike price $K_2$.

- The option pays off only if the exchange rate proves to be above $K_2$.
- The probability of this is higher for the implied probability distribution than for the lognormal distribution. Expect the implied distribution to give a relatively high price for this option.
- A relatively high price should lead to a relatively high implied volatility.

Next consider a deep in-the-money put option with a low strike price $K_1$.

- The option pays off only if the exchange rate proves to be below $K_1$.
- The probability of this is higher for implied probability distribution than for the lognormal distribution.
- Expect the implied distribution to give a relatively high price, and a relatively high implied volatility for this option too.
Empirical Tests of Volatility Smile

So the simile used by traders shows that they believe that the lognormal distribution understates the probability of extreme movements in exchange rates.

Consider the following data. We can measure the number of days in which the daily change of the exchange rate exceeds a certain number of standard deviations.

<table>
<thead>
<tr>
<th></th>
<th>Real world</th>
<th>Lognormal model</th>
</tr>
</thead>
<tbody>
<tr>
<td>&gt; 1 SD</td>
<td>25.04</td>
<td>31.73</td>
</tr>
<tr>
<td>&gt; 2 SD</td>
<td>5.27</td>
<td>4.55</td>
</tr>
<tr>
<td>&gt; 3 SD</td>
<td>1.34</td>
<td>0.27</td>
</tr>
<tr>
<td>&gt; 4 SD</td>
<td>0.29</td>
<td>0.01</td>
</tr>
<tr>
<td>&gt; 5 SD</td>
<td>0.08</td>
<td>0.00</td>
</tr>
<tr>
<td>&gt; 6 SD</td>
<td>0.03</td>
<td>0.00</td>
</tr>
</tbody>
</table>

We can see there are much fatter tails in the real world - hence more likely to have a large movements.
Why?

Why is there a smile in the foreign currency option? We need two conditions to hold for the lognormal distribution to hold:

- Volatility of the asset is constant
- Price of the asset changes smoothly with no jumps

Neither of these two assumptions hold for an exchange rate.

Volatility of an exchange rate is not constant and exchange rates frequently jump. Both of these tend to increase the likelihood of extreme events.

The percentage impact of a nonconstant volatility on prices becomes more pronounced as the maturity of the option is increased, but the volatility smile created by nonconstant volatility usually becomes less pronounced as the maturity of the option increases.

Business Snapshot Traders in the mid 80’s understood that there are heavier tails and purchased cheap put /call options and waited. These options occurred with greater frequency than lognormal and were too cheap (from Black-Scholes pricing). These traders made a lot of money! By late 80’s the volatility smile was introduced.
Volatility Smiles for Equity Options

Before the crash of 1987, stocks were generally assumed to follow the lognormal distribution.

After the crash, a volatility smile for equity options was introduced by Rubinstein and Jackwerth-Rubinstein.

**Figure 16.3** Volatility smile for equities.

The volatility smile or **volatility skew**, has the form of a downward sloping parabola.
- Volatility to price a **low-strike-price** option (deep-out-of-the-money put or deep-in-the-money call) is significantly higher than that used to price a **high-strike-price** option (deep-in-the-money put or deep-out-of-the-money call).
- The volatility smile for equity options corresponds to the implied probability distribution given by below:

![Graph showing implied and lognormal distributions](image)

compared to the corresponding lognormal distribution.
Why consistent?

- Consider a deep-out-of-the-money call option with a strike price of $K_2$. This has a lower price when the implied distribution is used than when the lognormal distribution is used.
- This is because the option pays off only if the stock price proves to be above $K_2$, and the probability of this is lower for the implied probability distribution than for the lognormal distribution.
- Thus expect the implied distribution to give a relatively low price for the option. A low price leads to a relatively low implied volatility which is what is observed.

Consider now a deep-out-of-the-money put option with strike price $K_1$.

- The option pays off only when the stock price is below $K_1$. The probability of this is higher for implied probability distribution.
- Expect the implied distribution to give a relatively higher price, and a relatively high price implies higher implied volatility.
Why the smile?

Reasons for the equity volatility smile

- **Fear of a crash.** Traders are concerned about the possibility of a crash, so they price the option accordingly.

- **Leverage.** As a company’s equity declines in value, the equity becomes more risky and its volatility increases. As a company’s equity increases in value, the equity becomes less risky and its volatility decreases.
Computing the Probability Distributions

Recall that the European call option on an asset with strike price $K$ and maturity $K$ is given by

$$c = e^{rT} \int_{S_T = K}^{\infty} (S_T - K) g(S_T) dS_T$$

where $r$ is the constant interest rate, $S_T$ is the asset price at time $T$, and $g$ is the risk-neutral probability density function of $S_T$.

Differentiating with respect to $K$ yields

$$\frac{\partial c}{\partial K} = -e^{rT} \int_{S_T = K}^{\infty} g(S_T) dS_T$$

and again

$$\frac{\partial^2 c}{\partial K^2} = e^{-rT} g(K)$$

Thus we get

$$g(K) = e^{rT} \frac{\partial^2 c}{\partial K^2} \approx e^{rT} \frac{c(K-\delta) - 2c(K) + c(K+\delta)}{\delta^2}$$

This implies that we can estimate the probability curve from the volatility smile curve.
Traders also consider the volatility term structure when pricing options. In other words the volatility used to price an at-the-money option depends on the maturity of the option.

- Volatility tends to be an increasing function of maturity when short-dated volatilities are historically low, since there is expectation that volatility will increase.
- Volatility tends to be a decreasing function of maturity when short-dated volatilities are historically high, since there is expectation that volatility will decrease.

**Volatility surfaces** combine volatility smiles with the volatility term structure to tabulate the volatilities appropriate for pricing an option with any strike price and any maturity.

Consider an example

<table>
<thead>
<tr>
<th>Strike Price</th>
<th>0.90</th>
<th>0.95</th>
<th>1.00</th>
<th>1.05</th>
<th>1.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 month</td>
<td>14.2</td>
<td>13.0</td>
<td>12.0</td>
<td>13.1</td>
<td>14.5</td>
</tr>
<tr>
<td>3 month</td>
<td>14.0</td>
<td>13.0</td>
<td>12.0</td>
<td>13.1</td>
<td>14.2</td>
</tr>
<tr>
<td>6 month</td>
<td>14.1</td>
<td>13.3</td>
<td>12.5</td>
<td>13.4</td>
<td>14.3</td>
</tr>
<tr>
<td>1 year</td>
<td>14.7</td>
<td>14.0</td>
<td>13.5</td>
<td>14.0</td>
<td>14.8</td>
</tr>
<tr>
<td>2 year</td>
<td>15.0</td>
<td>14.4</td>
<td>14.0</td>
<td>14.5</td>
<td>15.1</td>
</tr>
<tr>
<td>5 year</td>
<td>14.8</td>
<td>14.6</td>
<td>14.4</td>
<td>14.7</td>
<td>15.0</td>
</tr>
</tbody>
</table>
Some of the implied probability entries are computed using market data and Black-Scholes.

The rest are found via simple linear interpolation.

When a new option has to be valued, traders look up the appropriate volatility in the table.

**Example:** Consider the value of a 9-month option with a strike price of 1.05. The trader would interpolate between 13.4 and 14.0 in the table to obtain a volatility of

$$\frac{1}{2}(14.0 + 13.4) = 13.7\%$$

The volatility of 13.7% would then be used in the Black-Scholes formula or binomial tree.

- The shape of the volatility smile depends on the option maturity. The smile tends to become less pronounced as the option maturity increases.
- If $T$ is the time to maturity and $F_0$ is the forward price of the asset. Then sometimes traders define the volatility smile as the relationship between implied probability and

$$\frac{1}{\sqrt{T}} \ln \frac{K}{F_0}$$

rather than the relationship between implied volatility and $K$. The smile is much less dependent on the time to maturity.
Role of the Model

How important is the pricing model if traders are prepared to use a different volatility for every option?

We can think of Black-Scholes as an interpolation tool used by traders to ensure that an option is priced consistently with the market prices of other actively traded options.

If traders used a different model, then volatility surfaces and the shape of the smile would change, but the dollar prices found in the market should not change appreciably.
Volatility Smiles makes the calculation of Greek Letters more difficult. There are a couple of rules to follow:

- **Sticky Strike Rule**: Implied volatility remains constant from one day to the next.
- This implies that the Greek letters calculated using the Black-Scholes assumptions are correct from one day to the next, provided the volatility used for an option is its current implied volatility.
- **Stick Delta Rule**: Relationship between an option price and \( S/K \) today will apply tomorrow.
- As the price of the underlying asset changes, the implied volatility of the option is assumed to change to reflect how much the option is in or out of the money.
- Sticky Delta Rule implies the formulas for Greek letters need to be modified. For example the delta of a call option is

\[
\frac{\partial c_{BS}}{\partial S} + \frac{\partial c_{BS}}{\partial \sigma_{imp}} \frac{\partial \sigma_{imp}}{\partial S}
\]

by the chain rule.
Consider the impact of this on the delta of an equity option.

We know that from the volatility smile, that volatility is a decreasing function of strike price $K$.

Alternatively it can be regarded as an increasing function of $S/K$. Under sticky delta, the volatility increases as the asset price increases. Thus

$$\frac{\partial \sigma_{imp}}{\partial S} > 0$$

Thus delta is **higher** than that given by Black-Scholes assumptions.

A much more precise approach to the Greeks with regard to the volatility smiles is the **implied volatility function** model or the **implied tree model**. More on this next semester.
Consider now an unusual situation which can greatly affect the shape of a volatility smile.

- Suppose that a pharmaceutical stock is currently at $50 and an announcement on a pending lawsuit against it is expected in a few days. The news is expected to send the stock either up by $8 or down by $8.
- The probability distribution of the stock price in one month should consist of a mixture of two lognormal distributions, the first corresponding to favorable news, the second to unfavorable news.

**Figure 16.5** Effect of a single large jump. The solid line is the true distribution; the dashed line is the lognormal distribution.
The solid line show the mixtures of the two lognormal distributions, whereas the dashed line is the single lognormal distribution.

- The true probability distribution is bimodal. We can see this as an example of the binomial tree method:

\[ u = \frac{58}{50} = 1.16 \quad d = 0.84 \quad a = e^{rT} = 1.0101 \]

so

\[ p = \frac{a - d}{u - d} = \frac{1.0101 - 0.84}{1.16 - 0.84} = 0.5314 \]
Now we compute for different strike prices the value of the European call or European put.

- Consider the European call with a strike price of 44. Then from binomial tree operation:

\[
c = e^{-rT} \left[ pf_u + (1 - p) f_d \right]
\]

\[
e^{-0.12 \times \frac{1}{12}} \left[ 0.5314 \times (58 - 44) + (1 - 0.5314) \times 0 \right]
\]

\[
= 7.37
\]

Then we solve for the volatility in Black-Scholes and get \( \sigma = 58.8\% \).

- Consider the European call with a strike price of 48. Then from binomial tree operation:

\[
c = e^{-rT} \left[ pf_u + (1 - p) f_d \right]
\]

\[
e^{-0.12 \times \frac{1}{12}} \left[ 0.5314 \times (58 - 48) + (1 - 0.5314) \times 0 \right]
\]

\[
= 5.26
\]

Then we solve for the volatility in Black-Scholes and get \( \sigma = 69.5\% \).
Continue with other strike prices and get the implied volatilities, which yields the chart:

<table>
<thead>
<tr>
<th>Strike price</th>
<th>Call price</th>
<th>Put price</th>
<th>Implied volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td>42</td>
<td>8.42</td>
<td>0.00</td>
<td>0.0</td>
</tr>
<tr>
<td>44</td>
<td>7.37</td>
<td>0.93</td>
<td>58.8</td>
</tr>
<tr>
<td>46</td>
<td>6.31</td>
<td>1.86</td>
<td>66.6</td>
</tr>
<tr>
<td>48</td>
<td>5.26</td>
<td>2.78</td>
<td>69.5</td>
</tr>
<tr>
<td>50</td>
<td>4.21</td>
<td>3.71</td>
<td>69.2</td>
</tr>
<tr>
<td>52</td>
<td>3.16</td>
<td>4.64</td>
<td>66.1</td>
</tr>
<tr>
<td>54</td>
<td>2.10</td>
<td>5.57</td>
<td>60.0</td>
</tr>
<tr>
<td>56</td>
<td>1.05</td>
<td>6.50</td>
<td>49.0</td>
</tr>
<tr>
<td>58</td>
<td>0.00</td>
<td>7.42</td>
<td>0.0</td>
</tr>
</tbody>
</table>
Graphing the implied volatility as a function of Strike Price yields a volatility frown.

Therefore, nonstandard volatility smiles can result from nonstandard expected behavior of a stock process.
Summary

- Black-Scholes model and its extensions assume that the probability distribution of the underlying asset at any given future time is lognormal.
- This assumption is not made by traders.
- Traders assume a probability distribution of an equity price has a heavier left tail and a less heavy right tail than the lognormal distribution.
- Traders also assume that the probability distribution of an exchange rate has a heavier right tail and heavier left tail than the lognormal distribution.
- Traders use volatility smiles to allow for nonlognormality.
- The volatility smile defines the relationship between the implied volatility of an option and its strike price.
- For equity options, the volatility smile tends to be downward sloping.
  - Out-of-the-money puts and in-the-money calls tend to have high implied volatility.
  - Out-of-the-money calls and in-the-money puts tend to have low implied volatility.
- Foreign currency options, the volatility smile is $U$-shaped. Both out-of-money and in-the-money options have higher implied volatilities than at-the-money options.
- Often traders use volatility term structure. The implied volatility of an option depends on the duration of the option.
- When volatility smiles and volatility term structures are used together, we get a volatility surface.
- This defines volatility as a function of both strike price and time to maturity.
Homework: Due Dec. 5, 5PM.
Final Exam: December 10th in class.