Review for final exam
Problem 14.42:

Solution: In this case \( S_0 = 0.75, K = 0.75, r = 0.05, r_f = 0.09, \sigma = 0.04, \) and \( T = 0.75. \) The option can be valued by using the BS formula for currency options:

\[
d_1 = \frac{\ln \frac{0.75}{0.75} + \left(0.07 - 0.09 + \frac{0.04^2}{2}\right)0.75}{0.04\sqrt{0.75}} = -0.4157
\]

\[
d_2 = d_1 - 0.04\sqrt{0.75} = -0.4503
\]

and \( N(d_1) = 0.3388, \) and \( N(d_2) = 0.3262. \)

The value of the call option \( c \) is given by

\[
c = 0.75e^{-0.09 \times 0.75} \times 0.3388 - 0.75e^{-0.07 \times 0.75} \times 0.3262
\]

\[
= 0.0054
\]

So the call costs 0.0054 cents.
The put-call parity implies

\[ p + S_0 e^{r T} = c + K e^{-r T} \]

so

\[ p = 0.0054 + 0.75 e^{-0.07 \times 0.75} - 0.75 e^{-0.09 \times 0.75} = 0.0160 \]

The option to buy 0.75 USD with 1.00 CDM is the same as an option to sell one Canadian dollar for 0.75 USD. In other words it is a put option on the Canadian dollar and its price is 0.0160 USD.
Problem 15.26: Consider the situation in Prob. 15.25. Suppose that a second traded option with a delta of 0.1, a gamma of 0.5 and a vega of 0.6 is available. How could the portfolio be made delta, gamma, and vega neutral.

Solution: The delta of the portfolio is

\[-1000 \times 0.5 - 500 \times 0.8 - 2000 \times (-0.4) - 500 \times 0.7 = -450\]

The gamma of the portfolio is

\[-1000 \times 2.2 - 500 \times 0.6 - 2000 \times 1.3 - 500 \times 1.8 = -6000\]

The vega of the portfolio is

\[-1000 \times 1.8 - 500 \times 0.2 - 2000 \times 0.7 - 500 \times 1.4 = -4000\]

Then let \(w_1\) be the position of the first traded option and \(w_2\) be the position in the second traded option.
Require

\[ 6000 = 1.5w_1 + 0.5w_2 \]
\[ 4000 = 0.8w_1 + 0.6w_2 \]

Solving for the quantities yield \( w_1 = 3200 \) and \( w_2 = 2400 \). The portfolio has a delta of

\[ -450 + 3200 \times 0.6 + 2400 \times 0.1 = 1710 \]

The portfolio is now delta, gamma, vega neutral taking an additional short position in \( £1710 \).
Problem 15.28: The formula for the price $c$ of a European call futures option in terms of the futures price is given as

$$c = e^{-rT} [F_0 N(d_1) - KN(d_2)]$$

where

$$d_1 = \frac{\ln \frac{F_0}{K} + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}}$$

and $d_2 = d_1 - \sigma \sqrt{T}$.

1. Show $F_0 N'(d_1) = KN'(d_2)$:

$$F N'(d_1) = \frac{F}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}}$$

and

$$KN'(d_2) = KN'(d_1 - \sigma \sqrt{T}) = \frac{K}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2} + d_1 \sigma \sqrt{T} - \frac{\sigma^2 T}{2}}$$

Because $d_1 \sigma \sqrt{T} = \ln \frac{F}{K} + \frac{\sigma^2 T}{2}$ the second equation reduces to

$$KN'(d_2) = \frac{K}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2} + \ln \frac{F}{K}} = \frac{F}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}}$$
The result follows.

2. Show that the delta of the call price with respect to the futures price is $e^{-rT}N(d_1)$:

$$\frac{\partial c}{\partial F} = e^{-rT}N(d_1) + e^{-rT}N'(d_1)\frac{\partial d_1}{\partial F} - e^{-rT}KN'(d_2)\frac{\partial d_2}{\partial F}$$

Because $\frac{\partial d_1}{\partial F} = \frac{\partial d_2}{\partial F}$ it follows that from (a)

$$\frac{\partial c}{\partial F} = e^{-rT}N(d_1)$$

3. Show that the vega of the call price is $F_0\sqrt{T}N'(d_1)e^{-rT}$:

$$\frac{\partial c}{\partial \sigma} = e^{-rT}FN'(d_1)\frac{\partial d_1}{\partial \sigma} - e^{-rT}KN'(d_2)\frac{\partial d_2}{\partial \sigma}$$

Because $d_1 = d_2 - \sigma\sqrt{T}$ then

$$\frac{\partial d_1}{\partial \sigma} = \frac{\partial d_2}{\partial \sigma} + \sqrt{T}$$

It follows from (a) that

$$\frac{\partial c}{\partial \sigma} = e^{-rT}FN'(d_1)\sqrt{T}$$
4. Prove the formula for the rho of a call futures option. Why is the formula not the same as for a call futures option.

\[
\frac{\partial c}{\partial r} = -Te^{-rT} [FN(d_1) - KN(d_2)] = -cT
\]

Because \( q = r \) in the futures option, there are two components to rho. One arises from differentiation with respect to \( r \), the other arises from differentiation with respect to \( q \).
Wiener Processes and Itô’s Lemma

More sophisticated approach to modeling the behavior of assets underlying derivatives - view motion as a stochastic process.

A stochastic process is a process where future evolution is described by probability distributions.

Two types: discrete-time stochastic process changes values at discrete time steps. A continuous-time stochastic process changes value at any time.

Stochastic process can be continuous variable or discrete variable. A continuous-variable process can take any value within a certain range. (motion of a particle in fluid). A discrete-variable process can take only certain prescribed values. (coin flips)

Markov Process is a stochastic process where only present value of a variable is relevant for predicting the future. Coin flips are Markovian. If we flip the coin 30 times and comes up heads 30 straight times, next flip still 50/50 chance.
Normal Distribution

Let $\phi(\mu, \sigma)(x)$ denote the normal distribution. Then $\phi$ satisfies

$$
\phi(\mu, \sigma)(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}
$$

Probability of sum of
Normal Distribution

\[ N(\mu, \sigma) = \int_{-\infty}^{x} \phi(\mu, \sigma)(s) \, ds \quad \Phi(\mu, \sigma)(+\infty) = 1 \]

Consider a random number \( x \in (0, 1) \) then most likely within the middle of the curve, if we undo it.

Sums of two normal distributions mean zero is a normal distribution with mean zero and variance that’s the sum of the two variances. Proof: next time.
Consider a Markov stochastic process. Suppose that the current value is 10 and the change in its value during 1 year is $\phi(0, 1)$.

After two years? The change in two years is a sum of two one year Markov stochastic process with mean zero and standard deviation 1.

Therefore, the sum is a normal distribution with mean zero and variance $1 + 1 = 2$. Thus the standard deviation is $\sqrt{2}$.

Consider now the change in the variable during 6 months. The variance of the change in the value of the variable during 1 year equals the variance of the change during the first 6 months plus the variance of the change during the second 6 months.

We assume they are the same. Then variance of change during a 6-month period must be 0.5. Thus, the standard deviation of the change is $\sqrt{0.5}$. Thus 6-month distribution is $\phi(0, \sqrt{0.5})$.

Consider a small time step $\Delta t = \frac{1}{N}$, during which each period is an independent normal distribution. Then sum of the variances are equal to 1, so each variance should be $\Delta t$.

The standard deviation then is $\sqrt{\Delta t}$.

Uncertainty is proportional to square root of time.
We continue letting $\Delta t \to 0$ carefully! This is called the Wiener process or Brownian motion. It is a Markov stochastic process with mean zero and variance 1.0 per year. Therefore, it has

1. Change $\Delta z$ during a small period of time $\Delta t$ is

$$\Delta z = \epsilon \sqrt{\Delta t}$$

where $\epsilon$ has a standardized normal distribution $\phi(0, 1)$. Therefore, $\Delta z$ has a normal distribution with
- mean of $\Delta z = 0$
- standard deviation of $\Delta z = \sqrt{\Delta t}$
- variance of $\Delta z = \Delta t$.

2. Values of $\Delta z$ for any two different short intervals of time $\Delta t$ are independent. Therefore, $z$ follows a Markov process.
Measure the value of $z(T') - z(0)$ over a long period of time $T$.

View as a sum of $N$ small changes over small time changes $\Delta t$, i.e.

$$N = \frac{T}{\Delta t} \implies z(T) - z(0) = \sum_{i=1}^{N} \epsilon_i \sqrt{\Delta t}$$

where $\epsilon_i$ for $i \in \{1, \ldots, N\}$ are distributed $\phi(0, 1)$. The $\epsilon_i$’s are independent of each other.

Then $z(T) - z(0)$ is normally distributed with

- mean of $[z(T) - z(0)] = 0$
- variance of $[z(T) - z(0)] = N \Delta t = T$
- standard deviation of $[z(T) - z(0)] = \sqrt{\Delta T}$. 
Wiener Process

Self-similar structure. Lies in $C^{0,\frac{1}{2}}$. 
Generalized Wiener Process

The mean change per unit time for a stochastic process is known as the drift rate.

The variance per unit time for a stochastic process is known as the variance rate.

A generalized Wiener process for a variable $x$ can be defined in terms of $dz$ as

$$dx = adt + bdz$$

where $adt$ is the expected drift rate of $a$ per unit time.

Holds since $dx = adt \implies \frac{dx}{dt} = a$. Therefore,

$$x = x_0 + at$$

After time $T$ the variable $x$ travels $T$ units.
Generalized Wiener Process

The term $b \, dz$ regarded as noise added to the system, which is $b$ times a Wiener process.

In a small time interval $\Delta t$, the change $\Delta x$ in the variable of $x$ is given by

$$\Delta x = a \Delta t + b \epsilon \sqrt{\Delta t}$$

where $\epsilon$ has a standard normal distribution.

- mean of $\Delta x = a \Delta t$
- standard deviation of $\Delta x = b \sqrt{\Delta t}$
- variance of $\Delta x = b^2 \Delta t$.

The same argument show that the change in the value of $x$ in any time interval $T$ is normally distributed with

- mean of $x = aT$
- standard deviation of $x = b \sqrt{T}$
- variance of $x = b^2 T$. 
**Generalized Wiener Process**

**Figure 12.2** Generalized Wiener process with $a = 0.3$ and $b = 1.5$. 

\[ \text{Value of variable, } x \]

\[ \text{Generalized Wiener process} \]

\[ dx = a \, dt + b \, dz \]

\[ dx = a \, dt \]

\[ \text{Wiener process, } dz \]
A generalized Wiener process in which the parameters $a$ and $b$ are functions of the value of the underlying variable $x$ and $t$. An Itô process can be written as

$$dx = a(x, t)dt + b(x, t)dz$$

Both the expected drift rate and the variance rate of an Itô process are liable to change over time. In a small time interval between $t$ and $t + \Delta t$, the variable changes from $x$ to $x + \Delta x$, where

$$\Delta x = a(x, t)\Delta t + b(x, t)\epsilon\sqrt{\Delta t}$$

Thus $b^2$ is the variance and $a$ is the mean during the interval between $t$ and $t + \Delta t$. 
Process for a stock price

Discuss the process that models stock movements for a nondividend paying stock:

- Expected return $= \frac{\text{Expected drift}}{\text{Stock price}}$ is constant
- If $S$ is the stock price a time $t$, then the expected drift rate in $S$ should be assumed to be $\mu S$ for some constant parameter $\mu$.
- So in short period of time $\Delta t$ the expected increase in $S$ should be $\mu S \Delta t$.

If volatility of the stock is zero then model implies

$$\Delta S = \mu S \Delta t$$

In the limit $\Delta t \to 0$,

$$dS = \mu S dt$$

or

$$\frac{dS}{S} = \mu dt$$

Then

$$S_T = S_0 e^{\mu T}$$
Process for a stock price

Including volatility then expect: variability of the percentage return in a short period of time $\Delta t$ is the same regardless of the stock price. This suggests that the standard deviation of the change in a short period of time $\Delta t$ should be proportional to the stock price and leads to

$$dS = \mu S dt + \sigma S dz$$

or

$$\frac{dS}{S} = \mu dt + \sigma dz$$

(1)

We use (1) to price stocks. Here $\sigma$ is the volatility and $\mu$ is the expected return rate.

Limiting case of the random walk we saw with binomial trees.
If $S$ is the stock price at a particular time and $\Delta S$ is the increase in the stock price in the next small interval of time,

$$\frac{\Delta S}{S} = 0.15 \Delta t + 0.30 \epsilon \sqrt{\Delta t}$$

where $\epsilon$ has a standard normal distribution. Consider a time interval of 1 week or 0.0192 year and suppose that the initial stock price is $100. Then $\Delta t = 0.0192$, $S = 100$, and

$$\Delta S = 100 (0.00288 + 0.0416 \epsilon)$$

or

$$\Delta S = 0.288 + 4.16 \epsilon$$

showing that the price increase has a normal distribution with mean $0.288$ and standard deviation $2.16$. 
Discrete-Time Model

Discrete-time version of the model is

\[
\frac{\Delta S}{S} = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t}
\]  

(2)

so the change in the stock value over a short period of time is

\[
\Delta S = \mu S \Delta t + \sigma S \epsilon \sqrt{\Delta t}
\]

- Variable \( \Delta S \) is the change in the stock price \( S \) over a small interval of time \( \Delta t \) and \( \epsilon \) has a standard normal distribution (normal distribution with \( \sigma = 1 \) and \( \mu = 0 \)).
- \( \mu \) is the expected rate of return by the stock in a short period of time \( \Delta t \).
- \( \sigma \) is the volatility of the stock price.
Left-hand-side of (2) is the return provided by the stock in a short period of time.

- Term $\mu \Delta t$ is the expected value of the return
- Term $\sigma \epsilon \sqrt{\Delta t}$ is the stochastic component of the return. Variance is $\sigma^2 \Delta t$ (consistent with the definition of volatility defined earlier).

Then $\Delta S / S$ is normally distributed with mean $\mu \Delta t$ and standard deviation $\sigma \sqrt{\Delta t}$, so

$$ \frac{\Delta S}{S} \sim \phi(\mu \Delta t, \sigma \sqrt{\Delta t}) $$
Itô’s Lemma

An Itô process is one in which the drift and the volatility depend on both $x$ and $t$. Suppose $x$ is an Itô’s process then

$$dx = a(x, t)dt + b(x, t)dz$$

where $dz$ is a Wiener process and $a, b$ are functions of $x$ and $t$. Then $x$ has a variance $b^2$.

Itô’s Lemma states that a function $G$ of $x$ and $t$ follows the following process:

$$dG = \left( \frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b dz$$

In particular $G$ is an Itô process with drift rate

$$\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2$$

and variance

$$\left( \frac{\partial G}{\partial x} \right)^2 b^2$$
Itô’s Lemma: Modeling stock movements

We argued that a reasonable model of stock movements should be

$$dS = \mu S dt + \sigma S dz$$

with $\mu$ and $\sigma$ constants.

From Itô’s Lemma we can consider a process $G$ that depends on $t$ and $S$. Then

$$dG = \left( \frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S dz$$

so both $S$ and $G$ are affected by $dz$ - the noise in the system.
Lognormal Property

Recall that our model requires

\[ dS = \mu S dt + \sigma S dz \]

with \( \mu \) and \( \sigma \) constants.

Define \( G = \ln S \) then

\[
\frac{\partial G}{\partial S} = \frac{1}{S} \quad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2} \quad \frac{\partial G}{\partial t} = 0
\]

by Itô’s Lemma we have

\[
dG = \left( \frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S dz
\]

\[
= \left[ \frac{1}{S} \mu S + 0 + \frac{1}{2} \frac{-1}{S^2} \sigma^2 S^2 \right] dt + \frac{1}{S} \sigma S dz
\]

\[
= \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dz
\]

Therefore, \( G \) follows a generalized Wiener process with

- Drift = \( \mu - \frac{\sigma^2}{2} \)
- Variance = \( \sigma^2 \)
Lognormal Property

Therefore, the change in $\ln S$ between 0 and a future time $T$ is normally distributed with mean $(\mu - \frac{\sigma^2}{2})T$ and variance $\sigma^2T$. Hence:

$$\ln S_T - \ln S_0 \approx \phi \left[ \left( \mu - \frac{\sigma^2}{2} \right) T, \sigma \sqrt{T} \right]$$

or

$$\ln S_T \approx \phi \left[ \ln S_0 + \left( \mu - \frac{\sigma^2}{2} \right) T, \sigma \sqrt{T} \right]$$

This implies that log of the stock price is normally distributed:

A variable has a lognormal distribution if the natural log of the variable is normally distributed.

The standard deviation of the logarithm of the stock price is $\sigma \sqrt{T}$. 
We are now in position to derive the Black-Scholes or Black-Scholes-Merton differential equation. We build the model via a riskless portfolio, as we did for binomial trees. As for binomial trees, we carry some stock along with shorting the option. The amount of stock changes \textit{instantaneously}. Special assumptions are required:

1. The stock price follows the process defined earlier for \( \mu \) and \( \sigma \):

\[
\frac{dS}{S} = \mu dt + \sigma dz
\]

2. Short selling of securities with full use of proceeds is permitted
3. There are no transactions costs or taxes. All securities are perfectly divisible
4. There are no dividends during the life of the derivative
5. There are no riskless arbitrage opportunities
6. Security trading is continuous
7. The risk-free rate of interest, \( r \), is constant and the same for all maturities
Recall our process for a continuous stock movement modeled on an Itô process with expected gain \( \mu \) and volatility \( \sigma \).

\[
dS = \mu S \, dt + \sigma S \, dz
\]

Let \( f \) be the price of a call option that depends on \( S \). The variable \( f \) depends, then \( S \) and \( t \). Then

\[
df = \left( \frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \, dt + \frac{\partial f}{\partial S} \sigma S \, dz
\]

Recall the discrete-time analogues as

\[
\Delta S = \mu S \Delta t + \sigma S \sqrt{\Delta t}
\]

and so the discrete version of Itô’s Lemma is:

\[
\Delta f = \left( \frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t + \frac{\partial f}{\partial S} \sigma S \sqrt{\Delta t}
\]
We now build a portfolio that will **eliminate** the stochasticity of the process. The appropriate portfolio (as we will see) is

- -1 option
- \( \frac{\partial f}{\partial S} \) shares (recall \( \Delta = \frac{f_u - f_d}{S_0 u - S_0 d} \) in the binomial tree)

which changes continuously over time. Let \( \Pi \) be the value of the portfolio then

\[
\Pi = -f + \frac{\partial f}{\partial S}
\]

and \( \Delta \Pi \) be the value of the portfolio in the time interval \( \Delta t \) then

\[
\Delta \Pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S
\]
Derivation of Black-Scholes-Merton Differential Equation

Then

$$\Delta \Pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S$$

$$= - \left[ \left( \frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right) \Delta t + \frac{\partial f}{\partial S} \sigma S \Delta z \right]$$

$$+ \frac{\partial f}{\partial S} \left[ \mu S \Delta t + \sigma S \Delta z \right]$$

$$= - \left[ \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right] \Delta t$$

Note that $\Delta \Pi$ does not depend on $dz$, therefore there is no risk during time $\Delta t$! Thus the portfolio must instantaneously earn the same rate of return as other short-term risk-free securities. If it earned more than this return, arbitrageurs could make a riskless profit by borrowing money to buy the portfolio; if it earned less they could make a riskless profit by shorting the portfolio and buying risk-free securities. Thus:

$$\Delta \Pi = r \Pi \Delta t$$
Derivation of Black-Scholes-Merton Differential Equation

where \( r \) is the risk-free rate. Then

\[
- \left[ \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right] \Delta t = \left[ -f + \frac{\partial f}{\partial S} S \right] \Delta t
\]

so

\[
\frac{\partial f}{\partial t} + r S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = r f
\]

Equation (3) is the Black-Scholes partial differential equation. Any solution corresponds to the price of a derivative overlying a particular stock.

In order to specify further what the derivative is, we use a boundary condition to constrain it. Boundary conditions for European call options:

\[
f = \max\{S - K, 0\}
\]

when \( t = T \). Boundary conditions for European put options:

\[
f = \max\{K - S, 0\}
\]

when \( t = T \). The portfolio created is riskless only for infinitesimally short periods.
Risk-neutral valuation

Variables that factor into Black-Scholes-Merton are the current stock price, stock price volatility, and the risk-free rate of interest. All independent of risk of stock movements. This would mean that the Black-Scholes would depend on $\mu$ too.

If risk preferences of the investor do not enter the equation, then they cannot affect the solution.

Consider a derivative that provides a payoff at one particular time. It can be valued using risk-neutral valuation by using the following procedure:

- Assume that the expected return from the underlying asset is the risk-free interest rate $r$ (assume $\mu = r$)
- Calculate the expected payoff from the derivative
- Discount the expected payoff at the risk-free interest rate

Risk-neutral valuation is merely an artificial device for obtaining solutions to the Black-Scholes PDE. The solutions that are obtained are valid in all worlds, not just risk-neutral.

When we move from a risk-neutral world to a risk-averse world, two things occur: (a) the expected growth rate in the stock price changes and (b) the discount rate that must be used for any payoffs from the derivative. The two changes always offset each other exactly.
The Black-Scholes formulas for the price at time 0 of a European call option on a non-dividend-paying stock and for a European put option on a non-dividend paying stock are

\[ c = S_0 N(d_1) - Ke^{-rT}N(d_2) \]

and

\[ p = KN(-d_2) - S_0e^{-rT}N(-d_1) \]

where

\[ d_1 = \frac{\ln \frac{S_0}{K} + \left( r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} \]

\[ d_2 = \frac{\ln \frac{S_0}{K} + \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T} \]

and \( N(x) \) is the cumulative probability distribution function.
The variables $c$ and $p$ are the European call and put prices, $S_0$ is the current stock price at time 0, $K$ is the strike price, $r$ is the continuously compounded risk-free rate, $\sigma$ is the stock price volatility, and $T$ is the time to maturity of the option. Why?
Consider a European call option. The expected value of the option at maturity in a risk-neutral world is

\[ \hat{E}[\max\{S_T - K\}, 0] \]

where \( \hat{E} \) is the expected value in a risk-neutral world.

From the risk-neutral argument, the European call option price \( c \) is the expected valued discounted at the risk-free rate of interest, i.e.

\[ c = e^{-rT} \hat{E}[\max\{S_T - K\}, 0] \] (4)

Can check that (4) does indeed solve (3). We now compute the Black-Scholes formulas.
Recall that we wish to solve the Black-Scholes-Merton PDE

\[
\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf
\]

subject to the boundary conditions

\[f = \max\{S - K, 0\}\]

when \(t = T\). If the solution of the European call option is shown to be

\[f = e^{-rT} \hat{E}\left[\max\{S_T - K\}, 0\right]\]

then the Black-Scholes Formula holds.

We now prove (6) is indeed the solution.
Volatility

Estimating Volatility from Historical Data

Define

- \( n + 1 \): Number of observations
- \( S \): Stock price at end of the \( i \)th interval, with \( i = 0, 1, \ldots, n \).
- \( \tau \): Length of time interval in years

and let

\[
u_i = \ln \left( \frac{S_i}{S_{i-1}} \right)
\]

for \( i = 0, 1, \ldots, n \).

Then the sample standard deviation of \( u_i \) is given by

\[
s = \sqrt{\frac{1}{n - 1} \sum_{i=1}^{n} (u_i - \bar{u})^2}
\]
Volatility

or

\[ s = \sqrt{\frac{1}{n - 1} \sum_{i=1}^{n} u_i^2 - \frac{1}{n(n - 1)} \left( \sum_{i=1}^{n} u_i \right)^2} \]

From our stock process argument, the standard deviation of \( u_i \) is \( \sigma \sqrt{\tau} \). The variable \( s \) is an estimate of \( \sigma \sqrt{\tau} \), then \( \sigma \) can be estimate as \( \hat{\sigma} \), where

\[ \hat{\sigma} = \frac{s}{\sqrt{\tau}}. \]

The standard error of this calculation can be shown to be approximately \( \frac{\hat{\sigma}}{\sqrt{2n}} \).
Dividends

Used the assumption that no dividends are payed to establish Black-Scholes equation.

Assume the amount and timing of the dividends during the life of an option can be predicted with certainty. For short-life options this is not an unreasonable assumption. For long-life options it is usual to assume that the dividend yield rather than the cash dividend payments are known.

By the time the option matures, the dividends will have been paid and the riskless component will no longer exist. The Black-Scholes formula is therefore correct if \( S_0 \) is equal to the risky component of the stock price and \( \sigma \) is the volatility of the process followed by the risky component.

Operationally, this means that the Black-Scholes formula can be used provided that the stock price is **reduced by the present value of all dividends during the life of the option**.

The discounting being done from the ex-dividend dates at the risk-free rate. A dividend is counted as being during the life of the option if its ex-dividend date occurs during the life of the option.

**Easier to use dividend yield with Black-Scholes**
Replacing \( S_0 \) by \( S_0 e^{-qT} \) in the Black-Scholes formulas, then the price \( c \) of a European call and price \( p \) of a European put on a stock providing a dividend yield at rate \( q \) as

\[
c = S_0 e^{-qT} N(d_1) - Ke^{-rT} N(d_2)
\]

and

\[
p = Ke^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1).
\]

Since \( \ln \frac{S_0 e^{-qT}}{K} = \ln \frac{S_0}{K} - qT \) then \( d_1 \) and \( d_2 \) are

\[
d_1 = \frac{\ln \frac{S_0}{K} + (r - q + \frac{\sigma^2}{2}) T}{\sigma \sqrt{T}}
\]

\[
d_2 = \frac{\ln \frac{S_0}{K} + (r - q - \frac{\sigma^2}{2}) T}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T}
\]

and \( N(x) \) is the cumulative probability distribution function.
Portfolio Insurance

Use Stock Index Options to build insurance in an equity portfolio.

Recall: \textbf{beta} refers to how a portfolio mirrors another asset.

Assume that a well-diversified portfolio has a beta of 1.0 when compared to a stock index, i.e. the returns on the portfolio exactly mirror the stock index.

Each contract on S&P 500 is 100 times the index value.

The value of the portfolio is protected against the possibility of the index falling below \( K \) if for each \( 100S_0 \) dollars in the portfolio, the manager buys one put option contract with strike price \( K \).
Insurance when portfolio’s Beta is not 1.0

Use a capital asset pricing model:

\[
\text{Expected excess return of a portfolio over the risk-free interest rate equals}
\]
\[
\text{beta times the excess return of a market index over the risk-free interest rate}
\]

In particular for every 100$S_0$ dollars in the portfolio, a total of \textbf{beta} put contracts should be purchased.
Currency Options

Currency options are traded on the Philadelphia Stock Exchange. Useful to hedge an incoming Forward currency contract.

Valuation: As before let $S_0$ denote the spot exchange rate. $S_0$ is the value of one unit of the foreign currency in US dollars.

The owner of foreign currency receives a yield equal to the risk-free interest rate $r_f$ in the foreign currency. We replace $q$ with $r_f$ in our Black-Scholes formulas with dividend yield

$$c = S_0 e^{-r_f T} N(d_1) - K e^{-r T} N(d_2)$$

and

$$p = K e^{-r T} N(-d_2) - S_0 e^{-r_f T} N(-d_1).$$

with $d_1$ and $d_2$ as

$$d_1 = \frac{\ln \frac{S_0}{K} + \left( r - r_f + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}$$

$$d_2 = \frac{\ln \frac{S_0}{K} + \left( r - r_f - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T}$$
Note that $F_0 = S_0 e^{(r-r_f)T}$ then we can rewrite the equations as:

$$c = e^{-rT} [F_0 N(d_1) - K N(d_2)]$$

and

$$p = e^{-rT} [K N(-d_2) - S_0 N(-d_1)].$$

with $d_1$ and $d_2$ as

$$d_1 = \frac{\ln \frac{F_0}{K} + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}}$$

$$d_2 = \frac{\ln \frac{F_0}{K} - \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T}$$
Interest rate futures contract option

Interest rate futures option contracts work in the same way as the other futures options contracts.

The payoff from a call is \( \max\{F - K, 0\} \), where \( F \) is the futures price at the time of exercise and \( K \) is the strike price.

In addition to the cash payoff, the option holder obtains a long position in the futures contract when the option is exercised and the option writer obtains a corresponding short position.

Why use them?

- Options on futures contracts are more liquid (easy to buy/sell) than the underlying contract. Easier to trade the option than the actual contracts.
- Futures price is known immediately from trading on the futures exchanges, whereas the spot price of the asset not so readily available.
- Market for the futures options larger than the futures market.
Put-call parity for options on futures contracts

Consider the following argument for the put-call parity for options on futures contracts:

• Portfolio A: a European call futures option plus amount of cash equal to $Ke^{-rT}$
• Portfolio B: a European put futures option plus a long futures contract plus amount of cash equal to $F_0e^{-rT}$.

Consider the different scenarios for the futures price $F_T$ at maturity:

• Portfolio A: At time $T$, the cash becomes $K$. If $F_T > K$, the call option is exercised and the portfolio is worth

$$K + (F_T - K) = F_T.$$ 

If $F_T \leq K$ then the option is not exercised and the portfolio has value $K$. Thus portfolio A is worth

$$\max\{F_T, K\}.$$
• Portfolio B: At time $T$, the cash becomes $F_0$. The put option provides a payoff of $\max\{K - F_T, 0\}$. The futures contract is worth $F_T - F_0$. The value of the portfolio at time $T$ is worth

$$F_0 + (F_T - F_0) + \max\{K - F_T, 0\} = \max\{K, F_T\}$$

Since both portfolios at exercise, they have the same present value. Hence

$$Ke^{-rT} + c = F_0e^{-rT} + p$$

For an American option on a futures contract we have

$$F_0e^{-rT} - K \leq C - P \leq F_0 - Ke^{-rT}$$
Pricing European Options on Futures Contracts

Since the pricing model with $q = r$ is

$$c = e^{-rT} [S_0 N(d_1) - KN(d_2)]$$

and

$$p = e^{-rT} [KN(-d_2) - S_0 N(-d_1)] .$$

Since $\ln \frac{S_0 e^{-qT}}{K} = \ln \frac{S_0}{K} - qT$ then $d_1$ and $d_2$ are

$$d_1 = \frac{\ln \frac{S_0}{K} + \frac{\sigma^2}{2}T}{\sigma \sqrt{T}}$$

$$d_2 = \frac{\ln \frac{S_0}{K} - \frac{\sigma^2}{2}T}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T}$$

where $\sigma$ is the volatility of the futures price.
Black-Scholes formula for valuing European options on a non-dividend-paying stock can be extended to cover European options on a stock providing a known dividend yield.

- An index is analogous to a stock providing a dividend yield. The dividend yield is the average dividend yield on the stocks composing the index.
- A foreign currency is analogous to a stock providing a dividend yield where the dividend yield is the foreign risk-free interest rate.
- A futures price is analogous to a stock providing a dividend yield where the dividend yield is equal to the domestic risk-free rate.

The extension of Black-Scholes can be used to value European options on indices, foreign currencies, and futures contracts.

- For the stock index options: upon exercise of an index call option, the holder receives the amount by which the index exceeds the strike price at close of trading. Similarly, upon exercise of an index put option, the holder receives the amount by which the strike price exceeds the index at close of trading. Index options can be used for portfolio insurance. If the portfolio has beta of 1.0, it is appropriate to buy one put option for each 100$S_0$ dollars in the portfolio, where $S_0$ is the value of the index.
Otherwise, beta put options should be purchased for every $100S_0$ dollars in the portfolio, where beta is the portfolio calculated using the capital asset pricing model. The strike price of the put option purchased should reflect the level of insurance required.

- Currency options are traded both on exchanges and OTC. They can be used by corporate treasurers to hedge foreign exchange exposure. For example, a US corporate treasurer can hedge by buying put options that mature at that time.

- Futures options require the delivery of the underlying futures contract upon exercise. When a call is exercised, the holder acquires a long futures position plus a cash amount equal to the excess of the futures price over the strike price. Similarly, when a put is exercised, the holder acquires a short position plus a cash amount equal to the excess of the strike price over the futures price. The futures contract that is delivered typically expires slightly later than the option.
Hedging Options positions via Naked & Covered Positions

- In a **naked position** the investor does nothing to hedge against losses. In our example, this approach does well so long as the stock remains below $50. Then

- Alternatively the investor house can take a **covered position**. This involves buying 100,000 shares as soon as the option has been sold. If the option is exercised, the strategy works well. If the stock drops then there is a large loss. By the put-call parity this is similar to

\[
c + Ke^{-rT} = p + S_0 \implies -c + S_0 = -p + Ke^{-rT}
\]

so it is the same as writing a put option. Therefore, the covered position is bad if the stock price goes down.
Stop-Loss Strategy

The stop-loss strategy involves the following:

- Consider a bank that has written a call option with strike price $K$.
- The bank buys one unit of stock as soon as the price rises above $K$ and selling it as soon as its price is less than $K$.
- Point is to hold a naked position whenever the stock is less than $K$ and a covered position whenever the stock price is greater than $K$.
- The scheme is designed to ensure that at time $T$ the institution owns the stock if the option closes in the money and does not own it if the option closes out of the money.
- Strategy seems to produce payoffs that are the same as the payoffs on the option.
Two problems:

- Cash flows to the hedger occur at different times and must be discounted.
- Purchases and sales cannot be made at exactly the same price $K$. Crucial point...If the stock purchases are made at $K + \epsilon$ and sold at $K - \epsilon$ then every purchase and sale incurs a loss of $2\epsilon$.

If the stock prices change continuously (as it is modeled on a Brownian motion) then we expect the curve $S$ to cross our line $S = K$ an infinite number of times! Our profit will go away due to excessive number of transactions.
Delta Hedging

Instead of designing a portfolio with a stop-loss strategy, a different strategy is to design a *delta hedge*.

- Recall that $\Delta$ of an option is the rate of change of the option price with respect to the price of the underlying asset.
- If the $\Delta$ of a call option on a stock is 0.4, then when the stock changes by a small amount, the option value changes by 40% of that amount.
- We have that $\Delta = \frac{\partial c}{\partial S}$ where $c$ is the price of the call option and $S$ is the stock price.

Instead build a portfolio of $\Delta$ shares to compensate for each -1 option.
Delta Hedging, cont.

Since delta changes over time, the investor’s position remains delta hedged (delta neutral) for relatively short periods of time.

A hedge is rebalanced, or adjusted periodically, to remain delta neutral.

We will describe a dynamic-hedging scheme that rebalances the portfolio periodically to ensure a delta-neutral portfolio.

This is in contrast to static hedging schemes where the hedge is set up and left alone. Such schemes are called hedge and forget schemes.

We will use Black-Scholes analysis to help devise a good delta hedge scheme. Recall that the Black-Scholes portfolio that is riskless is

\[ -1 : \text{option} \]
\[ +\Delta : \text{shares of stock} \]
Delta of European Stock Options

A European call option on a non-dividend-paying stock is

\[ \Delta(\text{call}) = N(d_1) \]

and a European put option on a non-dividend-paying stock is

\[ \Delta(\text{put}) = N(d_1) - 1 \]

Figure 15.3 Variation of delta with stock price for (a) a call option and (b) a put option on a non-dividend-paying stock.

Keeping a delta hedge for a long position in a European call option involves maintaining a short position of \( N(d_1) \) shares at any given time.
Note that the $\Delta$ for a European put option is negative, so that a long position in a put option should be hedged with a long position in the underlying stock, and a short position in a put option should be hedged with a short position in the underlying stock.
Delta of Other European Options

For European call options on an asset paying a yield $q$,

$$\Delta(\text{call}) = e^{-qT}N(d_1)$$

where $d_1$ is defined by

$$d_1 = \frac{\ln \frac{S_0}{K} + (r - q + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}}$$

and for European puts

$$\Delta(\text{put}) = e^{-qT}[N(d_1) - 1]$$

for the same $d_1$.

If the asset is a currency, we replace $q$ with $r_f$, the foreign risk-free interest rate. If the asset is a futures contract, they are correct with $q$ equal to the risk-free interest rate $r$ and $S_0 = F_0$ in the definition of $d_1$. 
Dynamic Delta Hedging

Consider the operation of a delta hedging for our first example.

\[ S_0 = 49 \quad K = 50 \quad r = 0.05 \quad \sigma = 0.20 \quad T = 0.3846 \quad \mu = 0.13 \]
on 100,000 shares of stocks. The European call option has been written for $300,000. The initial \( \Delta \) is calculated:

\[
d_1 = \frac{\ln \frac{S_0}{K} + \left( r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}
\]

\[
= \frac{\ln \frac{49}{50} + \left( 0.05 + \frac{0.02^2}{2} \right) 0.3846}{0.2 \sqrt{0.3846}}
\]

and

\[ N(d_1) = 0.522 \]
• Once option has been written, the investor has to buy

\[ 100,000 \times 0.522 = 52,200 \text{ shares} \]

for a cost of \( 52,200 \times 49 = 2,557,800 \). The interest rate is 5%, so after one week the interest costs

\[ 2,557,800e^{0.05 \times \frac{1}{52}} = 2,500 \]

• Suppose now that the stock drops to $48.12. The delta declines to 0.458. The hedge needs to be

\[ 100,000 \times 0.458 = 45,800 \text{ shares} \]

Therefore, the bank needs to sell 6400 shares. The strategy realizes $308,000 in cash, and the borrowings become \( 2,557,800 + 2,500 - 308,000 = 2,252,300 \). The interest over this period is

\[ 2,252,300 \times e^{0.05 \times \frac{1}{52}} = 2,200 \]

• Suppose now that the stock drops to 47.37. The delta declines to 0.400. The hedge needs to be

\[ 100,000 \times 0.400 = 40,000 \text{ shares} \]

Therefore, the bank needs to sell 5800 shares. The strategy realizes \( 5800 \times 47.37 = $274,746 \). The borrowings become

\[ 2,252,300 + 2,200 - 274,746 = 1,979,754 \]. The interest over the period is

\[ 1,979,754 \times e^{0.05 \times \frac{1}{52}} = 1,900 \]

• And on...
Table 15.2  Simulation of delta hedging. Option closes in the money and cost of hedging is $263,300.

<table>
<thead>
<tr>
<th>Week</th>
<th>Stock price</th>
<th>Delta</th>
<th>Shares purchased</th>
<th>Cost of shares purchased ($000)</th>
<th>Cumulative cost including interest ($000)</th>
<th>Interest cost ($000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>49.00</td>
<td>0.522</td>
<td>52.200</td>
<td>2,557.8</td>
<td>2,557.8</td>
<td>2.5</td>
</tr>
<tr>
<td>1</td>
<td>48.12</td>
<td>0.458</td>
<td>(6,400)</td>
<td>(308.0)</td>
<td>2,252.3</td>
<td>2.2</td>
</tr>
<tr>
<td>2</td>
<td>47.37</td>
<td>0.400</td>
<td>(5,800)</td>
<td>(274.7)</td>
<td>1,979.8</td>
<td>1.9</td>
</tr>
<tr>
<td>3</td>
<td>50.25</td>
<td>0.596</td>
<td>19,600</td>
<td>984.9</td>
<td>2,966.6</td>
<td>2.9</td>
</tr>
<tr>
<td>4</td>
<td>51.75</td>
<td>0.693</td>
<td>9,700</td>
<td>502.0</td>
<td>3,471.5</td>
<td>3.3</td>
</tr>
<tr>
<td>5</td>
<td>53.12</td>
<td>0.774</td>
<td>8,100</td>
<td>430.3</td>
<td>3,905.1</td>
<td>3.8</td>
</tr>
<tr>
<td>6</td>
<td>53.00</td>
<td>0.771</td>
<td>(300)</td>
<td>(15.9)</td>
<td>3,893.0</td>
<td>3.7</td>
</tr>
<tr>
<td>7</td>
<td>51.87</td>
<td>0.706</td>
<td>(6,500)</td>
<td>(337.2)</td>
<td>3,559.5</td>
<td>3.4</td>
</tr>
<tr>
<td>8</td>
<td>51.38</td>
<td>0.674</td>
<td>(3,200)</td>
<td>(164.4)</td>
<td>3,398.5</td>
<td>3.3</td>
</tr>
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<td>9</td>
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<td>598.9</td>
<td>4,000.7</td>
<td>3.8</td>
</tr>
<tr>
<td>10</td>
<td>49.88</td>
<td>0.550</td>
<td>(23,700)</td>
<td>(1,182.2)</td>
<td>2,822.3</td>
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</tr>
<tr>
<td>11</td>
<td>48.50</td>
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<td>(13,700)</td>
<td>(664.4)</td>
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<tr>
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<td>643.5</td>
<td>2,806.2</td>
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<td>50.37</td>
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<td>246.8</td>
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<td>(900)</td>
<td>(46.7)</td>
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<td>560.4</td>
<td>4,502.6</td>
<td>4.3</td>
</tr>
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<td>11,300</td>
<td>620.0</td>
<td>5,126.9</td>
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</tr>
<tr>
<td>18</td>
<td>54.62</td>
<td>0.990</td>
<td>1,200</td>
<td>65.5</td>
<td>5,197.3</td>
<td>5.0</td>
</tr>
<tr>
<td>19</td>
<td>55.87</td>
<td>1.000</td>
<td>1,000</td>
<td>55.9</td>
<td>5,258.2</td>
<td>5.1</td>
</tr>
<tr>
<td>20</td>
<td>57.25</td>
<td>1.000</td>
<td>0</td>
<td>0.0</td>
<td>5,263.3</td>
<td>5.1</td>
</tr>
</tbody>
</table>
In this simulation, the stock price climbs.

As it becomes evident that the option will be exercised at the maturity date, and delta approaches 1.0.

By week 20, the hedger has a fully covered position. The hedger receives $5 million for the stock held, so that the total cost of writing the option and hedging it is $263,300.
Delta of a Portfolio

The delta of a portfolio of options or other derivatives dependent on a single asset whose price is $S$ is

$$\frac{\partial \Pi}{\partial S}$$

where $\Pi$ is the value of the portfolio.

- The delta of the portfolio can be calculated from the deltas of the individual options in the portfolio. If a portfolio consists of a quantity $w_i$ of option, the delta of the portfolio is given by

$$\Delta = \sum_{i=1}^{n} w_i \Delta_i$$

where $\Delta_i$ is the delta of the $i$th option.

- The formula can be used to calculate the position in the underlying asset necessary to make the delta of the portfolio zero. When this position has been taken, the portfolio is referred to as being **delta neutral**
**Example:** Suppose a bank has the following three positions in options in Australian dollars

- A long position in 100,000 call options with strike price 0.55 and an expiration date in 3 months. The delta of each option is 0.533.
- A short position in 200,000 call options with strike price 0.56 and an expiration date in 5 months. The delta of each option is 0.468.
- A short position in 50,000 put options with strike price 0.56 and an expiration date in 2 months. The delta of each option is -0.508.

The delta of the whole portfolio is

\[100,000 \times 0.533 - 200,000 \times 0.468 - 50,000 \times (-0.508) = -14,900\]

Therefore, the entire portfolio can be made delta neutral with a long position of 14,900 Australian dollars.

Delta neutrality can be achieved also by a 6-month forward contract. Suppose the risk-free rate is 8% per annum in Australia and 5% in the US. The delta of a forward contract maturing at time \(T\) on one Australian dollar is \(e^{-r_f T} = e^{-0.08 \times 0.05} = 0.9608\).

The long position in Australian dollar forward contracts for delta neutrality is therefore

\[\frac{14,900}{0.9608} = 15,508.\]
Transaction Costs

Maintaining a delta-neutral position in a single option and underlying asset, in the way we used, is prohibitively expensive because of costs due to transactions on each trade.

For a large portfolio of options, delta neutrality is more feasible. One trade in the underlying asset is necessary to zero out delta for the whole portfolio.

The hedging transactions costs are absorbed by the profits on many different trades.
Theta

The theta $\Theta$ of a portfolio of options is the rate of change of the value of the portfolio with respect to the passage of time with all else remaining the same.

Theta is sometimes referred as the time decay of the portfolio. For a European call option on a non-dividend-paying stock, then

$$\Theta(c) = \frac{\partial c}{\partial t} = -\frac{S_0 N'(d_1) \sigma}{2\sqrt{T}} - rK e^{-rT} N(d_2)$$

where $d_1$ and $d_2$ are defined as before and

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

For a European put option on the stock

$$\Theta(p) = -\frac{S_0 N'(d_1) \sigma}{2\sqrt{T}} + rK e^{-rT} N(-d_2)$$
For a European call on a dividend paying stock with rate $q$ is

$$\Theta(\text{call}) = -\frac{S_0 N'(d_1) e^{-qT} \sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2)$$

where

$$d_1 =$$

for a European put on a dividend paying stock with rate $q$ is

$$\Theta(\text{call}) = -\frac{S_0 N'(d_1) e^{-qT} \sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2)$$

Theta is different type of hedge parameter as delta, but is useful in delta-neutral portfolios.
The **gamma**, $\Gamma$ of a portfolio of options on an underlying asset is the rate of change of the portfolio’s delta with respect to the price of the underlying asset.

It is the second partial derivative of the portfolio with respect to asset price:

$$\Gamma = \frac{\partial^2 \Pi}{\partial S^2}$$

If gamma is small, delta changes slowly, and adjustments to keep a portfolio delta neutral need to be made only relatively infrequently.

If gamma is large in absolute terms, delta is highly sensitive to the price of the underlying asset.

It is then quite risky to leave a delta-neutral portfolio unchanged for any length of time.

When the stock price moves from $S$ to $S'$, delta hedging assumees that the option price moves from $C$ to $C'$, when in fact it moves from $C$ to $C''$. The difference between $C'$ and $C''$ leads to a hedging error.

This error depends on the curvature of the relationship between the option price and the stock price. Gamma measures this curvature.
Suppose that $\Delta S$ is the price change of an underlying asset during a small interval of time $\Delta t$ and $\Delta \Pi$ is the corresponding price change in the portfolio. We will see that:

$$\Delta \Pi = \Theta \Delta t + \frac{1}{2} \Gamma (\Delta S)^2$$

for a delta-neutral portfolio, where $\Theta$ is the theta of the portfolio.
For European call options on non-dividend-paying stocks, we have

\[ \Gamma = \frac{N'(d_1)}{S \sigma \sqrt{T - t}} \]

At the initial time we have

\[ \Gamma = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}} \]
Making a Portfolio Gamma Neutral

- A position in the underlying asset itself or a forward contract on the underlying asset both have zero gamma and cannot be used to change the gamma of a portfolio.
- What is required is a position in an instrument (such as an option) that is not linearly dependent on the underlying asset.
- Suppose that a delta-neutral portfolio has a gamma equal to $\Gamma$, and a traded option has a gamma equal to $\Gamma_T$.
- If the number of traded options added to the portfolio is $w_T$, the gamma of the portfolio is $w_T \Gamma_T + \Gamma$.
- Thus the position in the traded option necessary to make the portfolio gamma neutral is $-\frac{\Gamma}{\Gamma_T}$.
- Including the traded option is likely to change the delta of the portfolio, so the position in the underlying asset cannot be changed continuously when delta hedging is used.
- Delta neutrality provides protection against larger movements in this stock price between hedge rebalancing.
Vega

So far have assumed that the volatility of the asset underlying a derivative is constant.

Volatilities change over time. In particular the value of a derivative is liable to change because of movements in volatility as well as because of changes in the asset price and the passage of time.

We set vega of a portfolio of derivatives, \( \mathcal{V} \), is the rate of change of the value of the portfolio with respect to the volatility of the underlying asset:

\[
\mathcal{V} = \frac{\partial \Pi}{\partial \sigma}
\]

- If vega is high in absolute terms, then the portfolio’s value is very sensitive to small changes in volatility.
- If vega is low in absolute terms, then volatility changes have little impact on the value of the portfolio.
Note that a position in the underlying asset has zero vega, but the vega of a portfolio can be changed by adding a position in a traded option.

- If $\mathcal{V}$ is the vega of the portfolio and $\mathcal{V}_T$ is the vega of a traded option, a position of $-\frac{\mathcal{V}}{\mathcal{V}_T}$ in the traded option makes the portfolio instantaneously vega neutral.
- However, a portfolio that is gamma neutral will not generally be vega neutral and vice-versa.
- If a hedger requires a portfolio to be both gamma and vega neutral, at least two traded derivatives dependent on the underlying asset must usually be used.

**Example:** Consider a portfolio that is delta neutral, with a gamma of -5000 and a vega of -8000. A traded option has a gamma of 0.5, a vega of 2.0, and delta of 0.6.

The portfolio can be made vega neutral by including a long position in 4000 traded options. This increases the delta to 2400 and require that 2400 units of the asset be sold to maintain delta neutrality. The gamma of the portfolio would change from -5000 to -3000.

To make the portfolio gamma and vega neutral, we suppose that there is a second traded option with gamma of 0.8, vega of 1.2, and delta of 0.5.

If $w_1$ and $w_2$ are the quantities of the two traded options included in the portfolio, we require that

\[
0 = -5000 + 0.5w_1 + 0.8w_2
\]
\[
0 = -8000 + 2.0w_1 + 1.2w_2
\]
Solving these yield $w_1 = 400$ and $w_2 = 6000$. The portfolio can be made gamma and vega neutral by including 400 of the first traded option and 6000 of the second traded option.

The delta of the portfolio after the addition of the positions in the two traded options is $400 \times 0.6 + 6000 \times 0.5 = 3240$. Thus we need to sell 3240 units of the asset to maintain delta neutrality.
Calculating vega

For a European call or put on a non-dividend-paying stock, vega is given by

\[ \mathcal{V} = S_0 \sqrt{T} N'(d_1) \]

where

\[ d_1 = \frac{\ln \frac{S}{K} + (r + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}}. \]

For a European call or put on a dividend-paying stock with yield \( q \), the vega is

\[ \mathcal{V} = S_0 \sqrt{T} N'(d_1) e^{-qT} \]

where

\[ d_1 = \frac{\ln \frac{S}{K} + (r - q + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}}. \]

When the asset is a stock index, \( q \) is the dividend yield. When it is a currency contract, then set \( q \) to be the risk-free foreign rate \( r_f \). When it is a futures contract, \( S_0 = F_0 \) and \( q = r \).
Vega

• Confusing to compute the vega of a portfolio since we derived Black-Scholes by assuming constant volatility.
• \( \nu \) calculated from stochastic volatility models with constant volatility works well.
• \( \nu \) neutrality protects from large changes in the volatility \( \sigma \), whereas \( \Gamma \) neutrality protects against large swings in the price of the underlying asset between hedge rebalancing.
Rho

The rho of a portfolio of options is the rate of change of the value of the portfolio with respect to the interest rate

\[ \rho = \frac{\partial \Pi}{\partial r} \]

which measures the sensitivity of the value of the portfolio to interest rate changes.

For a European call option on non-dividend-paying stock:

\[ \rho = K T e^{-r T} N(d_2) \]

where \( d_2 \) is defined as before. For a European put option

\[ \rho = -K T e^{-r T} N(-d_2) \]

The formulas also hold for European calls and puts for dividend paying stocks with \( d_2 \) adjusted.
In general a portfolio manager wishes to acquire a put option to protect against large declines while achieving gains if the market appreciates.

One approach is to buy put options on a market index. Another approach is to create the put synthetically.

To create a synthetic put option, one maintains a position in the underlying asset so that the delta of the position is equal to the delta of the required option.

This can be more attractive than buying the put from the market:

- Options markets do not always have the liquidity to absorb trades that managers of large funds would like to have access to.
- Fund managers often require strike prices and exercise dates that are different from those available from the exchange-traded markets.
How to synthetically create the put?

The option can be created by trading the portfolio or by trading in index futures contracts.

• Consider the first approach - creating a put option by trading the portfolio. Recall the delta of a European put on the portfolio is

\[
\Delta = e^{-qT} \left[ N(d_1) - 1 \right]
\]  

(7)

where

\[
d_1 = \frac{\ln \frac{S_0}{K} + (r - q + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}}
\]

with \(S_0\) the value of the portfolio, \(K\) the strike price, \(r\) the risk-free rate, \(q\) the dividend yield on the portfolio, \(\sigma\) the volatility of the portfolio, and \(T\) the life of the option.

Assume that the volatility of the portfolio is \textbf{beta} times the volatility of a well-diversified market index.

Therefore, to create the put option synthetically, the fund manager should ensure that at any give time a proportion

\[
e^{-qT} \left[ N(d_1) - 1 \right]
\]

of the stocks in the original portfolio has been sold and the proceeds invested in riskless assets.
As the value of the original portfolio declines, the delta of the put given by (7) becomes more negative and the proportion of the original portfolio sold must be increased.

As the value of the original portfolio increases, the delta of the put becomes less negative and the proportion of the original portfolio sold must be decreased (and shares purchased)

- This strategy to create portfolio insurance entails dividing funds between the stock portfolio on which the insurance is required and riskless assets.
- As the value of the stock portfolio increases, riskless assets are sold and the position in the stock portfolio is increased.
- As the value of the stock portfolio decreases, the position in the stock portfolio is decreased and riskless assets are purchased.
- Insurance costs arise as the fact that selling occurs after a decline in the market and buying occurs after a rise in the market.

Note that the portfolio insurance created by synthetic means (selling portfolio at downswings and purchasing on upswings), can accentuate trends:

If a market declines, portfolio managers sell stock or sell index futures contracts. Both actions accentuate the decline. Sale of stock is liable to drive down market indices further.

Whether portfolio insurance schemes affect volatility depends on how easily a market can absorb the trades that are generated by portfolio insurance.
Summary

- The Delta $\Delta$ of an option is the rate of change of its price with respect to the price of the underlying asset. Delta hedging involves creating a position with zero delta or a delta-neutral position.
- Because the delta of the underlying asset is 1.0, one way is to take a position of $-\Delta$ in the underlying asset for each long option being hedged. Position needs to be adjusted frequently.
- Once an option position has been made, one can examine the $\Gamma$ of the portfolio - the rate of change of its delta with respect to the price of the underlying asset.
- Gamma measures the curvature of the relationship between the option price and the asset price. The impact of this curvature on the performance of the hedge can be reduced by making the portfolio gamma neutral.
- If $\Gamma$ is the gamma of the position being hedged, this reduction is usually achieved by a position in a traded option with a total gamma of $-\Gamma$.
- Both $\Delta$ and $\Gamma$ are calculated assuming that volatility is constant. However, in practice volatility does change.
- The $\mathcal{V}$ of a portfolio measures the rate of change of its value with respect to volatility.
• Trader can make hedge position vega neutral by taking an offsetting position. To make the portfolio both Gamma and Vega neutral, the trader needs two different derivatives on the same asset.

• Other variables that measure the risk in a portfolio are the rho and theta which measure sensitivity of the portfolio to rate changes and time changes, respectively.

• In practice traders usually maintain delta neutrality daily and monitor Gamma and Vega for large swings.

• Traders can sometimes build put options synthetically for large equity portfolios. They do so by either trading the portfolio or by trading index futures on the portfolio. Trading the portfolio entails splitting the portfolio between equities and risk-free securities. As the market declines more is invested in risk-free securities. As the market increases, more is invested in equities.
Implied Volatility

There are two ways to think about volatility:

- From price changes, we can compute the volatility via standard deviation.
- Another method is to consider data used in Black-Scholes:

\[
c = S_0 N(d_1) - Ke^{-rT}N(d_2)
\]

with \[
d_1 = \ln \frac{S_0}{K} + \left( r + \frac{\sigma^2}{2} \right) T
\]
\[
d_2 = \ln \frac{S_0}{K} + \left( r - \frac{\sigma^2}{2} \right) T
\]
\[
\sigma \sqrt{T} = d_1 - \sigma \sqrt{T}
\]

If we know \( S_0, \, K, \, r, \) and \( c \), then we can solve, implicitly, for \( \sigma \). The resulting \( \sigma \) is called the **implied volatility**.
Put-Call Parity and Volatility

Recall the put-call parity

\[ c + Ke^{-rT} = p + S_0e^{-qT} \]

which is based on a simple arbitrage argument.

- It does not require any information on probabilities of underlying stock movements or volatility!
- True if the stock process is lognormal or not.

Therefore, implied volatility of a European call is always the same as the implied volatility of a European put.

As a consequence the correct volatility to use for the European call should be the same as for the European put. Approximately true for the American options too...
We now consider our first **volatility smile**. This is a graph of volatility as a function of strike price. We assumed for Black-Scholes that this is a constant function...

On the other hand traders use the following volatility smile

**Figure 16.1** Volatility smile for foreign currency options.

- Volatility is relatively low for at-the-money options.
- Volatility is relatively high for the more in-the-money or out-of-the-money the strike price is.
Volatility Smile for Foreign Currency Options

The associated probability distribution should no longer be lognormal, since the crucial ingredient to $S$ being lognormal was

$$\frac{dS}{S} = \mu dt + \sigma \epsilon dz$$

The implied distribution turns out to be

The distribution with the same mean and same standard deviation

- has fat tails
- is steeper
Why is there a smile in the foreign currency option? We need two conditions to hold for the lognormal distribution to hold:

- Volatility of the asset is constant
- Price of the asset changes smoothly with no jumps

Neither of these two assumptions hold for an exchange rate.

Volatility of an exchange rate is not constant and exchange rates frequently jump. Both of these tend to increase the likelihood of extreme events.

The percentage impact of a nonconstant volatility on prices becomes more pronounced as the maturity of the option is increased, but the volatility smile created by nonconstant volatility usually becomes less pronounced as the maturity of the option increases.
Before the crash of 1987, stocks were generally assumed to follow the lognormal distribution.

After the crash, a volatility smile for equity options was introduced by Rubinstein and Jackwerth-Rubinstein.

**Figure 16.3** Volatility smile for equities.

The volatility smile or **volatility skew**, has the form of a downward sloping parabola.
• Volatility to price a **low-strike-price** option (deep-out-of-the-money put or deep-in-the-money call) is significantly higher than that used to price a **high-strike-price** option (deep-in-the-money put or deep-out-of-the-money call).

• The volatility smile for equity options corresponds to the implied probability distribution given by below:

\[
\begin{align*}
&K_1 & K_2 \\
&\text{Implied} & \text{Lognormal}
\end{align*}
\]

compared to the corresponding lognormal distribution.
Why the smile?

Reasons for the equity volatility smile

- **Fear of a crash.** Traders are concerned about the possibility of a crash, so they price the option accordingly.
- **Leverage.** As a company’s equity declines in value, the equity becomes more risky and its volatility increases. As a company’s equity increases in value, the equity becomes less risky and its volatility decreases.
Traders also consider the volatility term structure when pricing options.

In other words the volatility used to price an at-the-money option depends on the maturity of the option.

- Volatility tends to be an increasing function of maturity when short-dated volatilities are historically low, since there is expectation that volatility will increase.
- Volatility tends to be a decreasing function of maturity when short-dated volatilities are historically high, since there is expectation that volatility will decrease.

**Volatility surfaces** combine volatility smiles with the volatility term structure to tabulate the volatilities appropriate for pricing an option with any strike price and any maturity.

Some of the implied probability entries are computed using market data and Black-Scholes.

The rest are found via simple linear interpolation.

When a new option has to be valued, traders look up the appropriate volatility in the table.
• The shape of the volatility smile depends on the option maturity. The smile tends to become less pronounced as the option maturity increases.

• If $T$ is the time to maturity and $F_0$ is the forward price of the asset. Then sometimes traders define the volatility smile as the relationship between implied probability and

\[
\frac{1}{\sqrt{T}} \ln \frac{K}{F_0}
\]

rather than the relationship between implied volatility and $K$. The smile is much less dependent on the time to maturity.
Summary

- Black-Scholes model and its extensions assume that the probability distribution of the underlying asset at any given future time is lognormal.
- This assumption is not made by traders.
- Traders assume a probability distribution of an equity price has a heavier left tail and a less heavy right tail than the lognormal distribution.
- Traders also assume that the probability distribution of an exchange rate has a heavier right tail and heavier left tail than the lognormal distribution.
- Traders use volatility smiles to allow for nonlognormality.
- The volatility smile defines the relationship between the implied volatility of an option and its strike price.
- For equity options, the volatility smile tends to be downward sloping.
  - Out-of-the-money puts and in-the-money calls tend to have high implied volatility.
  - Out-of-the-money calls and in-the-money puts tend to have low implied volatility.
- Foreign currency options, the volatility smile is U-shaped. Both out-of-money and in-the-money options have higher implied volatilities than at-the-money options.
- Often traders use volatility term structure. The implied volatility of an option depends on the duration of the option.
- When volatility smiles and volatility term structures are used together, we get a volatility surface.
- This defines volatility as a function of both strike price and time to maturity.