Lecture 5. Interest Rates & Bonds
A stock index tracks changes in the value of a hypothetical portfolio of stocks.

- Weight of a stock in the portfolio equals the proportion of the portfolio invested in the stock.

- Percentage increase in the stock index over a small period of time is set by the percentage increase in the value of the hypothetical portfolio.

- If one stock rises (drops) sharply, then the stock is given more (less) weight.

- Underlying portfolio is adjusted automatically.
Hedging an Equity Portfolio

Stock index futures are useful for hedging a well-diversified stock portfolio. Let

\[ P \equiv \text{Current value of the portfolio} \]
\[ A \equiv \text{Current value of the stocks underlying one futures contract} \]

If the portfolio exactly mirrors the index, the optimal hedge ratio \( h^* = 1 \). Then the number of futures contracts should be

\[ N^* = \frac{P}{A} \]

Example: Suppose a portfolio worth $1 million mirrors the S&P 500. The current value of the index is 1,000, and each futures contract is worth $250 times the index. Then

\[ P = 1,000,000 \quad A = 250,000. \]
Beta

If the portfolio does not exactly mirror the index, then we use $\beta$ to determine the appropriate hedge ratio. In general

$$\beta = h^* \text{ therefore } N^* = \frac{\beta P}{A}$$

assuming that the maturity of the futures contract is close to the maturity of the hedge (ignoring daily settlement)
Example of $\beta$ usage

Suppose

Value of S & P 500 index $\equiv 1000$

Value of Portfolio $\equiv \$5,000,000$

Dividend yield on index $\equiv 1\%$ per annum

$\beta$ of portfolio $\equiv 1.5$

- 1% dividend is paid by the index for investing in the index

- Assume the futures contract on the S&P 500 with 4 months to maturity is used to hedge portfolio over next 3 months

- Assume current futures price of the contract is 1,010

- One contract for delivery is $250$ times the index.
Example of $\beta$ usage, cont.

- We set $A = 250 \times 1000 = 250,000$ (current value of stocks underlying futures contract)

- By $N^* = \beta \frac{P}{A}$ we have $N^* = 1.5 \times \frac{5,000,000}{250,000} = 30$ contracts.

Consider scenarios: S & P index drops to 900 in 3 months and the futures price is 902, then

- Gain from short futures position is

  $30 \times (1010 - 902) \times 250 = 810,000$

- Loss on the index is 10 %.

- Dividend pays 1% per year or 0.25% in the quarter.

- Index loss plus Dividend yields a loss of $-9.75\%$ in the 3-month period.
Example of $\beta$ usage, cont.

- Since the $\beta = 1.5$ then pricing yields

  expect. returns on portfolio $= 1.5 \times \text{return on index}$

- Expect return on portfolio is

  $$1.5 \times (-9.75) = -14.625$$

  The expected value of the portfolio (including dividends) is

  $$5,000,000 \times (1 - 0.14625) = 4,268,750$$

  the end gain on hedge is

  $$4,268,750 + 810,000 = 5,078,750$$
Rolling the hedge forward

• If the hedge expiration happens after the delivery date of the futures contract, need to roll the hedge

• Accomplished by closing out the futures contract and taking the same futures contract with a later delivery date

Consider a sequence of futures contracts listed 1, 2, 3, \ldots, n the company uses:

\begin{align*}
  t_1 & \quad \text{Short futures contract 1} \\
  t_2 & \quad \text{Close contract 1 & Short futures contract 2} \\
  t_3 & \quad \text{Close contract 2 & Short futures contract 3} \\
  \vdots & \quad \vdots \\
  t_n & \quad \text{Close contract n-1 & Short futures contract n}
\end{align*}
Rolling the hedge, example

Consider a company in April 2004 that it will need to sell 100,000 barrels of oil to sell in June 2005. It decides to hedge with a short position with hedge ratio of 1.0.

- Current spot price is $19

- Company decides to roll its hedge position forward at 6-month intervals.

<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>Oct. 2004 Futures price</td>
<td>18.20</td>
<td>17.40</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mar. 2005 Futures price</td>
<td></td>
<td>17.00</td>
<td>16.50</td>
<td></td>
</tr>
<tr>
<td>July 2005 Futures price</td>
<td></td>
<td></td>
<td>16.30</td>
<td>15.90</td>
</tr>
<tr>
<td>Spot price</td>
<td>19.00</td>
<td></td>
<td></td>
<td>16.00</td>
</tr>
</tbody>
</table>

- Company shorts 100 October 2004 contracts...rolls the contracts in Sept. 2004 into March 2005 contract...rolls the contract in Feb. 2005 into July 2005 contracts
Rolling the hedge, example

Continuing the example:

<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Oct. 2004 Futures price</td>
<td>18.20</td>
<td>17.40</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mar. 2005 Futures price</td>
<td>17.00</td>
<td>16.50</td>
<td></td>
<td></td>
</tr>
<tr>
<td>July 2005 Futures price</td>
<td></td>
<td>16.30</td>
<td>15.90</td>
<td></td>
</tr>
<tr>
<td>Spot price</td>
<td>19.00</td>
<td></td>
<td></td>
<td>16.00</td>
</tr>
</tbody>
</table>

- In September 2004, the October contracts have a $18.20 - 17.40 = 0.80$ per barrel profit.
- In February 2005, the March contracts have a $17.00 - 16.50 = 0.50$ per barrel profit.
- In June 2005, the July contracts have a $16.30 - 15.90 = 0.40$ per barrel profit.

Therefore, the profit from the rolling hedge strategy yields a

$$ (18.20 - 17.40) + (17.00 - 16.50) + (16.30 - 15.90) = 1.70 $$

per barrel profit. On the other hand the price of oil has decreased to $16.00. Therefore, the company earns

$$ 100,000 \times 1.70 + 100,000 \times 16.00 = $1.7 \text{ million} $$
Interest Rates

An interest rate defines the amount of money a borrower promises to pay a lender. Types of rates:

- **Risk-free**: rates investors earn on Treasury bills and Treasury bonds. Example: US Treasury rates. Assumed that the government will not default on the bill. *Traders may argue that not risk-free (read Snapshot 4.1, page 77)*

- **LIBOR - London Interbank Offer Rate**: by a bank is the rate at which the bank is prepared to make a large deposit with other banks. Usually 1-month, 3-month, 6-month, and 12-month LIBOR in major currencies.

- Banks require good (AA) credit ratings to accept a LIBOR quote.

- LIBOR rates are **not** free of risk, but considered close to risk-free.

- **LIBID - London Interbank Bid Rate**: rate at which a bank will accept deposits from other banks. Slightly lower than LIBOR
Interest rates, cont.

- **Repo agreement**: or repurchase agreement contract where an investment dealer who owns securities agree to see then to another company now and buy them back later at a slightly higher price.

- Interest rate earned is the called the **repo rate**

- **Overnight repo**: most common repo agreement. Negotiated each day.
Investor deposits $1000 in a bank with annual interest rate of 10% then after one year:

\[ 1000 \times (1 + 0.1) = 1100 \]

If instead the interest rate is 10% compounded twice a year: then every six months earn 5%

\[ 1000 \times (1 + 0.05) \times (1 + 0.05) = 1102.50 \]

If the interest rate is 10% compounded \( n \) times a year, then every \( \frac{365}{n} \) days, the investor earns \( \frac{10}{n} \)%:

\[ 1000 \times (1 + \frac{10}{n})^{n} \]

For example if \( n = 365 \) then

\[ 1000 \times (1 + \frac{10}{365})^{365} \]
Interest rates

Terminal value for an initial deposit of $A$ at a rate of $R$ over $n$ years is

$$A(1 + R)^n$$

Terminal value for an initial deposit of $A$ at a rate of $R$ over $n$ years, compounded $m$ times per year is

$$A \left(1 + \frac{R}{m}\right)^{mn}$$

Note $\lim_{m \to \infty} \left(1 + \frac{R}{m}\right)^m = e^R$ with $e \approx 2.71828$. An investment compounded continuously for $n$ years at a rate of $R$ is

$$Ae^{Rn}$$

Continuous compounding is very close to daily ($m = 365$) compounding.
Continuous Compounding

- **Compounding** a sum of money at a continuously compounded rate $R$ for $n$ years involves multiplying by $e^{Rn}$.

- **Discounting** a sum of money at a continuously compounded rate $R$ for $n$ years involves multiplying by $e^{-Rn}$.

- Derivatives use continuously compounded interest rates more frequently than other types.

Converting continuously compounded rate $R_C$ to $m$-times compounding per year rate $R_m$:

$$Ae^{R_C n} = A \left( 1 + \frac{R_m}{m} \right)^{mn}$$

$$R_C = m \ln \left( 1 + \frac{R_m}{m} \right)$$

$$R_m = m \left( e^{\frac{R_C}{m}} - 1 \right)$$
Zero Rates

- An $n$-year zero-coupon interest rate is the rate of interest on an investment that starts today and last $n$ years.

- All interest and principle is realized at the end of the $n$th year (no intermediate payments).

- Also called $n$-spot rate or $n$-year zero rate.
Bond Pricing

- Bonds usually provide **coupons** or interest payments, periodically.

- Principal (called **face or par value**) is payed at the end of the life.

- Price of a bond can be calculated as the present value of all cash that will be received by the owner of the bond.
Consider the following Treasury zero rates:

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Zero rate(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>5.0</td>
</tr>
<tr>
<td>1.0</td>
<td>5.8</td>
</tr>
<tr>
<td>1.5</td>
<td>6.4</td>
</tr>
<tr>
<td>2.0</td>
<td>6.8</td>
</tr>
</tbody>
</table>

- Consider a 2-year Treasury bond with a principal of $100 provides coupons at a rate of 6% semiannually.

- Present value is determined from the rates above, discounted:

\[
3e^{-0.05 \times 0.5} + 3e^{-0.058 \times 1.0} + 3e^{-0.064 \times 1.5} + 103e^{-0.068 \times 2.0} = 98.39
\]

- 98.39 is the theoretical value of the bond at present.
The yield on a bond is the discount rate that, when applied to all cash flows, gives a bond price equal to the market price.

Consider the previous case with a theoretical value of 98.39. The yield, $y$, is calculated via

$$3e^{-y 	imes 0.5} + 3e^{-y 	imes 1.0} + 3e^{-y 	imes 1.5} + 103e^{-y 	imes 2.0} = 98.39.$$ 

This is a fourth order polynomial in $e^{-y/2}$. Easy to calculate with a computer:

$$y = 6.76\%$$
Par Yield

- The **par yield** for a certain maturity is the **coupon rate** that causes the bond prices to equal its par value.

- Consider our 2 year bond with \( c \)-annual coupon rate (so \( \frac{c}{2} \) semi-annually). Then

\[
\frac{c}{2} e^{-0.05 \times 0.5} + \frac{c}{2} e^{-0.058 \times 1.0} + \frac{c}{2} e^{-0.064 \times 1.5} + \left( 100 + \frac{c}{2} \right) e^{-0.068 \times 2.0} = 100
\]

- This is a linear equation. \( c = 6.87\% \). Thus the 2-year par yield is 6.87% per year with semiannual compounding.

- In general let \( d \) be the present value of $1 received at the maturity of the bond, \( A \) be the value of an annuity that pays one dollar on each coupon payment date, \( m \) be the number of coupon payments per year. Then the par yield \( c \) satisfies

\[
100 = A \frac{c}{m} + 100d \quad \text{so} \quad c = \frac{(100 - 100d) m}{A}
\]
Determining Treasury Zero Rates

To calculate the Treasury Zero Rates, bootstrap using data. Calculation via an example:

<table>
<thead>
<tr>
<th>Bond Principle</th>
<th>Time to Maturity</th>
<th>Annual Coupon</th>
<th>Bond Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.25</td>
<td>0</td>
<td>97.5</td>
</tr>
<tr>
<td>100</td>
<td>0.50</td>
<td>0</td>
<td>94.9</td>
</tr>
<tr>
<td>100</td>
<td>1.00</td>
<td>0</td>
<td>90.0</td>
</tr>
<tr>
<td>100</td>
<td>1.50</td>
<td>8</td>
<td>96.0</td>
</tr>
<tr>
<td>100</td>
<td>2.00</td>
<td>12</td>
<td>101.6</td>
</tr>
</tbody>
</table>

With half coupons paid semiannually. Use \( R_C = m \ln \left( 1 + \frac{R_m}{m} \right) \) to determine \( R_C \).

• 3-month bond: return of 2.5% on an investment of $97.5. Then

\[
\frac{4 \times 2.5}{97.5} = 0.10256
\]

per annum. Then

\[
R_C = 4 \ln \left( 1 + \frac{0.10256}{4} \right) = 0.10127 \text{ per annum}
\]
Determining Treasury Zero Rates, cont.

- 6-month bond: return of 5.1% on an investment of $94.9. Then

\[
\frac{2 \times 5.1}{94.9} = 0.10748
\]

per annum. Then

\[
R_C = 2 \ln \left( 1 + \frac{0.10748}{2} \right) = 0.10469 \text{ per annum}
\]

- 1-year bond: return of 10% on an investment of $90. Then

\[
\frac{1 \times 10}{90} = 0.11111
\]

per annum. Then

\[
R_C = \ln (1 + 0.11111) = 0.10536 \text{ per annum}
\]
• 18-month bond: three coupon payments of $4 per 6 months.
  – At 6 months, $4 coupon. Discount rate from above is 10.469%.
  – At 1 year, $4 coupon. Discount rate from above is 10.536%.
  – For the payment of $104 at 1.5 years, denote the discount rate $R$. The present value is $96, so:

  \[ 4e^{-0.10469 \times 0.5} + 4e^{-0.10536 \times 1.0} + 104e^{-R \times 1.5} = 96 \]

  or

  \[ e^{-R \times 1.5} = 0.85196 \quad R = -\frac{\ln 0.85196}{1.5} = 0.10681 \]

• 2-year bond: four coupon payments of $6 per 6 months. Follow the same procedure:

  \[ 6e^{-0.10469 \times 0.5} + 6e^{-0.10536 \times 1.0} + 6e^{-0.10681 \times 1.0} + 106e^{-R \times 2.0} = 101.6 \]

  or $R = 0.10808$ or 10.808%.
Determining Treasury Zero Rates, cont.

Bond data.

<table>
<thead>
<tr>
<th>Bond Principle</th>
<th>Time to Maturity</th>
<th>Annual Coupon</th>
<th>Bond Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.25</td>
<td>0</td>
<td>97.5</td>
</tr>
<tr>
<td>100</td>
<td>0.50</td>
<td>0</td>
<td>94.9</td>
</tr>
<tr>
<td>100</td>
<td>1.00</td>
<td>0</td>
<td>90.0</td>
</tr>
<tr>
<td>100</td>
<td>1.50</td>
<td>8</td>
<td>96.0</td>
</tr>
<tr>
<td>100</td>
<td>2.00</td>
<td>12</td>
<td>101.6</td>
</tr>
</tbody>
</table>

Resulting Zero Coupon Rates

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Zero Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>10.127</td>
</tr>
<tr>
<td>0.50</td>
<td>10.469</td>
</tr>
<tr>
<td>1.00</td>
<td>10.536</td>
</tr>
<tr>
<td>1.50</td>
<td>10.681</td>
</tr>
<tr>
<td>2.00</td>
<td>10.808</td>
</tr>
</tbody>
</table>
Remarks: why not linear response?

- **Liquidity Preference Theory**: forward rates should be higher than expected future rates. Why?
- Investors prefer to preserve liquidity, so want short-term bonds.
- Borrowers prefer to borrow at fixed rates for long periods of time.
- In order to match depositors with borrowers and avoid interest rate problems, financial intermediaries raise long-term interest rates relative to expected future short-term interest rates.

Other explanations? **Expectation Theory** or **Market Segmentation Theory**.
The **Forward interest rates** are the interest rates implied by current zero rates for periods of times in the future. Rates are assume to compound continuously. Example:

- Suppose an investment returns 3% per year for a one year investment. Then $100 \mapsto 100e^{0.03 \times 1} = \$103.05$ after one year.

- Suppose the investment returns 4% per year for a two year investment. Then $100 \mapsto 100e^{0.04 \times 2} = \$108.33$ after two years.

- The forward rate is implied by the zero rates for the period of time between the end of the first year and end of the second year

  \[ 100e^{0.04 \times 2} = 108.33 = 100e^{0.03 \times 1} e^{R \times 1} \]

so \( R = 0.05 \). This is the averaged rate.
Forward Rates, cont.

• If the 3 year rate is 4.6%, then the 3-year forward rate comes from

\[ 100e^{0.046 \times 3} = 100e^{0.03 \times 1}e^{0.05 \times 1}e^{R \times 1} \]

so \( R = 0.138 - 0.08 = 0.058. \)

• If \( R_1 \) and \( R_2 \) are the zero rates for maturities \( T_1 \) and \( T_2 \) then the forward interest rate \( R_F \) yields

\[ R_F = \frac{R_2T_2 - R_1T_1}{T_2 - T_1} = R_2 + (R_2 - R_1) \frac{T_1}{T_2 - T_1} \]

• As \( T_2 \to T_1 \), then

\[ R_T \approx R + T \frac{dR}{dT} \]

is the instantaneous forward rate for a maturity of \( T. \)
A forward rate agreement (FRA) is an OTC agreement that a certain interest rate will apply to either borrowing or lending a certain principal during a specified period of time. Borrowing or lending usually done at LIBOR.

\[ R_K \equiv \text{Rate of interest agreed in the FRA} \]

\[ R_F \equiv \text{Forward LIBOR rate for period } T_1 \text{ to } T_2 \text{ today} \]

\[ R_M \equiv \text{Actual LIBOR rate for period } T_1 \text{ to } T_2 \text{ observed} \]

\[ L \equiv \text{Principal underlying contract} \]

Consider an example with compounding frequency reflecting their maturity. Company A would earn \( R_M \) from a LIBOR loan. The FRA means it will earn \( R_K \). Extra interest is

\[ R_K - R_M \]

Interest is set at time \( T_1 \) and paid at time \( T_2 \). Extra interest rate leads to a cash flow of

\[ L(R_K - R_M)(T_2 - T_1) \]

On the other hand there is a cash flow from Company B to Company A at \( T_2 \) of

\[ L(R_M - R_K)(T_2 - T_1) \]
FRA’s are usually settled at time $T_1$ rather than $T_2$, so they are discounted from time $T_2$ to $T_1$. For Company X, the payoff is

$$\frac{L(R_K - R_M)(T_2 - T_1)}{1 + R_M(T_2 - T_1)}$$

and the payoff at $T_1$ for Company Y is

$$\frac{L(R_M - R_K)(T_2 - T_1)}{1 + R_M(T_2 - T_1)}$$

More when we discuss swaps.
Duration

The **duration** of a bond is a measure of how long the holder of the bond has to wait to receive cash payments (averaged).

- An $n$-year zero coupon bond has a duration of $n$ years (since payment at the end).
- A coupon-bearing bond has a duration $< n$.

Suppose a bond gives cash flows $c_i$ at time $t_i$ for times $t_1, \ldots, t_n$. The price $B$ and yield $y$, continuously compounded, satisfies

$$B = \sum_{i=1}^{n} c_i e^{-yt_i}$$

The duration $D$ is defined as

$$D = \frac{1}{B} \sum_{i=1}^{n} t_i c_i e^{-yt_i} = \sum_{i=1}^{n} t_i \left[ \frac{c_i e^{-yt_i}}{B} \right]$$
Duration

Note this is a filter: sum of weights is one. Consider a change in the yield $\Delta y$ then the bond value changes via:

$$\Delta B = \sum_{i=1}^{n} c_i e^{-y_{2ti}} - \sum_{i=1}^{n} c_i e^{-y_{1ti}}$$

$$= \sum_{i=1}^{n} c_i \left[ e^{-y_{2ti}} - e^{-y_{1ti}} \right]$$

$$\approx \sum_{i=1}^{n} c_i (-t_i \Delta y) e^{-y_{ti}}$$

$$= -(\Delta y) BD$$

This gives us the duration relationship with yield changes

$$\frac{\Delta B}{B} = -D \Delta y$$
Consider a three year 10% coupon bond with face value $100.

Suppose the yield is 12% per annum with continuous compounding.

To compute present value of the bonds, discount (i.e. $5e^{-0.12 \times 0.5} = 4.709$, etc.)

To get weights, compute $\frac{4.709}{94.213} = 0.050$, etc.

<table>
<thead>
<tr>
<th>Time</th>
<th>Cash flow</th>
<th>Present value</th>
<th>Weight</th>
<th>Time $\times$ Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>5</td>
<td>4.709</td>
<td>0.050</td>
<td>0.025</td>
</tr>
<tr>
<td>1.0</td>
<td>5</td>
<td>4.435</td>
<td>0.047</td>
<td>0.047</td>
</tr>
<tr>
<td>1.5</td>
<td>5</td>
<td>4.176</td>
<td>0.044</td>
<td>0.066</td>
</tr>
<tr>
<td>2.0</td>
<td>5</td>
<td>3.933</td>
<td>0.042</td>
<td>0.083</td>
</tr>
<tr>
<td>2.5</td>
<td>5</td>
<td>3.703</td>
<td>0.039</td>
<td>0.098</td>
</tr>
<tr>
<td>3.0</td>
<td>105</td>
<td>73.256</td>
<td>0.778</td>
<td>2.333</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>94.213</td>
<td>1.000</td>
<td>2.653</td>
</tr>
</tbody>
</table>

We see, summing up the weights $\times$ times, a duration of 2.653.
Consider a small change in the interest rates, measured in **basis points**. Each basis point is 0.01% per annum. Suppose in our example the yield on the bond increases by 20 basis points, then the change in the bond value is

\[ \Delta B = -BD\Delta y = -(94.213) \times (2.653) \times (0.002) = -0.500 \]

Therefore, the bond price **drops** to

\[ 94.213 - 0.500 = 93.713 \]

Check: if the yield rate is 12.2% then the discount rate is

\[
5e^{-0.122\times0.5} + 5e^{-0.122\times1.0} + 5e^{-0.122\times1.5} \\
+ 5e^{-0.122\times2.0} + 5e^{-0.122\times2.5} + 105e^{-0.122\times3.0} \\
= 93.713
\]

**Note:** **Accurate for** \( \Delta y \ll 1 \)
Modified duration

When $y$ is not continuously compounded, then $B = \sum c_i (1 + y)^{-t_i}$ and we can define a new type of duration: $D = \frac{1}{B} \sum c_i t_i (1 + y)^{-t_i}$. Note

$$\Delta B = \sum c_i (1 + y_2)^{-t_i} - (1 + y_1)^{-t_i}$$

$$\approx \sum c_i (\Delta y)(-t_i)(1 + y)^{-(t_i+1)}$$

$$= -BD \frac{\Delta y}{1 + y}$$

This gives the approximate

$$\Delta B = -\frac{BD\Delta y}{1 + y}$$
For \( y \) with \( m \)-times compounding per year: we have

\[
\Delta B = -\frac{BD\Delta y}{1 + \frac{y}{m}}
\]

This defines a modified duration

\[
D^* = \frac{D}{1 + \frac{y}{m}}
\]

Then

\[
\Delta B = -BD^* \Delta y
\]

Again

**Accurate for** \( \Delta y \ll 1 \)**
Duration, discussion

What if there is a large change in the yield interest rate, $\Delta y$? Then duration formula not as accurate:

$$\Delta B = -BD\Delta y$$

- Expand to the next order via Taylor’s theorem.

$$B(y + h) \approx B(y) + h\frac{dB(y)}{dy} + \frac{h^2}{2} \frac{d^2B(y)}{dy^2}.$$  

- Here $B(y) = \sum c_i e^{-yt_i}$

$$\Delta B \approx -(\Delta y) \sum c_i t_i e^{-yt_i} + \frac{(\Delta y)^2}{2} \sum c_i t_i^2 e^{-yt_i}$$

$$= -BD\Delta y + \frac{BC}{2} (\Delta y)^2$$

where

$$C = \frac{1}{2} \frac{dB^2}{dy^2}$$

defines the convexity.
Convexity

So a better measure is

\[
\frac{\Delta B}{B} = -D\Delta y + \frac{C}{2} (\Delta y)^2
\]

where

\[
C = \frac{1 d B^2}{2 dy^2}
\]

Convexity tends to be greatest when the portfolio provides payments evenly over a long period of time. Least when the payments are concentrated around one point in time.
Bond Portfolios

• The duration of a bond portfolio can be defined as a weighted average of the durations of the individual bonds in the portfolio.

• Then $\Delta B = -BD\Delta y$ also holds where $B$ is the total value of the portfolio.

• This assumes all bonds change by the same amount when $\Delta y$ changes.
Homework

Due Sept. 26, 5PM.

- 3.2, 3.8, 3.16, 4.11

- Graded: 3.23, 3.25, 4.24

Midterm Date, Wednesday, Nov. 7th