Lecture 11

Credit Risk

Options, Futures, Derivatives / March 3, 2008
Credit Risk

Credit Risk arises from the probability that borrowers and counterparts in derivatives transactions may default.

We attempt to quantify the risk associated to credit risk.

Credit Ratings

- Moody’s and S&P provide ratings that describe the creditworthiness of corporate bonds.
- Bonds with Aaa rating are considered to have little to no chance of default.
- Moody’s subdivides categories such as Aa to Aa1, Aa2, Aa3, etc.
- S&P subdivides categories such as AA to AA+, AA, AA−, etc.
- Only Aaa or AAA are not subdivided.
Historical Default Probabilities

We can consider the historical default rates of the certain class of corporate bonds:

For example:

- A bond with an initial credit rating of $A$ has a 0.23% chance of defaulting by the end of the third year.
- A bond with an initial credit rating of $Caa$ has a 69.83% chance of defaulting by the end of the seventh year.

<table>
<thead>
<tr>
<th>Table 20.1</th>
<th>Average cumulative default rates (%), 1970–2003. (Source: Moody’s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Term$ (years)</td>
<td>1</td>
</tr>
<tr>
<td>Aaa</td>
<td>0.00</td>
</tr>
<tr>
<td>Aa</td>
<td>0.02</td>
</tr>
<tr>
<td>A</td>
<td>0.02</td>
</tr>
<tr>
<td>Baa</td>
<td>0.20</td>
</tr>
<tr>
<td>Ba</td>
<td>1.26</td>
</tr>
<tr>
<td>B</td>
<td>6.21</td>
</tr>
<tr>
<td>Caa</td>
<td>23.65</td>
</tr>
</tbody>
</table>
We can compute the probability of default for a particular year from the table.

- The probability of a \( Ba \) bond defaulting in the third year is
  
  \[
  6.00 - 3.48 = 2.52\%
  \]

- The probability of a \( Caa \) bond defaulting between the fifth and seventh year is
  
  \[
  69.36 - 60.83 = 8.53\%
  \]

The default probabilities are an increasing function of time, since over time a company with strong credit may deteriorate. Companies with poor credit may default in a short period of time.

We define two quantities:

- **Unconditional default probability**: This is the probability of default during a particular year as seen from the initial year.
  
  **Example**: The unconditional default probability of a \( Caa \) bond defaulting in the third year is
  
  \[
  48.02 - 37.20 = 10.82\%
  \]
• **Default intensities** or **Hazard rates**: This is the probability of default during a particular year as seen from the initial year conditioned on no default occurring earlier.

  **Example**: The probability that a *Caa* bond will last past year two is

  
  \[100 - 37.20 = 62.80\%\]

  Therefore, the default intensity is

  \[\frac{0.1082}{0.6280} = 17.23\%\]

If we instead compute the default intensity \( \lambda(t) \) at time \( t \) over a shorter length of time \( \Delta t \). Then \( \lambda(t) \Delta t \) is the probability of default between time \( t \) and time \( t + \Delta t \) conditional on no earlier default.
If $V(t)$ is the cumulative probability of the company surviving to time $t$, then

$$V(t + \Delta t) - V(t) = -\lambda(t)V(t)\Delta t$$

Taking limits we get

$$\frac{dV(t)}{dt} = e^{-\int_0^t \lambda(\tau)d\tau}$$

Define $Q(t)$ as the probability of default by time $t$. It follows that

$$Q(t) = 1 - e^{-\int_0^t \lambda(\tau)d\tau}$$

or

$$Q(t) = 1 - e^{-\lambda(t)t}$$ (1)

where $\lambda(t)$ is the average default intensity between time 0 and time $t$. 

Recovery Rates

- When a company goes bankrupt, those that are owed money by the company file claims against the assets of the company.
- Either the company reorganizes and creditors agree to partial payments or the assets are liquidated and the used to meet outstanding claims.
- Recovery rates for a bond is normally defined as the bond’s market value immediately after a default, as a percent of the face value.
- Senior secured debt holders receive 51 cents to the dollar owed on average; whereas junior subordinated debt holders receive 25 cents to the dollar owed on average.

<table>
<thead>
<tr>
<th>Class</th>
<th>Average recovery rate %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Senior secured</td>
<td>51.6</td>
</tr>
<tr>
<td>Senior unsecured</td>
<td>36.1</td>
</tr>
<tr>
<td>Senior subordinated</td>
<td>32.5</td>
</tr>
<tr>
<td>Subordinated</td>
<td>31.1</td>
</tr>
<tr>
<td>Junior subordinated</td>
<td>24.5</td>
</tr>
</tbody>
</table>

- Recovery rates are significantly negatively correlated with default rates.
It found that the following relationship provides a good fit to the data:

\[
\text{Average recovery rate} = 50.3 - 6.3 \times \text{Average default rate}
\]

where both the average recovery rate and the average default rate are measured as percentages.

**Estimating Default Probabilities from Bond Prices**

- Probabilities of default can be estimated from the prices of bonds it has issued.
- Assume that the reason a corporate bond sells for less than a risk-free bond is the possibility of default.
- Consider an approximate calculation. Suppose that a bond yields 200 basis points more than a similar risk-free bond and that the expected recovery rate in the event of default is 40%.
- The holder of a corporate bond must be expecting to lose 200 basis points (2% per year) from defaults. Given the recovery rate of 40%, this leads to an estimate of the probability of a default per year conditional on no earlier default of \( \frac{0.02}{1 - 0.4} = 2.22\% \).
- In general

\[
h = \frac{s}{1 - R}
\]

(2)

where \( h \) is the default intensity per year, \( s \) is the spread of the corporate bond yield over the risk-free rate, and \( R \) is the expected recovery rate.
A more exact calculation, suppose that the corporate bond we have been considering lasts for 5 years, provides a coupon 6% per annum (paid semiannually) and that the yield on the corporate bond is 7% per annum (with continuous compounding).

- The yield on a similar risk-free bond is 5% (with continuous compounding).
- The yields imply that the price of the corporate bond is 95.34 and the price of the risk-free bond is 104.09.
- The expected loss from default over the 5-year life of the bond is therefore 104.09 - 95.34 = 8.75.
- Suppose that the probability of default per year (assumed in this simple example to be the same each year) is $Q$.

Consider for example the loss of default at 3.5 years. The expected value of the risk-free bond at time 3.5 years (using forward interest rates) is

$$3 + 3e^{-0.05 \times 0.5} + 3e^{-0.05 \times 1.0} + 103e^{-0.05 \times 1.5} = 103.46$$

Given the default of recovery rates in the previous section, the amount recovered if there is a default is 40, so that the loss given default is 103.46 - 40 or $64.34.$
The present value of this loss is 54.01. The expected loss is therefore $54.01Q$.

Next consider for example the loss of default at 4.5 years. The expected value of the risk-free bond at time 3.5 years (using forward interest rates) is

\[ 3 + 103e^{-0.05 \times 0.5} = 104.34 \]

Given the default of recovery rates in the previous section, the amount recovered if there is a default is 40, so that the loss given default is 104.34-40 or $63.46$. The present value of this loss is 50.67. The expected loss is therefore 50.67.

<table>
<thead>
<tr>
<th>Time (years)</th>
<th>Default probability</th>
<th>Recovery amount ($)</th>
<th>Risk-free value ($)</th>
<th>Loss given default ($)</th>
<th>Discount factor</th>
<th>PV of expected loss ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>$Q$</td>
<td>40</td>
<td>106.73</td>
<td>66.73</td>
<td>0.9753</td>
<td>65.08$Q$</td>
</tr>
<tr>
<td>1.5</td>
<td>$Q$</td>
<td>40</td>
<td>105.97</td>
<td>65.97</td>
<td>0.9277</td>
<td>61.20$Q$</td>
</tr>
<tr>
<td>2.5</td>
<td>$Q$</td>
<td>40</td>
<td>105.17</td>
<td>65.17</td>
<td>0.8825</td>
<td>57.52$Q$</td>
</tr>
<tr>
<td>3.5</td>
<td>$Q$</td>
<td>40</td>
<td>104.34</td>
<td>64.34</td>
<td>0.8395</td>
<td>54.01$Q$</td>
</tr>
<tr>
<td>4.5</td>
<td>$Q$</td>
<td>40</td>
<td>103.46</td>
<td>63.46</td>
<td>0.7985</td>
<td>50.67$Q$</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td><strong>288.48$Q$</strong></td>
</tr>
</tbody>
</table>

The total expected loss is 288.48$Q$. Setting this equal to 8.75, we obtain a value of $Q$ equal to 3.03%. The calculations we have give assume that the default probability is the same in each year and that defaults take place at just one time during the year. We can extend the calculations to assume that defaults take place at just one time during the year.
• The calculations assume that the default probability is the same each year and that defaults take place at just one time during the year. We can extend the calculations to assume that defaults can take place more frequently.

• Instead of assuming a constant unconditional probability of default we can assume a constant default intensity or assume a particular pattern for the variation of default probabilities with time.
With several bonds we can estimate several parameters describing the term structure of default probabilities.

Suppose we have bonds maturing in 3, 5, 7, and 10 years.

- We could use the first bond to estimate the default probability per year for the first 3 years.
- We could use the second bond to estimate default probability for years 4 and 5.
- Etc.
- The approach is analogous to the bootstrap procedure for calculating the zero-coupon yield curve.
A key issue when bond prices are used to estimate default probabilities is the meaning of the terms "risk-free rate" and "risk-free bond".

From equation (2), the spread $s$ is the excess of the corporate bond yield over the yield on a similar risk-free bond.

In our previous table, the risk-free value of the bond must be calculated using the risk-free rate.

Benchmark risk-free rate this is usually used in quoting corporate bond yields is the yield on similar Treasury bonds.

Traders usually use LIBOR/swap rates as proxies for risk-free rates instead of the Treasury rates when valuing derivatives. Also use these rates as risk-free rates when calculating default probabilities.

When determining default prob. from bond prices, the spread $s$ in (2) is the spread of the bond yield over the LIBOR/swap rate.

The risk-free discount rates used in the table before are LIBOR/swap zero rates.

Credit default swaps (to be discussed next week) can be used to imply the risk-free rate assumed by traders. The rate used appears to be approximately equal to the LIBOR/swap rate minus 10 basis points on average.
In practice traders often use **asset swap** spreads as a way of extracting default probabilities from bond prices. This is because asset swap spreads provide a direct estimate of the spread of bond yields over the LIBOR/swap curve.

Consider how an **asset swap** works. Consider the situation where an asset swap spread for a particular bond is quoted as 150 basis points. There are three possible situations:

1. The bond sells for its par value of 100. The swap then involves one side (company A) paying the coupon on the bond and the other side (company B) paying LIBOR plus 150 basis points.
2. The bond sells below its par value, say, for 95. The swap is then structured so that in addition to the coupons company A pays $5 per $100 of notional principal at the outset.
3. The underlying bond sells above par, say, for 108. Company B would then make a payment of $8 per $100 of principal at the outset.

The effect of this is that the present value of the asset swap spread is the amount by which the price of the corporate bond is exceeded by the price of a similar risk-free bond where the risk-free rate is assumed to be given by the LIBOR/swap curve.
Consider again our example where the LIBOR/swap zero curve is flat at 5%.

- Suppose that instead of knowing the bond’s price we know that the risk-free bond exceeds the value of the corporate bond is the present value of 150 basis points per year for 5 years.
- Assuming semiannual payments, this is $6.55 per $100 of principal.
- Total loss in this case would be set equal to $6.55. This means that the default probability per year $Q$ would be $\frac{6.55}{288.48} = 2.27\%$. 
Comparison of Default Probability Estimates

The default probabilities estimate from historical data are much less than those derived from bond prices.

<table>
<thead>
<tr>
<th>Rating</th>
<th>Historical default intensity</th>
<th>Default intensity from bonds</th>
<th>Ratio</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aaa</td>
<td>0.04</td>
<td>0.67</td>
<td>16.8</td>
<td>0.63</td>
</tr>
<tr>
<td>Aa</td>
<td>0.06</td>
<td>0.78</td>
<td>13.0</td>
<td>0.72</td>
</tr>
<tr>
<td>A</td>
<td>0.13</td>
<td>1.28</td>
<td>9.8</td>
<td>1.15</td>
</tr>
<tr>
<td>Baa</td>
<td>0.47</td>
<td>2.38</td>
<td>5.1</td>
<td>1.91</td>
</tr>
<tr>
<td>Ba</td>
<td>2.40</td>
<td>5.07</td>
<td>2.1</td>
<td>2.67</td>
</tr>
<tr>
<td>B</td>
<td>7.49</td>
<td>9.02</td>
<td>1.2</td>
<td>1.53</td>
</tr>
<tr>
<td>Caa</td>
<td>16.90</td>
<td>21.30</td>
<td>1.3</td>
<td>4.40</td>
</tr>
</tbody>
</table>

For companies that start with a particular rating, the average annual default intensity over 7 years calculated from (a) historical data and (b) bond prices.
The calculation of default intensities, using historical data are based on (1) and our table of cumulative defaults on bond types. We have for example:

\[ \bar{\lambda}(7) = -\frac{1}{7} \ln [1 - Q(7)] \]

where \( \bar{\lambda}(t) \) is the average default intensity (or hazard rate) by time \( t \) and \( Q(t) \) is the cumulative probability of default by time \( t \).

- The values of \( Q(7) \) are taken directly from our table.
- Consider, for example, an A-rated company. The value of \( Q(7) \) is 0.0091. The average 7-year default intensity is 0.13% since

\[ \bar{\lambda}(7) = -\frac{1}{7} \ln(0.9909) = 0.0013 \]

- The calculations on bond prices are based on (2) and bond yields from Merrill Lynch.
- The recovery rate is assumed to be 40% and the risk-free interest rate is assume to be the 7-year swap rate minus 10 basis points (the 10 basis point subtraction is from empirical data)
- For example, for A-rated bonds, the average Merrill Lynch yield was 6.274%. The average swap rate was 5.605%, so that the average risk-free rate was 5.505%. This gives the average 7-year default probability as

\[ \frac{0.06274 - 0.05505}{1 - 0.4} = 0.0128 \]

or 1.28%.
Remarks

From our chart we see

- The ratio of default probability backed out of bond prices to the default calculated from historical data tends to decline as the credit quality declines with the ratio very high for investment grade companies.
- The difference between the two default probabilities tends to increase as credit quality declines.

Real-World vs. Risk-Neutral Probabilities

Default probabilities implied from bond yields are risk-neutral probabilities of default. To explain why, consider the calculations of default probabilities of $Q$ table.

- These calculations assume that expected default losses can be discounted at the risk-free rate
- The risk-neutral valuation principle shows that this is a valid procedure providing the expected losses are calculated in a risk-neutral world. This means that the default probability $Q$ must be a risk-neutral probability.

On the other hand default probabilities implied from historical data are real-world default probabilities or **physical probabilities**. The expected excess return between real-world and risk-neutral default probabilities.
Real-World vs. Risk-Neutral Probabilities

Why do we see such big differences between real-world and risk-neutral default probabilities?

1. Corporate bonds are relatively illiquid and bond traders demand an extra return to compensate for this.

2. The subjective default probabilities of bond traders may be much higher than those given in our first table. Bond traders may be allowing for depression scenarios much worse than anything seen during the period from 1970 to 2003.

3. Bonds do not default independently of each other. This is the most important reason for the results in our last table. There are periods of time when default rates are very low and periods of time when they are very high. This gives rise to systematic risk (i.e. risk that cannot be diversified away) and bond traders should require an expected excess return for bearing the risk. The variation in default rates from year to year may be due to overall economic conditions or it may be because a default by one company has a ripple effect resulting in defaults by other companies. (The latter is referred to by researchers as credit contagion).

4. Bond returns are highly skewed with limited upside. As a result it is much more difficult to diversity risks in a bond portfolio than in an equity portfolio. A very large number of different bonds must be held. In practice many bond portfolios are far from fully diversified. As a result bond traders may require an extra return for bearing unsystematic risk in addition to the systematic risk mentioned above.
Using Equity Prices to Estimate Default Probabilities

In order to get the default probabilities we rely on the company’s credit rating. Unfortunately, credit ratings are revised relatively infrequently. This leads analysts to argue that equity prices can provide more up-to-date information for estimating default probabilities.

Merton proposed a model where a company’s equity is an option on the assets of the company. Suppose that a firm has one zero-coupon bond outstanding and that the bond matures at time $T$. Define

- $V_0$ Value of company’s assets today
- $V_T$ Value of company’s assets at time $T$
- $E_0$ Value of company’s equity today
- $E_T$ Value of company’s equity at time $T$
- $D$ Amount of debt interest and principal due to be repaid at time $T$
- $\sigma_V$ Volatility of assets (assumed constant)
- $\sigma_E$ Instantaneous volatility of equity
Consider two cases:

- If $V_T < D$, it is (at least in theory) rational for the company to default on the debt at time $T$. The value of the equity is then zero.
- If $V_T > D$, the company should make the debt repayment at time $T$ and the value the equity at this time is $V_T - D$.

Merton’s model, therefore, gives the value of the firm’s equity at time $T$ as

$$E_T = \max\{V_T - D, 0\}$$

This shows that the equity is a call option on the value of the assets with a strike price equal to the repayment required on the debt.

Black-Scholes formula gives the value of the equity today as

$$E_0 = V_0 N(d_1) - De^{-rT}N(d_2)$$

where

$$d_1 = \ln \frac{V_0}{D} + (r + \frac{\sigma^2 V}{2})T \quad \sigma \sqrt{T}$$

$$d_2 = d_1 - \sigma \sqrt{T}$$

The value of the debt today is $V_0 - E_0$. 
• The risk-neutral probability that the company will default on the debt is

\[ N(-d_2) \]

• To calculate this, we require \( V_0 \) and \( \sigma_V \). Neither of these are directly observable. However if the company is publicly traded, we can observe \( E_0 \).

• This means (3) provides one condition that must be satisfied by \( V_0 \) and \( \sigma_V \).

• We can also estimate \( \sigma_E \). From Itô’s lemma

\[
\sigma_E E_0 = \frac{\partial E}{\partial V} \sigma_V V_0
\]

or

\[
\sigma_E E_0 = N(d_1) \sigma_V V_0 \tag{4}
\]

This provides another equation that must be satisfied by \( V_0 \) and \( \sigma_V \).

This provides two nonlinear equations (3) and (4) to get \( V_0 \) and \( \sigma_V \).
The value of a company’s equity is $3 million and the volatility of the equity is 80%.

- The debt that will have to be paid in 1 year is $10 million. The risk-free rate is 5% per annum. Thus
  \[ E_0 = 3 \quad \sigma_E = 0.80 \quad r = 0.05 \quad T = 1 \quad D = 10 \]
- Solving (3) and (4) yields
  \[ V_0 = 12.40 \quad \sigma_V = 0.2123 \]

  The parameter \( d_2 = 1.1408 \), so that the probability of default is \( N(-d_2) = 0.127 \) or 12.7%.

- The market value of the debt is \( V_0 - E_0 \) or 9.40. The present value of the promised payment on the debt is
  \[ 10e^{-0.05 \times 1} = 9.51 \]

- The expected loss on the debt is therefore, \( \frac{9.51 - 9.40}{9.51} = 1.2\% \) of its no-default value.
- Comparing this with the probability of default gives the expected recovery in the event of a default as \( \frac{12.7 - 1.2}{12.7} = 91\% \).
Merton Model

- Basic Merton model can be extended in many ways. One can assume that a default occurs whenever the values of the assets falls below a barrier level (more in the coming weeks)
- How does it match with true default probabilities? Merton model does produce a good ranking of default probabilities.
Credit exposure on a derivatives transaction is more complicated than that on a loan.

This is because the claim that will be made in the event of a default is more uncertain.

Consider a financial institution that has one derivatives contract outstanding with a counterparty. consider three situations:

1. Contract is always a liability to the financial institution
2. Contract is always an asset to the financial institution
3. Contract can become either an asset or a liability to the financial institution

Example of a derivatives contract in the first category is a short option position; an example of the second category is a long option position; an example of the third category is a forward option
• Derivatives in first category have no credit risk to the financial institution. If the counterparty goes bankrupt, there will be no loss. The derivative is one of the counterparty’s assets. It is likely to be retained, closed out, or sold to a third party. The result is no loss (or gain) to the financial institution

• Derivatives in second category always have credit risk to the financial institution. If the counterparty goes bankrupt, a loss is likely to be experienced. The derivative is one of the counterparty’s liabilities. The financial institution has to make a claim against the assets of the counterparty and may receive some percentage of the value of the derivative.

• Derivatives in the third category may or may not have credit risk. If the counterparty defaults when the value of the derivative is positive to the financial institution, a claim will be made against the assets of the counterparty and a loss is likely to be experienced. If the counterparty defaults when the value is negative to the financial institution, no loss is made because the derivative will be retained, closed out, or sold to a third party.
**Question**: How should a financial institution adjust the value of a derivative to allow for counterparty credit risk?

Consider a derivative that has a value of $f_0$ today assuming no defaults. Let us suppose that defaults can take place at times $t_1, \ldots, t_n$ and that the value of the derivative to the financial institution (assuming no defaults) at time $t_i$ is $f_i$. Define the risk-neutral probability of default at time $t_i$ as $q_i$ and the expected recovery rate as $R$.

The exposure at time $t_i$ is the financial institution’s potential loss. This is $\max\{f_i, 0\}$. Assume that the expected recovery in the event of a default is $R$ times the exposure. Assume also that the recovery rate and the probability of default is independent of the value of the derivative. The risk-neutral expected loss from default at time $t_i$ is

$$q_i (1 - R) \hat{E} [\max\{f_i, 0\}]$$

where $\hat{E}$ is the expected value in a risk-neutral world.
Take present values leads to the cost of defaults being

\[ \sum_{i=1}^{n} u_i v_i \]  

(5)

where \( u_i \) equals \( q_i (1 - R) \) and \( v_i \) is the value today of an instrument that pays off the exposure on the derivative under consideration at time \( t_i \).

Consider again the three categories of derivatives mentioned earlier.

- The first category (where the derivative is always a liability to the financial institution) is easy to deal with. The value of \( f_i \) is always negative and so the total expected loss from defaults given by (5) is always zero. The financial institution needs to make no adjustments for the cost of defaults.

- For the second category (where the derivative is always an asset to the financial institution) \( f_i \) is always positive. The expression \( \max\{f_i, 0\} \) is always equal to \( f_i \). Since \( v_i \) is the present value of \( f_i \), it always equals \( f_0 \). The expected loss from default is therefore \( f_0 \) times the total probability of default during the life of the derivative times \( 1 - R \).

- For the third category of derivatives, the sign of \( f_i \) is uncertain. The variable \( v_i \) is a call option on \( f_i \) with a strike price of zero. One way of calculating \( v_i \) is to simulate the underlying market variable over the life of the derivative. Sometimes approximate analytic calculations are possible.
Notes:

- The analyses we have presented assume that the probability of default is independent of the value of the derivative.
- This is likely to be a reasonable approximation in circumstances when the derivative is a small part of the portfolio of the counterparty or when the counterparty is using the derivative for hedging purposes.
Credit Risk Mitigation

In many cases discussed overstates the credit risk in a derivatives transaction. This is because there are a number of clauses that derivatives dealers include in their contracts to mitigate credit risk.

**Netting**: a clause that has become standard in over-the-counter derivatives contracts. Such clause states that if a company defaults on one contract it has with a counterparty then it must default on all outstanding contracts with the counterparty.

Consider, for example, a financial institution that has three contracts outstanding with a particular counterparty worth +$10 million, +$30 million, and -$25 million to the financial institution. Suppose the counterparty runs into financial problems and defaults on its outstanding obligations. To the counterparty the three contracts have the values of -$10 million, -$30 million, and +$25 million, respectively.

Without netting the counterparty would default on the first two contracts and retain the third for a loss to the financial institution of $40 million. With netting it is compelled to default on all three contracts for a loss to the financial institution of $15 million.
Collateralization: Suppose that a company and a financial institution have entered into a number of derivatives contracts. A typical collateralization agreement specifies that the contracts be marked to market periodically using a preagreed formula.

If the total value of the contracts to the financial institution is above a certain threshold level on a certain day, it can ask the company to post collateral. The amount of collateral posted when added to the collateral already posted by the company is equal to the difference between the value of the contract to the financial institution and the threshold level. This is a form of marking to market of the contract.

Downgrade Triggers: This is a clause stating that if the credit rating of the counterparty falls below a certain level, say $BBB$, the financial institution has the option to close out a derivatives contract at its market value.
A **default correlation** is used to describe the tendency for two companies to default at about the same time.

There are a number of reasons why default correlations exist.

- Companies in the same industry or the same geographic region tend to be affected similarly by external events and as a result may experience financial difficulties at the same time.
- Economic conditions generally cause average default rates to be higher in some years than in other years. A default by one company may cause a default by another.
- Default correlation means that credit risk cannot be complete diversified away and is the major reason why risk-neutral default probabilities are greater than real-world default probabilities.
- Default correlation is important to the determination of probability distributions for default losses from a portfolio of exposures to different counterparties.
Gaussian Copula Model for Time to Default

This is a popular reduced form default correlation model.

The model quantifies the correlation between times to default for two different companies. The model implicitly assumes that all companies will default eventually.

In any application of the model we are typically interested in the possibility of defaults over the next 1 year, 5 years, or 10 years. We are therefore only interested in the left tail of the distribution of the time to default.

The model can be used in conjunction with either real-world or risk-neutral default probabilities.

The left tail of the real-world probability distribution for the time to default of a company can be estimated from data produced by rating agencies such as that in our original table of defaults. The left tail of the risk-neutral probability distribution of the time to default can be estimate from bond prices using the approach using $Q$'s.
Define $t_1$ as the time to default of company 1 and $t_2$ as the time to default of company 2. If the probability distributions of $t_1$ and $t_2$ were normal, we could assume that the joint probability distribution of $t_1$ and $t_2$ is bivariate normal.

As it happens the probability distributions of a company's time to default is not even approximately normal. This is where a Gaussian copula model comes in. We transform $t_1$ and $t_2$ into new variables $x_1$ and $x_2$ using

$$x_1 = N^{-1}[Q_1(t_1)] \quad x_2 = N^{-1}[Q_2(t_2)]$$

where $Q_1$ and $Q_2$ are the cumulative probability distributions for $t_1$ and $t_2$, respectively, and $N^{-1}$ is the inverse of the cumulative normal distribution.

- The 5-percentile point in the probability distribution for $t_1$ is transformed to $x_1 = -1.645$ which is the 5-percentile point in the standard normal distribution.
- The 10-percentile point in the probability distribution for $t_1$ is transformed to $x_1 = -1.282$ which is the 10-percentile point in the standard normal distribution.

The $t_2$ to $x_2$ transformation is similar.
By construction $x_1$ and $x_2$ have normal distributions with mean zero and unit standard deviation. We assume that the joint distribution of $x_1$ and $x_2$ is bivariate normal with correlation $\rho_{12}$. This assumption is referred to as using a **Gaussian copula**.

The assumption is convenient because it mean that the joint probability distribution of $t_1$ and $t_2$ is fully defined by the cumulative default probability distributions $Q_1$ and $Q_2$ for $t_1$ and $t_2$, together with a single correlation parameter $\rho_{12}$.

The attraction of the Gaussian copula model is that it can be extended to many companies.

Suppose that we are considering $n$ companies and that $t_i$ is the time to default of the $i$th company.

We transform each $t_i$ into a new variable $x_i$ that has a standard normal distribution. The transformation is the percentile-to-percentile transformation

$$x_i = N^{-1} [Q_i(t_i)]$$

where $Q_i$ is the cumulative probability distribution for $t_i$. We ten assume that the $x_i$ are multivariate normal. The default correlation between $t_i$ and $t_j$ is measured as the correlation between $x_i$ and $x_j$. This is referred to a the copula correlation.

The Gaussian copula approach is a useful way representing the correlation structure between variable that are not normally distributed. It allows the correlation structure of the variables to be estimated separately from their marginal (unconditional) distributions. Although the variables themselves are not multivariate normal, the approach assumes that after a transformation is applied to each variable they are multivariate normal.
**Example:** Suppose that we wish to simulate defaults during the next 5 years in 10 companies. The copula default correlations between each pair of companies is 0.2. For each company the default during the next 1, 2, 3, 4, 5 years is 1%, 3%, 6%, 10%, 15%, respectively. When a Gaussian copula is used we sample from a multivariate normal distribution to obtain the $x_i, (1 \leq i \leq 10)$ with a pairwise correlation between the $x_i$ being 0.2. We then convert the $x_i$ and $t_i$, a time to default.

When the sample from the normal distribution is

- less than $N^{-1}(0.01) = -2.33$, a default takes place within the first year;
- when the sample is between -2.33 and $N^{-1}(0.03) = -1.88$, a default takes place during the second year;
- when the sample is between -1.88 and $N^{-1}(0.06) = -1.55$, a default takes place during the third year;
- when the sample is between -1.55 and $N^{-1}(0.10) = -1.28$, a default takes place during the fourth year;
- when the sample is between -1.28 and $N^{-1}(0.15) = -1.04$, a default takes place during the fifth year;
- when the sample is greater than -1.04, there is no default during the 5 years.
Using Factors to Define the Correlation Structure

To avoid defining a different correlation between $x_i$ and $x_j$ for each pair of companies $i$ and $j$ in the Gaussian copula model, a one-factor model is often used. The assumption is that

$$x_i = a_i M + \sqrt{1 - a_i^2} Z_i$$

(6)

here $m$ is a common factor affecting defaults for all companies and $Z_i$ is a factor affecting only company $i$. The variables $M$ and the $Z_i$ have independent standard normal distributions. the $a_i$ are constant parameters between $-1$ and $+1$. The correlation between $x_i$ and $x_j$ is $a_i a_j$.

Suppose that the probability that company $i$ will default by a particular time $T$ is $Q_i(T)$. Under the Gaussian copula model, a default happens when $N(x_i) < Q_i(T)$ or $x_i < N^{-1}[Q_i(T)]$. 
From (6) we get the condition

\[ a_i M + \sqrt{1 - a_i^2} Z_i < N^{-1} [Q_i(T)] \]

or

\[ Z < \frac{N^{-1} [Q_i(T)] - a_i M}{\sqrt{1 - a_i^2}} \]

Conditional on the value of the factor \( M \), the probability of default is therefore

\[ Q_i(T|M) = N \left( \frac{N^{-1} [Q_i(T)] - a_i M}{\sqrt{1 - a_i^2}} \right) \] (7)

A particular case of the one-factor Gaussian model is where the probability distributions of default are the same for all \( i \) and the correlations between \( x_i \) and \( x_j \) is the same for all \( i \) and \( j \).

Suppose that \( Q_i(T) = Q_j(T) \) for all \( i \) and that the common correlations is \( \rho \), so that \( a_i = \sqrt{\rho} \) for all \( i \). Equation (7) becomes

\[ Q(T|M) = N \left( \frac{N^{-1} [Q(T)] - \sqrt{\rho} M}{\sqrt{1 - \rho}} \right) \] (8)
An alternative correlation measure used by rating agencies is the **binomial correlation measure**. For two companies A and B this is the coefficient of correlation between.

1. A variable that equals 1 if company A defaults between times 0 and $T$, and 0 otherwise;
2. A variable that equals 1 if company B defaults between times 0 and $T$, and 0 otherwise.

The measure is

$$
\beta_{AB}(T) = \frac{P_{AB}(T) - Q_A(T)Q_B(T)}{\sqrt{[Q_A(T) - Q_A(T)^2][Q_B(T) - Q_B(T)^2]}}
$$

(9)

where $P_{AB}(T)$ is the joint probability of A and B defaulting between time 0 and time $T$, $Q_A(T)$ is the cumulative probability that company A will default by time $T$, and $Q_B(T)$ is the cumulative probability probability that company B will default by time $T$. Typically $\beta_{AB}(T)$ depends on $T$, the length of the time period considered. Usually it increases as $T$ increases.
From the definition of a Gaussian copula model, \( P_{AB}(T) = M[x_A(T), x_B(T); \rho_{AB}] \), where \( x_A(T) = N^{-1}(Q_A(T)) \) and \( x_B(T) = N^{-1}(Q_B(T)) \) are the transformed times to default for companies A and B. Here \( M(a, b; \rho) \) is the probability that, in a bivariate normal distribution where the correlation between the variables is \( \rho \), the first variable is less than \( a \) and the second variable is less than \( b \). It follows that

\[
\beta_{AB}(T) = \frac{M[x_A(T), x_B(T); \rho_{AB}] - Q_A(T)Q_B(T)}{\sqrt{[Q_A(T) - Q_A(T)^2][Q_B(T) - Q_B(T)^2]}}
\]

(10)

This shows that if \( Q_A(T) \) and \( Q_B(T) \) are known, \( \beta_{AB}(T) \) can be calculated from \( \rho_{AB} \) and vice versa. Usually \( \rho_{AB} \) is markedly greater than \( \beta_{AB}(T) \).

This illustrates the point that the magnitude of a correlation measure depends on how it is defined.

**Example:**

Suppose that the prob. of company A defaulting in a 1-year period is 1% and the prob. of company B defaulting in a 1-year period is also 1%. In this case \( x_A(1) = x_B(1) = N^{-1}(0.01) = -2.326 \). If \( \rho_{AB} = 0.20 \), \( M(x_A(1), x_B(1), \rho_{AB}) = 0.000337 \) and equation (10) shows that \( \beta_{AB} = 0.024 \) when \( T = 1 \).
Credit VaR

Credit value at risk can be defined analogously to the way we defined value at risk for market risks. For example, a credit VaR with a confidence level of 99.9% and a 1-year time horizon is the credit loss that we are 99.9% confident will not be exceeded over 1 year.

Consider a bank with a very large portfolio of similar loans. As an approximation we assume that the probability of default is the same for each loan and the correlation between each pair of loans is the same.

When the Gaussian copula model for time to default is used, the right-hand side of equation (8) is approximately equal to the percentage of defaults by time $T$ as a function of $M$. The factor $M$ has a standard normal distribution. We are $X\%$ certain that its value will be greater than $N^{-1}(1 - X) = -N^{-1}(X)$. We are therefore $X\%$ certain that the percentage of losses over $T$ years on a large portfolio will be less than $V(X, T)$ where

$$V(X, T) = N \left( \frac{N^{-1}[Q(T)] - \sqrt{\rho}N^{-1}(X)}{\sqrt{1 - \rho}} \right)$$

(11)

Here $Q(T)$ is the probability of default by time $T$ and $\rho$ is the copula correlation between any pair of loans.
A rough estimate of the credit VaR when $X\%$ confidence level is used and the time horizon is $T$ is therefore $L(1 - R) V(X, T)$, where $L$ is the size of the loan portfolio and $R$ is the recovery rate. The contribution of a particular loan of size $L_i$ to the credit VaR is $L_i(1 - R) V(X, T)$.

**Example:**

Suppose that a bank has a total of $100$ million of retain exposures. The 1-year probability of default averages 2% and the recovery rate averages 60%. The copula correlations parameter is estimated as 0.1. In this case,

$$V(0.999, 1) = N \left( \frac{N^{-1}(0.02) - \sqrt{0.1} N^{-1}(0.999)}{\sqrt{1 - 0.1}} \right) = 0.128$$

showing that the 99.9% worst case default rate is 12.8%. The 1-year 99.9% credit VaR is therefore

$$100 \times 0.128 \times (1 - 0.6) = 5.13 \text{ million}$$
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