Credit Risk & Credit Derivatives
We can consider the historical default rates of the certain class of corporate bonds:

**Table 20.1**  Average cumulative default rates (%), 1970–2003. (Source: Moody’s)

<table>
<thead>
<tr>
<th>Term (years)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>7</th>
<th>10</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aaa</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.04</td>
<td>0.12</td>
<td>0.29</td>
<td>0.62</td>
<td>1.21</td>
<td>1.55</td>
</tr>
<tr>
<td>Aa</td>
<td>0.02</td>
<td>0.03</td>
<td>0.06</td>
<td>0.15</td>
<td>0.24</td>
<td>0.43</td>
<td>0.68</td>
<td>1.51</td>
<td>2.70</td>
</tr>
<tr>
<td>A</td>
<td>0.02</td>
<td>0.09</td>
<td>0.23</td>
<td>0.38</td>
<td>0.54</td>
<td>0.91</td>
<td>1.59</td>
<td>2.94</td>
<td>5.24</td>
</tr>
<tr>
<td>Baa</td>
<td>0.20</td>
<td>0.57</td>
<td>1.03</td>
<td>1.62</td>
<td>2.16</td>
<td>3.24</td>
<td>5.10</td>
<td>9.12</td>
<td>12.59</td>
</tr>
<tr>
<td>Ba</td>
<td>1.26</td>
<td>3.48</td>
<td>6.00</td>
<td>8.59</td>
<td>11.17</td>
<td>15.44</td>
<td>21.01</td>
<td>30.88</td>
<td>38.56</td>
</tr>
<tr>
<td>B</td>
<td>6.21</td>
<td>13.76</td>
<td>20.65</td>
<td>26.66</td>
<td>31.99</td>
<td>40.79</td>
<td>50.02</td>
<td>59.21</td>
<td>60.73</td>
</tr>
<tr>
<td>Caa</td>
<td>23.65</td>
<td>37.20</td>
<td>48.02</td>
<td>55.56</td>
<td>60.83</td>
<td>69.36</td>
<td>77.91</td>
<td>80.23</td>
<td>80.23</td>
</tr>
</tbody>
</table>
We can compute the probability of default for a particular year from the table.

We define two quantities:

- **Unconditional default probability**: This is the probability of default during a particular year as seen from the initial year.
  
  **Example**: The unconditional default probability of a *Caa* bond defaulting in the third year is
  
  $48.02 - 37.20 = 10.82\%$

- **Default intensities** or **Hazard rates**: This is the probability of default during a particular year as seen from the initial year conditioned on no default occurring earlier.
  
  **Example**: The probability that a *Caa* bond will last past year two is
  
  $100 - 37.20 = 62.80\%$

  Therefore, the default intensity is
  
  $\frac{0.1082}{0.6280} = 17.23\%$

If we instead compute the default intensity $\lambda(t)$ at time $t$ over a shorter length of time $\Delta t$. Then $\lambda(t)\Delta t$ is the probability of default between time $t$ and time $t + \Delta t$ conditional on no earlier default.
If $V(t)$ is the cumulative probability of the company surviving to time $t$, then

$$V(t + \Delta t) - V(t) = -\lambda(t)V(t)\Delta t$$

Taking limits we get

$$V(t) = e^{-\int_0^t \lambda(\tau)d\tau}$$

Define $Q(t)$ as the probability of default by time $t$. It follows that

$$Q(t) = 1 - e^{-\int_0^t \lambda(\tau)d\tau}$$

or

$$Q(t) = 1 - e^{-\overline{\lambda(t)}t}$$

(1)

where $\overline{\lambda(t)}$ is the average default intensity between time 0 and time $t$. 
Recovery Rates

- When a company goes bankrupt, those that are owed money by the company file claims against the assets of the company.
- Either the company reorganizes and creditors agree to partial payments or the assets are liquidated and used to meet outstanding claims.
- Recovery rates for a bond is normally defined as the bond’s market value immediately after a default, as a percent of the face value.
- Senior secured debt holders receive 51 cents to the dollar owed on average; whereas junior subordinated debt holders receive 25 cents to the dollar owed on average.

<table>
<thead>
<tr>
<th>Class</th>
<th>Average recovery rate %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Senior secured</td>
<td>51.6</td>
</tr>
<tr>
<td>Senior unsecured</td>
<td>36.1</td>
</tr>
<tr>
<td>Senior subordinated</td>
<td>32.5</td>
</tr>
<tr>
<td>Subordinated</td>
<td>31.1</td>
</tr>
<tr>
<td>Junior subordinated</td>
<td>24.5</td>
</tr>
</tbody>
</table>

- Recovery rates are significantly negatively correlated with default rates.
Estimating Default Probabilities from Bond Prices

- Probabilities of default can be estimate from the prices of bonds it has issued.
- Assume that the reason a corporate bond sells for less than a risk-free bond is the possibility of default.
- Consider an approximate calculation. Suppose that a bond yields 200 basis points more than a similar risk-free bond and that the expected recovery rate in the event of default is 40%.
- The holder of a corporate bond must be expecting to lose 200 basis points (2% per year) from defaults. Given the recovery rate of 40%, this leads to an estimate of the probability of a default per year conditional on no earlier default of \( \frac{0.02}{1-0.4} = 2.22\% \).
- In general

\[
h = \frac{s}{1 - R}
\]

where \( h \) is the default intensity per year, \( s \) is the spread of the corporate bond yield over the risk-free rate, and \( R \) is the expected recovery rate.
The default probabilities estimate from historical data are much less than those derived from bond prices.

<table>
<thead>
<tr>
<th>Rating</th>
<th>Historical default intensity</th>
<th>Default intensity from bonds</th>
<th>Ratio</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aaa</td>
<td>0.04</td>
<td>0.67</td>
<td>16.8</td>
<td>0.63</td>
</tr>
<tr>
<td>Aa</td>
<td>0.06</td>
<td>0.78</td>
<td>13.0</td>
<td>0.72</td>
</tr>
<tr>
<td>A</td>
<td>0.13</td>
<td>1.28</td>
<td>9.8</td>
<td>1.15</td>
</tr>
<tr>
<td>Baa</td>
<td>0.47</td>
<td>2.38</td>
<td>5.1</td>
<td>1.91</td>
</tr>
<tr>
<td>Ba</td>
<td>2.40</td>
<td>5.07</td>
<td>2.1</td>
<td>2.67</td>
</tr>
<tr>
<td>B</td>
<td>7.49</td>
<td>9.02</td>
<td>1.2</td>
<td>1.53</td>
</tr>
<tr>
<td>Caa</td>
<td>16.90</td>
<td>21.30</td>
<td>1.3</td>
<td>4.40</td>
</tr>
</tbody>
</table>

For companies that start with a particular rating, the average annual default intensity over 7 years calculated from (a) historical data and (b) bond prices.
Remarks

From our chart we see

- The ratio of default probability backed out of bond prices to the default calculated from historical data tends to decline as the credit quality declines with the ratio very high for investment grade companies.
- The difference between the two default probabilities tends to increase as credit quality declines.
Real-World vs. Risk-Neutral Probabilities

Why do we see such big differences between real-world and risk-neutral default probabilities?

1. Corporate bonds are relatively illiquid and bond traders demand an extra return to compensate for this.
2. The subjective default probabilities of bond traders may be much higher than those given in our first table. Bond traders may be allowing for depression scenarios much worse than anything seen during the period from 1970 to 2003.
3. Bonds do not default independently of each other.
4. Bond returns are highly skewed with limited upside. As a result bond traders may require an extra return for bearing unsystematic risk in addition to the systematic risk mentioned above.
Using Equity Prices to Estimate Default Probabilities

In order to get the default probabilities we rely on the company’s credit rating. Unfortunately, credit ratings are revised relatively infrequently. This leads analysts to argue that equity prices can provide more up-to-date information for estimating default probabilities.

Merton proposed a model where a company’s equity is an option on the assets of the company. Suppose that a firm has one zero-coupon bond outstanding and that the bond matures at time $T$. Define

- $V_0$ Value of company’s assets today
- $V_T$ Value of company’s assets at time $T$
- $E_0$ Value of company’s equity today
- $E_T$ Value of company’s equity at time $T$
- $D$ Amount of debt interest and principal due to be repaid at time $T$
- $\sigma_V$ Volatility of assets (assumed constant)
- $\sigma_E$ Instantaneous volatility of equity
Consider two cases:

- If $V_T < D$, it is (at least in theory) rational for the company to default on the debt at time $T$. The value of the equity is then zero.
- If $V_T > D$, the company should make the debt repayment at time $T$ and the value the equity at this time is $V_T - D$.

Merton's model, therefore, gives the value of the firm's equity at time $T$ as

$$E_T = \max\{V_T - D, 0\}$$

This shows that the equity is a call option on the value of the assets with a strike price equal to the repayment required on the debt.

Black-Scholes formula gives the value of the equity today as

$$E_0 = V_0 N(d_1) - D e^{-rT} N(d_2)$$

where

$$d_1 = \frac{\ln \frac{V_0}{D} + (r + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}}$$

$$d_2 = d_1 - \sigma \sqrt{T}$$

The value of the debt today is $V_0 - E_0$. 
• The risk-neutral probability that the company will default on the debt is

\[ N(-d_2) \]

• To calculate this, we require \( V_0 \) and \( \sigma_V \). Neither of these are directly observable. However if the company is publicly traded, we can observe \( E_0 \).

• This means (3) provides one condition that must be satisfied by \( V_0 \) and \( \sigma_V \).

• We can also estimate \( \sigma_E \). From Itô's lemma

\[ \sigma_E E_0 = \frac{\partial E}{\partial V} \sigma_V V_0 \]

or

\[ \sigma_E E_0 = N(d_1) \sigma_V V_0 \] (4)

This provides another equation that must be satisfied by \( V_0 \) and \( \sigma_V \).

This provides two nonlinear equations (3) and (4) to get \( V_0 \) and \( \sigma_V \).
Example

The value of a company’s equity is $3 million and the volatility of the equity is 80%.

- The debt that will have to be paid in 1 year is $10 million. The risk-free rate is 5% per annum. Thus
  \[ E_0 = 3 \quad \sigma_E = 0.80 \quad r = 0.05 \quad T = 1 \quad D = 10 \]
- Solving (3) and (4) yields
  \[ V_0 = 12.40 \quad \sigma_V = 0.2123 \]
  The parameter \( d_2 = 1.1408 \), so that the probability of default is \( N(-d_2) = 0.127 \) or 12.7%.
- The market value of the debt is \( V_0 - E_0 \) or 9.40. The present value of the promised payment on the debt is
  \[ 10e^{-0.05 \times 1} = 9.51 \]
- The expected loss on the debt is therefore, \( \frac{9.51-9.40}{9.51} = 1.2\% \) of its no-default value.
- Comparing this with the probability of default gives the expected recovery in the event of a default as \( \frac{12.7-1.2}{12.7} = 91\% \).
Default Correlation

A **default correlation** is used to describe the tendency for two companies to default at about the same time.

There are a number of reasons why default correlations exist.

- Companies in the same industry or the same geographic region tend to be affected similarly by external events and as a result may experience financial difficulties at the same time.
- Economic conditions generally cause average default rates to be higher in some years than in other years. A default by one company may cause a default by another.
- Default correlation means that credit risk cannot be complete diversified away and is the major reason why risk-neutral default probabilities are greater than real-world default probabilities.
- Default correlation is important to the determination of probability distributions for default losses from a portfolio of exposures to different counterparties.
Gaussian Copula Model for Time to Default

This is a popular reduced form default correlation model.

The model quantifies the correlation between times to default for two different companies. The model implicitly assumes that all companies will default eventually.

In any application of the model we are typically interested in the possibility of defaults over the next 1 year, 5 years, or 10 years. We are therefore only interested in the left tail of the distribution of the time to default.

The model can be used in conjunction with either real-world or risk-neutral default probabilities.

The left tail of the real-world probability distribution for the time to default of a company can be estimated from data produced by rating agencies such as that in our original table of defaults. The left tail of the risk-neutral probability distribution of the time to default can be estimate from bond prices using the approach using $Q$'s.
Define $t_1$ as the time to default of company 1 and $t_2$ as the time to default of company 2. If the probability distributions of $t_1$ and $t_2$ were normal, we could assume that the joint probability distribution of $t_1$ and $t_2$ is bivariate normal.

As it happens the probability distributions of a company's time to default is not even approximately normal. This is where a Gaussian copula model comes in. We transform $t_1$ and $t_2$ into new variables $x_1$ and $x_2$ using

$$x_1 = N^{-1}[Q_1(t_1)] \quad x_2 = N^{-1}[Q_2(t_2)]$$

where $Q_1$ and $Q_2$ are the cumulative probability distributions for $t_1$ and $t_2$, respectively, and $N^{-1}$ is the inverse of the cumulative normal distribution.

- The 5-percentile point in the probability distribution for $t_1$ is transformed to $x_1 = -1.645$ which is the 5-percentile point in the standard normal distribution.
- The 10-percentile point in the probability distribution for $t_1$ is transformed to $x_1 = -1.282$ which is the 10-percentile point in the standard normal distribution.

The $t_2$ to $x_2$ transformation is similar.
By construction $x_1$ and $x_2$ have normal distributions with mean zero and unit standard deviation. We assume that the joint distribution of $x_1$ and $x_2$ is bivariate normal with correlation $\rho_{12}$. This assumption is referred to as using a **Gaussian copula**.

The assumption is convenient because it mean that the joint probability distribution of $t_1$ and $t_2$ is fully defined by the cumulative default probability distributions $Q_1$ and $Q_2$ for $t_1$ and $t_2$, together with a single correlation parameter $\rho_{12}$.

The attraction of the Gaussian copula model is that it can be extended to many companies.

Suppose that we are considering $n$ companies and that $t_i$ is the time to default of the $i$th company.

We transform each $t_i$ into a new variable $x_i$ that has a standard normal distribution. The transformation is the percentile-to-percentile transformation

$$x_i = N^{-1} [Q_i(t_i)]$$

where $Q_i$ is the cumulative probability distribution for $t_i$. We ten assume that the $x_i$ are multivariate normal. The default correlation between $t_i$ and $t_j$ is measured as the correlation between $x_i$ and $x_j$. This is referred to a the copula correlation.

The Gaussian copula approach is a useful way representing the correlation structure between variable that are not normally distributed. It allows the correlation structure of the variables to be estimated separately from their marginal (unconditional) distributions. Although the variables themselves are not multivariate normal, the approach assumes that after a transformation is applied to each variable they are multivariate normal.
**Example:** Suppose that we wish to simulate defaults during the next 5 years in 10 companies. The copula default correlations between each pair of companies is 0.2. For each company the default during the next 1, 2, 3, 4, 5 years is 1%, 3%, 6%, 10%, 15%, respectively. When a Gaussian copula is used we sample from a multivariate normal distribution to obtain the $x_i, (1 \leq i \leq 10)$ with a pairwise correlation between the $x_i$ being 0.2. We then convert the $x_i$ and $t_i$, a time to default.

When the sample from the normal distribution is

- less than $N^{-1}(0.01) = -2.33$, a default takes place within the first year;
- when the sample is between -2.33 and $N^{-1}(0.03) = -1.88$, a default takes place during the second year;
- when the sample is between -1.88 and $N^{-1}(0.06) = -1.55$, a default takes place during the third year;
- when the sample is between -1.55 and $N^{-1}(0.10) = -1.28$, a default takes place during the fourth year;
- when the sample is between -1.28 and $N^{-1}(0.15) = -1.04$, a default takes place during the fifth year;
- when the sample is greater than -1.04, there is no default during the 5 years.
Using Factors to Define the Correlation Structure

To avoid defining a different correlation between $x_i$ and $x_j$ for each pair of companies $i$ and $j$ in the Gaussian copula model, a one-factor model is often used. The assumption is that

$$x_i = a_i M + \sqrt{1 - a_i^2} Z_i$$  \hspace{1cm} (5)

here $m$ is a common factor affecting defaults for all companies and $Z_i$ is a factor affecting only company $i$. The variables $M$ and the $Z_i$ have independent standard normal distributions. The $a_i$ are constant parameters between $-1$ and $+1$. The correlation between $x_i$ and $x_j$ is $a_i a_j$.

Suppose that the probability that company $i$ will default by a particular time $T$ is $Q_i(T)$. Under the Gaussian copula model, a default happens when $N(x_i) < Q_i(T)$ or $x_i < N^{-1}[Q_i(T)]$. 
From (5) we get the condition

\[ a_i M + \sqrt{1 - a_i^2} Z_i < N^{-1} [Q_i(T)] \]

or

\[ Z < \frac{N^{-1} [Q_i(T)] - a_i M}{\sqrt{1 - a_i^2}} \]

Conditional on the value of the factor \( M \), the probability of default is therefore

\[ Q_i(T|M) = N \left( \frac{N^{-1} [Q_i(T)] - a_i M}{\sqrt{1 - a_i^2}} \right) \] (6)

A particular case of the one-factor Gaussian model is where the probability distributions of default are the same for all \( i \) and the correlations between \( x_i \) and \( x_j \) is the same for all \( i \) and \( j \).

Suppose that \( Q_i(T) = Q_j(T) \) for all \( i \) and that the common correlations is \( \rho \), so that \( a_i = \sqrt{\rho} \) for all \( i \). Equation (6) becomes

\[ Q(T|M) = N \left( \frac{N^{-1} [Q(T)] - \sqrt{\rho} M}{\sqrt{1 - \rho}} \right) \] (7)
Credit Derivatives

Credit derivatives allow companies to trade credit risks in the same way that they trade market risks.

Companies can enter into credit derivative contracts to protect themselves from credit defaults of their lenders/borrowers.
Credit Default Swaps

The most popular credit derivative is a **credit default swap** (CDS)

- A CDS contract provides insurance against the risk of a default by a particular company.
- The company is known as the **reference entity** and a default by the company is known as a **credit event**.
- The buyer of the insurance obtains the right to sell bonds issued by the company for their face value when a credit event occurs.
- The sellers of the insurance agrees to buy the bonds for their face value when a credit event occurs.
- The total face value of the bonds that can be sold is known as the credit default swap’s **notional principal**.

Features of a CDS

- Buyers of a CDS makes periodic payments to the seller until the end of the life of the CDS or until a credit event occurs.
- The payments are typically made in arrears every quarter, every half year, or every year.
- The settlement, in the event of a default, involves either physical delivery of the bonds or a cash payment.
Example of a CDS

Consider a typical deal. There are two parties entering into a 5-year CDS on March 1, 2004. Assume that the notional principal is $100 million and the buyer agrees to pay 90 basis points (0.9%) annually for protection against default by the reference entity.

- If the reference entity does not default, the buyer receives no payoff and pays $900,000 on March 1 of each years, 2005, . . . , 2009.
- If there is a credit event, a substantial payoff is likely. Suppose that the buyer notifies the seller of a credit event on June 1, 2007, (quarter of the year into 2008).
- If the contract specifies physical settlement, the buyer has the right to sell bonds issued by the reference entity with a face value of $100 million for $100 million.
- If the contract requires cash settlement, an independent calculation agent will poll dealers to determine the mid-market value of the cheapest-to-deliver bond a predesignated number of days after the credit event. Suppose this bond is worth $35 per $100 of face value. the cash payoff would be $65 million.
The regular quarterly, semiannual, or annual payments form the buyer of protection to the seller of protection cease when there is a credit event. However, because these payments are made in arrears, a final accrual payment by the buyer is usually required.

In the example a buyer would be required to pay to the seller the amount of the annual payment accrued between Mar. 1, 2007 and June 1, 2007.

- The total amount paid per year, as a percent of the notional principal is known as the CDS spread.
- Many large banks are market makers in the credit default swap market.
- When quoting on a new 5-year CDS on Ford, a market maker might bid 250 basis points and offer 260 basis points.
Midterm

Concentrate on:

- Chapter 13 - Chapter 19 with emphasis on Chapters 17-19.