Lecture 22

Interest Rate Derivatives
Consider a European call option on a variable whose value is $V$ - which does not have to be the price of a traded security. Define

\[ T : \text{Time to maturity of the option} \]
\[ F : \text{Forward price of } V \text{ for a contract with maturity } T \]
\[ F_0 : \text{Value of } F \text{ at time zero} \]
\[ K : \text{Strike price of the option} \]
\[ P(t, T) : \text{Price at time } t \text{ of a zero-coupon bond paying } $1 \text{ at time } T \]
\[ V_T : \text{Value of } V \text{ at time } T \]
\[ \sigma : \text{Volatility of } F \]

We value the option by:

1. Assuming $\ln V_T$ is normal with mean $F_0$ and standard deviation $\sigma \sqrt{T}$
2. Discounting the expected payoff at the $T$-year rate (equivalent to multiplying the expected payoff by $P(0, T)$.)
The payoff from the option at time $T$ is $\max\{V_T - K, 0\}$. The lognormal assumption for $V_T$ implies that the expected payoff is

$$E(V_T)N(d_1) - KN(d_2)$$

where $E(V_T)$ is the expected value of $V_T$ and

$$d_1 = \frac{\ln \frac{E(V_T)}{K} + \frac{\sigma^2}{2}T}{\sigma \sqrt{T}}$$

$$d_2 = \frac{\ln \frac{E(V_T)}{K} - \frac{\sigma^2}{2}T}{\sigma \sqrt{T}}$$

Because we are assuming that $E(V_T) = F_0$, the value of the option is

$$c = P(0, T) [F_0N(d_1) - KN(d_2)]$$

where

$$d_1 = \frac{\ln \frac{F_0}{K} + \frac{\sigma^2}{2}T}{\sigma \sqrt{T}}$$

$$d_2 = \frac{\ln \frac{F_0}{K} - \frac{\sigma^2}{2}T}{\sigma \sqrt{T}}$$
This is the Black model and, importantly, it does not assume geometric Brownian motion for the evolution of either $V$ or $F$. All that we require is that $V_T$ be lognormal at time $T$. The parameter $\sigma$ is usually referred to as the volatility of $F$ or the forward volatility of $V$. Its only role is to define the standard deviation of $\ln V_T$ by means of the relationship

$$\text{Standard deviation of } \ln V_T = \sigma \sqrt{T}$$

The volatility parameter does not necessarily say anything about the standard deviation of $\ln V$ at times other than $T$.

**Delayed Payoff:**

We can extend Black’s model to allow for the situation where the payoff is calculated from the value of the variable $V$ at time $T$, but the payoff is actually made at some later time $T^*$. The expected payoff is discounted from time $T^*$ instead of time $T$ so that

$$c = P(0, T^*) \left[ F_0 N(d_1) - K N(d_2) \right]$$

where

$$d_1 = \frac{\ln \frac{F_0}{K} + \frac{\sigma^2}{2} T}{\sigma \sqrt{T}}$$

$$d_2 = \frac{\ln \frac{F_0}{K} - \frac{\sigma^2}{2} T}{\sigma \sqrt{T}}$$
Validity of Black’s Model:

Black’s model is appropriate when interest rates are assumed to be either constant or deterministic. In this case, the forward price of $V$ equals its future price and $E(S_T) = F_0$ in a risk neutral world.

When interest rates are stochastic, there are two aspects of the derivation of the formulas that are open to question:

1. Why do we set $E(V_T)$ equal to the forward price $F_0$ of $V$? This is not the same as the futures price.
2. Why do we ignore the fact that interest rates are stochastic when discounting?

These two assumptions offset each other. Black’s model has a sounder basis and wider applicability than first guessed.
Embedded Bond Options

Bond option is an option to buy or sell a particular bond by a particular date for a particular price.

In addition to trading in the OTC market, bond options are frequently embedded in bonds when they are issued to make them more attractive to either the issuer or potential purchasers.

**Embedded Bond Options**

One example of a bond with an embedded bond option is a **callable bond**

- Here a bond contains a provision allowing the issuing firm to buy back the bond at a predetermined price at certain times in the future.
- The holder of such a bond has sold a call option. The strike price or call price in the option is the predetermined price that must be paid by the issuer to the holder.
- Callable bonds cannot usually be called for the first few years of their life (lockout period).
- After that the call price is usually a decreasing function of time.
  - For example in a 10-year callable bond, there might be no call privileges for the first 2 years.
  - The issuer might have the right to buy the bond back at a price of 110 in years 3 and 4 of its life, at a price of 107.5 in years 5 and 6, at a price of 106 in years 7 and 8, and at a price of 103 in years 9 and 10.
- The value of the call option is reflected in the quoted yields on bonds. Bonds with call features generally offer higher yields than bonds with no call features.
Another example of an embedded bond option is a **puttable bond**.

- This type of bond contains a provision that allow the holder to demand early redemption at a predetermined price at certain times in the future.
- The holder of the bond has purchased a put option on the bond as well as the bond itself.
- Because the put option increases the value of the bond to the holder, bonds with put features provide lower yields than bonds with no put features.
- Example of a puttable bond is a 10-year bond where the holder has the right to be repaid at the end of 5 years. (also called **retractable bond**).
European Bond Options

Many OTC bond options and some embedded bond options are European. The assumption made
is that the bond price at the maturity of the option is lognormal. Then our model can be used to
price the option with $F_0$ equal to the forward bond price $F_B$. The variable $\sigma$ is set equal to the
forward bond price volatility, $\sigma_B$. Note $\sigma_B$ is defined so that $\sigma_B \sqrt{T}$ is the standard deviation of
the logarithm of the bond price at the maturity of the option. The equations for pricing a
European bond option are

$$c = P(0, T) \left[ F_B N(d_1) - K N(d_2) \right]$$
$$p = P(0, T) \left[ K N(-d_2) - F_B N(-d_1) \right]$$

where

$$d_1 = \frac{\ln \frac{F_B}{K} + \frac{\sigma_B^2 T}{2}}{\sigma_B \sqrt{T}} \quad d_2 = d_1 - \sigma_B \sqrt{T}$$

We note that we can compute $F_B$ as

$$F_B = \frac{B_0 - I}{P(0, T)}$$

where $B_0$ is the bond price at time zero and $I$ is the present value of the coupons that will be paid
during the life of the option.
Example: Consider a 10-month European call option on a 9.75-year bond with a face value of $1000.

When the option matures, the bond will have 8 years and 11 months remaining.

Suppose that the current cash bond price is $960, the strike price is $1000, the 10-month risk-free interest rate is 10% per annum, the volatility of the forward bond price in 10 months is 9% per annum.

The bond pays a semiannual coupon of 10% and coupon payments of $50 are expected in 3 months and 9 months (the accrued interest is $25 and the quoted bond price is $935). Suppose that the 3-month and 9-month risk-free interest rates are 9.0% and 9.5% per annum, respectively. The present value of the coupon payments is, therefore,

\[ 50e^{-0.25 \times 0.09} + 50e^{-0.75 \times 0.095} = 95.45 \]

The bond forward price is therefore

\[ F_B = (960 - 95.45) e^{0.1 \times 0.8333} = 939.68 \]
Recall that the cash price for a bond is the quoted price for a bond is...

1. If the strike price is the cash price that would be paid for the bond on exercise, the parameters for the call are $F_B = 939.68$, $K = 1000$, $P(0, T) = e^{-0.1 \times (10/12)} = 0.9200$, $\sigma_B = 0.09$ and $T = \frac{10}{12}$. The price of the call option is $9.49$.

2. If the strike price is the quoted price that would be paid for the bond on exercise, 1 month's accrued interest must be added to $K$ because the maturity of the option is 1 month after a coupon date. This produces a value for $K$ of

$$1000 + 50 \times 0.166667 = 1008.33$$

The values for the other parameters in the call equation are unchanged $F_B = 939.68$, $P(0, T) = e^{-0.1 \times (10/12)} = 0.9200$, $\sigma_B = 0.09$ and $T = 0.8333$. The price of the call option is $7.97$. 
The standard deviation of the logarithm of a bond's price changes as we look further ahead. The standard deviation is zero today because there is no uncertainty about the bond's price today.

It is also zero at the bond's maturity because we know that the bond's price will equal its face value at maturity. Between today and the maturity of the bond, the standard deviation first increases and then decreases.

**Figure 26.1** Standard deviation of logarithm of bond price at future times.

The volatility $\sigma_B$ that should be used when a European option on the bond is valued is

$$ \frac{\text{Standard dev. of logarithm of bond price at maturity of option}}{\sqrt{\text{Time to maturity of option}}} $$
What happens when we keep the underlying bond fixed and increase the life of the option. A typical pattern for $\sigma_B$ as a function of the life of the option. In general $\sigma_B$ declines as the life of the option increases.

**Figure 26.2** Variation of forward bond price volatility $\sigma_B$ with life of option when bond is kept fixed.

Yield Volatilities

The volatilities that are quoted for bond options are often yield volatilities rather than price volatilities.

The duration is used by the market to convert a quoted yield volatility into a price volatility. Suppose that $D$ is the modified duration of the bond underlying the option at the option maturity as defined earlier.
The relationship between the change $\Delta F_B$ in the forward bond price $F_B$ and the change $\Delta y_F$ in the forward yield $y_F$ is

$$\frac{\Delta F_B}{F_B} \approx -D \Delta y_F$$

or

$$\frac{\Delta F_B}{F_B} \approx -D y_F \frac{\Delta y_F}{y_F}$$

Volatility is a measure of the standard deviation of percentage changes in the value of a variable.

The equation suggests that the volatility of the forward bond price $\sigma_B$ used in Black’s model can be approximately related to the volatility of the forward bond yield $\sigma_y$ by

$$\sigma_B = D y_0 \sigma_y \tag{1}$$

where $y_0$ is the initial value of $y_F$.

- When a yield volatility is quoted for a bond option, the implicit assumption is usually that it will be converted to a price volatility using (1)
- This volatility will then be used in conjunction with the call or put equation to obtain a price.
• Suppose that the bond underlying a call option will have a modified duration of 5 years at option maturity, the forward yield is 8%, and the forward yield volatility quoted by a broker is 20%.

• This means that the market price of the option corresponding to the broker quote is the price given by our equation when the volatility variable $\sigma_B$ is

$$5 \times 0.08 \times 0.2 = 0.08$$

or 8% per annum.

• The figure shows that forward bond volatilities depend on the option considered. Forward yield volatilities as we have just defined them are more constant. This is why traders prefer them...

**Theoretical Justification for the Model**: One alternative to using risk-neutral valuation is the assumption that the world is forward risk neutral with respect to a zero-coupon bond maturing at time $T$. We found:

1. The current value of any security is its expected value at time $T$ in this world multiplied by the price of a zero-coupon bond maturing at time $T$.
2. The expected value of any variable (except an interest rate) a time $T$ in this world equals its forward value.
The first of these results shows that the price of a call option with maturity $T$ years on a bond is

$$c = P(0, T)E_T \left[ \max\{B_T - K, 0\} \right]$$

(2)

where $B_T$ is the bond price at time $T$ and $E_T$ denotes expected value in a world that is forward risk neutral with respect to a zero-coupon bond maturity at time $T$. The second result implies that

$$E_T(B_T) = F_B$$

(3)

Assuming the bond price is lognormal with the standard deviation of the logarithm of the bond price equal to $\sigma_B \sqrt{T}$, we find from a calculation that (2) becomes

$$c = P(0, T) \left[ E_T(B_T)N(d_1) - K N(d_2) \right]$$

where

$$d_1 = \ln \frac{E_T(B_T)}{K} + \frac{\sigma_B^2 T}{2} \frac{1}{\sigma_B \sqrt{T}}$$

$$d_2 = \ln \frac{E_T(B_T)}{K} - \frac{\sigma_B^2 T}{2} \frac{1}{\sigma_B \sqrt{T}}$$

Using (3) this reduces to the Black’s model formula. This shows that we can use today’s $T$-year maturity interest rate for discounting provided that we also set the expected bond price equal to the forward bond price.
Another interest rate option offered in the OTC market is an interest rate cap.

- Consider first a floating-rate note where the interest rate is reset periodically equal to LIBOR.
- The time between resets is known as the tenor. Suppose the tenor is 3 months. The interest rate on the note for the first 3 months is the initial 3-month LIBOR rate; the interest rate for the next 3 months is set equal to the 3-month LIBOR rate prevailing in the market at the 3-month point; and so forth.

An interest rate cap is designed to provide insurance against the rate of interest on the floating-rate note rising above a certain level, known as the cap rate.

- Suppose that the principal amount is $10 million, the tenor is 3 months, the life of the cap is 3 years, and the cap rate is 4%.
- The cap provides insurance against the interest on the floating rate note rising above 4%.
Assume that there are no day-count issues and there is exactly 0.25 year between each payment date.

- Suppose that on a particular reset date the 3-month LIBOR interest rate is 5%. The floating rate note would require

\[
0.25 \times 0.05 \times \$10,000,000 = \$125,000
\]

of interest to be paid 3 months later.
- With a 3-month LIBOR rate of 4% the interest payment would be

\[
0.25 \times 0.04 \times \$10,000,000 = \$100,000
\]

- Therefore, the cap provides a payoff of $25,000.
- At each reset date during the life of the cap we observe LIBOR. If LIBOR is less than 4%, there is no payoff from the cap three months later.
- If LIBOR is greater than 4%, the payoff is one quarter of the excess applied to the principal of $10 million. Note that caps are usually defined so that the initial LIBOR rate, even if it is greater than the cap rate, does not lead to a payoff on the first reset date.
Cap as a Portfolio of Interest Rate Options:

Consider a cap with a total life of $T$, a principal of $L$, and a cap rate of $R_K$.

Suppose that the reset dates are $t_1, \ldots, t_n$ and define $t_{n+1} = T$. Define $R_k$ as the interest rate for the period between time $t_k$ and $t_{k+1}$ observed at time $t_k$. The cap leads to a payoff at time $t_{k+1}$ of

$$L\delta_k \max\{R_k - R_K, 0\}$$

where $\delta_k = t_{k+1} - t_k$. Both $R_k$ and $R_K$ are expressed with a compounding frequency equal to the frequency of resets.

Equation (4) is a call option on the LIBOR rate observed at time $t_k$ with the payoff occurring at time $t_{k+1}$. The cap is a portfolio of $n$ such options. LIBOR rates are observed at times $t_1, \ldots, t_n$ and the corresponding payoffs occur at times $t_2, t_3, \ldots, t_{n+1}$. The $n$ call options underlying the cap are known as caplets.
Cap as a Portfolio of Bond Options:

An interest rate cap can also be characterized as a portfolio of put options on zero-coupon bonds with payoffs on the puts occurring at the time they are calculated.

The payoff in (4) at time $t_{k+1}$ is equivalent to

$$\frac{L\delta_k}{1 + R_k\delta_k} \max\{R_k - R_K, 0\}$$

at time $t_k$. This reduces to

$$\max\{L - \frac{L(1 - R_K\delta_k)}{1 + R_k\delta_k}, 0\}$$

(5)

The expression

$$\frac{L(1 + R_K\delta_k)}{1 + R_k\delta_k}$$

is the value at time $t_k$ of a zero-coupon bond that pays off $L(1 + R_K\delta_k)$ at time $t_{k+1}$. The expression in (5) therefore the payoff from a put option with maturity $t_k$ on a zero-coupon bond with maturity $t_{k+1}$ when the face value of the bond is $L(1 + R_K\delta_k)$ and the strike price is $L$. It follows that an interest rate cap can be regarded as a portfolio of European put options on zero-coupon bonds.
Floors and Collars

Interest rate floors and interest rate collars (sometimes called floor-ceiling agreements) are defined analoguously to caps.

- A **floors** provides a payoff when the interest rate on the underlying floating-ratenote falls below a certain rate.
- With the notation already introduced a floor provides a payoff at time $t_{k+1}$ of
  \[ L\delta_k \max\{R_K - R_k, 0\} \]
  Analogously to an interest rate cap, an interest rate floor is a portfolio of put options on interest rates or a portfolio of call options on zero-coupon bonds.
- Each of the individual options comprising a floor is known as a **floorlet**.
- A **collar** is an instrument designed to guarantee that the interest rate on the underlying floating rate note always lies between two levels. A collar is a combination of a long position in a cap and a short position in a floor. The cost of entering into the collar is then zero.
Valuation of Caps and Floors:

The caplet corresponding to the rate observed at time $t_k$ provides a payoff at time $t_{k+1}$ of

$$L \delta_k \max\{R_k - R_K, 0\}$$

If the rate $R_k$ is assumed to be lognormal with volatility $\sigma_k$, then our delayed payoff equation:

$$c = P(0, T^*) [F_0 N(d_1) - K N(d_2)]$$

in Black’s model implies the value of the caplet is

$$L \delta_k P(0, t_{k+1}) [F_k N(d_1) - R_K N(d_2)] \tag{6}$$

where

$$d_1 = \frac{\ln \frac{F_k}{R_K} + \frac{\sigma_k^2 t_k}{2}}{\sigma_k \sqrt{t_k}}$$

$$d_2 = d_1 - \sigma_k \sqrt{t_k}$$

and $F_k$ is the forward rate for the period of time between $t_k$ and $t_{k+1}$. The corresponding floorlet is

$$L \delta_k P(0, t_{k+1}) [R_K N(-d_2) - F_k N(-d_1)]$$
**Example:** Consider a contract that caps the LIBOR interest rate on $10,000 at 8% per annum (with quarterly compounding) for 3 months starting in 1 year.

- This is a caplet and could be one element of a cap.
- Suppose that the LIBOR / swap zero curve is flat at 7% per annum with quarterly compounding and the volatility of the 3-month forward rate underlying the caplet is 20% per annum.
- The continuously compounded zero rate for all maturities is 6.9394%
- We find $F_k = 0.07$, $\delta_k = 0.25$, $L = 10,000$, $R_K = 0.08$, $t_k = 1.0$, $t_{k+1} = 1.25$, $P(0, t_{k+1}) = e^{-0.069394 \times 1.25} = 0.9169$ and $\sigma_k = 0.20$. Also

  $$d_1 = \frac{\ln \frac{0.07}{0.08} + 0.2^2 \times 1/2}{0.2 \times 1} = -0.5677 \quad d_2 = d_1 - 0.2 = -0.7677$$

  so the caplet price is

  $$0.25 \times 10,000 \times 0.9169 \times [0.07N(-0.5677) - 0.08N(-0.7677)] = 5.162$$
Swap options (swaptions) are options on interest rate swaps.

- The holder has the right to enter into a certain interest rate swap at a certain time in the future.
- Many large financial institutions that offer interest rate swap contracts also sell or buy swaptions.
- Example of a swap: Consider a company that knows that in 6 months it will enter into a 5-year floating-rate loan agreement and knows that it will wish to swap the floating interest payments for fixed interest rate.

Swaptions provide companies with a guarantee that the fixed rate of interest they will pay on a loan at some future time will not exceed some level.
Valuation of European Swap Options

The swap rate for a particular maturity at a particular time is the fixed rate that would be exchanged for LIBOR in a newly issued swap with that maturity.

- Suppose that the swap rate for an \( n \)-year swap starting at time \( T \) proves to be \( s_T \).
- By comparing the cash flows on a swap where the fixed rate is \( s_T \) to the cash flows on the swap where the fixed rate is \( s_K \), we see that the payoff from the swaption consists of a series of cash flows equal to
  \[
  \frac{L}{m} \max\{s_T - s_K, 0\}
  \]
- The cash flows are received \( m \) times per year for the \( n \) years of the life of the swap.
- Suppose that the swap payment dates are \( T_1, T_2, \ldots, T_{mn} \), measured in years from today. Here \( T_k \approx T + \frac{k}{m} \). Each cash flow is the payoff from a call option on \( s_T \) with strike price \( s_K \).
- The value of the cash flow received at time \( T_i \) is
  \[
  \frac{L}{m} P(0, T_i) [s_0 N(d_1) - s_K N(d_2)]
  \]
  where
  \[
  d_1 = \frac{\ln \frac{s_0}{s_K} + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}}
  \]
  \[
  d_2 = d_1 - \sigma \sqrt{T}
  \]
\(s_0\) is the forward swap rate calculated at time zero and \(\sigma\) is the volatility of the forward swap rate (so that \(\sigma \sqrt{T}\) is the standard deviation of \(\ln S_T\)).

- The total value of the swaption is

\[
\sum_{i=1}^{mn} \frac{L}{m} P(0, T_i) \left[ s_0 N(d_1) - s_K N(d_2) \right]
\]

- Defining \(A\) as the value of a contract that pays \(\frac{1}{m}\) at time \(T_i\), \(1 \leq i \leq mn\), the value of the swaption becomes

\[
LA \left[ s_0 N(d_1) - s_K N(d_2) \right]
\]

where

\[
A = \frac{1}{m} \sum_{i=1}^{mn} P(0, T_i)
\]

- If the swaption gives the holder the right to receive a fixed rate of \(s_K\) instead of paying it, the payoff from the swaption is

\[
\frac{L}{m} \max\{s_K - s_T, 0\}
\]

This put option on \(s_T\). As before, the payoffs are received at time \(T_i\), then we get the value of the swaption is

\[
LA \left[ s_K N(-d_2) - s_0 N(-d_1) \right]
\]
Example: Suppose that the LIBOR yield curve is flat at 6% per annum with continuous
compounding

Consider a swaption that gives the holder the right to pay 6.2% in a 3-year swap starting in 5 years.

the volatility of the forward swap rate is 20%. Payments are made semiannually and the principal
is $100.

Then

\[
A = \frac{1}{2} \left[ e^{-0.06 \times 5.5} + e^{-0.06 \times 6} + e^{-0.06 \times 6.5} + e^{-0.06 \times 7} + e^{-0.06 \times 7.5} + e^{-0.06 \times 8} \right] = 2.0035
\]

The rate 6% per annum with continuous compounding translates into 6.09% with semiannual
compounding.

It follows that \( s_0 = 0.0609, s_K = 0.062, T = 5, \) and \( \sigma = 0.2, \) so that

\[
d_1 = \frac{\ln \frac{0.0609}{0.062} + \frac{0.2^2}{2} \times 5}{0.2 \sqrt{5}} = 0.1836 \quad d_2 = d_1 - 0.2 \sqrt{5} = -0.2636
\]

The value of the swaption is

\[
100 \times 2.0035 \left[ 0.0609 \times N(0.1836) - 0.062 \times N(-0.2636) \right] = 2.07
\]
or $2.07.
Theoretical Justification for the Swap Option Model

We can show that Black’s model for swap options is internally consistent by considering a world that is forward risk neutral with respect to the annuity $A$. Our analysis shows

1. The current value of any security is the current value of the annuity multiplied by the expected value of

   \[
   \frac{\text{Security price at time } T}{\text{Value of the annuity at time } T}
   \]

   in this world.

2. The expected value of the swap rate at time $T$ in this world equals the forward swap rate.

We have shown that the value of the value of the swaption is

\[
LAE_A \left[ \max\{s_T - s_K, 0\} \right]
\]
We then find that this equals

\[ LA \left[ E_A(s_T)N(d_1) - s_K N(d_2) \right] \left[ \max\{s_T - s_K, 0\} \right] \]

and

\[
\begin{align*}
    d_1 &= \frac{\ln \frac{E_A(s_T)}{s_K} + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}} \\
    d_2 &= \frac{\ln \frac{E_A(s_T)}{s_K} - \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}}
\end{align*}
\]

and the second result shows that

\[ s_0 = E_A(s_T) \]
Interest Rate Derivatives: Short Rate

So far we have modeled the pricing of interest rate options on the assumption that the probability distribution of an interest rate, a bond price, or some other variable at a future point in time is lognormal.

They are widely used for valuing caps, European bond options, and European swap options.

Problems: do not provide a description of how interest rates evolve through time. Not as useful for valuing interest rate derivatives such as American-style swap options, callable bonds, and structured notes.

Alternatives involve term structure model - describes all zero-coupon interest rates.
Background

The short rate $r$ at a time $t$ is the rate that applies to an infinitesimally short period of time at time $t$.

Called the **instantaneous short rate**. Bond prices, option , prices, and other derivative prices depend only on the process followed by $r$ in a risk-neutral world.

The process for $r$ in the real world is irrelevant. The risk-neutral world we consider here will be the traditional risk-neutral world where, in a very short time period between $t$ and $t + \Delta t$, the investors earn on average $r(t)\Delta t$.

All processes for $r$ that we present will be processes in this risk-neutral world.

- The value at time $t$ of an interest rate derivative that provides a payoff of $f_T$ at time $T$ is
  \[ \hat{E} \left[ e^{-\bar{r}(T-t)} f_T \right] \]
  where $\bar{r}$ is the average value of $r$ in the time interval between $t$ and $T$, and $\hat{E}$ denotes the expected value in the traditional risk-neutral world.
Let $P(t, T)$ as the price at time $t$ of a zero-coupon bond that pays off $1$ at time $T$. Then

$$P(t, T) = \hat{E} \left[ e^{-\bar{r}(T-t)} \right]$$

(9)

If $R(t, T)$ is the continuously compounded interest rate at time $t$ for a term of $T - t$ then

$$P(t, T) = e^{-R(t,T)(T-t)}$$

(10)

so that

$$R(t, T) = -\frac{1}{T-t} \ln P(t, T)$$

(11)

and so

$$R(t, T) = -\frac{1}{T-t} \ln \hat{E} \left[ e^{-\bar{r}(T-t)} \right]$$

(12)

This equation enables the term structure of interest rates at any given time to be obtained from the value of $r$ at that time and the risk-neutral process for $r$. It shows that once we have fully defined the process for $r$, we have fully defined everything about the initial zero curve and its evolution through time.
Equilibrium Models

Equilibrium models usually start with assumptions about economic variables and derive a process for the short rate $r$. They then explore what the process for $r$ implies about bond prices and option prices.

In a one-factor equilibrium model, the process for $r$ involves only one source of uncertainty. Usually the risk-neutral process for the short rate is described by an Itô process of the form

$$dr = m(r)dt + s(r)dz$$

The instantaneous drift, $m$, and instantaneous standard deviation, $s$, are assumed to be functions of $r$, but are independent of time.

The assumption of a single factor is not as restrictive as it might appear. A one-factor model implies that all rates move in the same direction over any short time interval, but not that they all move by the same amount.

The shape of the zero curve can therefore change with the passage of time.
Consider three one-factor equilibrium models:

\[ m(r) = \mu r; \quad s(r) = \sigma r \quad \text{Rendleman-Bartter model} \]
\[ m(r) = a(b - r); \quad s(r) = \sigma \quad \text{Vasicek model} \]
\[ m(r) = a(b - r); \quad s(r) = \sigma \sqrt{r} \quad \text{Cox-Ingersoll-Ross model} \]

**Rendleman-Bartter Model:** Here the risk-neutral process for \( r \) is

\[ dr = \mu r dt + \sigma r dz \]

where \( \mu \) and \( \sigma \) are constants. This means that \( r \) follows geometric Brownian motion. The process for \( r \) is of the same type as that assumed for a stock price. It can be represented using a binomial tree similar to the one used for stocks.

The assumption that the short-term interest rate behaves like a stock price is a natural starting pint, but is less than ideal.

One important difference between interest rates and stock prices is that interest rates appear to be pulled back to some long-run average level over time. This phenomenon is known as **mean reversion**. When \( r \) is high, mean reversion tends to cause it to have a negative drift; when \( r \) is low, mean reversion tends to cause it to have a positive drift.
The Rendleman-Bartter model does not incorporate mean reversion.

**Figure 28.1** Mean reversion.

![Diagram of interest rate trend over time, showing high interest rate with a negative trend and low interest rate with a positive trend, with a reversion level indicated.]

There are compelling economic arguments in favor of mean reversion. When rates are high, the economy tends to low down, and there is low demand for funds from borrowers. As a result, rates decline. When rates are low, there tends to be a high demand for funds on the part of borrowers and rates tend to rise.
In the Vasicek Model the risk-neutral process for $r$ is

$$dr = a(b - r)dt + \sigma dz$$

where $a$, $b$, and $\sigma$ are constants. This model incorporates mean reversion. The short rate is pulled to a level $b$ at rate $a$. Superimposed upon this reversion is a normally distributed stochastic term $\sigma dz$.

The expression for the price at time $t$ of a zero-coupon bond that pays $1$ at time $T$ is

$$P(t, T) = A(t, T)e^{-B(t,T)r(t)}$$

where $r(t)$ is the value of $r$ at time $t$,

$$B(t, T) = \frac{1 - e^{-a(T-t)}}{a}$$

and

$$A(t, T) = \exp \left[ \frac{(B(t, T) - T + t) \left(a^2b - \frac{\sigma^2}{2}\right)}{a^2} - \frac{\sigma^2B(t, T)^2}{4a} \right]$$
When $a = 0$, $B(t, T) = T - t$, and $A(t, T) = \exp \left[ \frac{\sigma^2(T-t)^3}{6} \right]$.

From the fact that $R(t, T) = -\frac{1}{T-t} \ln P(t, T)$ then

$$R(t, T) = -\frac{1}{T-t} \ln A(t, T) + \frac{1}{T-t} B(t, T) r(t)$$

showing that the entire term structure can be determined as a function of $r(t)$ once $a, b, \sigma$ are chosen. Some possible shapes are:
Cox-Ingersoll-Ross Model

In Vasicek’s model the short-term interest rate $r$ can become negative. In Cox-Ingersoll-Ross model, the interest rate is always positive and mean-reverting. Here

$$dr = a(b - r)dt + \sigma \sqrt{r}dz$$

This has the same mean-reverting properties as Vasicek, but the standard deviation is proportional to $\sqrt{r}$. So as short-term interest rate increases, the standard deviation also increases.

Then

$$P(t, T) = A(t, T)e^{-B(t,T)r(t)}$$

where $r(t)$ is the value of $r$ at time $t$,

$$B(t, T) = \frac{2 \left(e^{-\gamma(T-t)} - 1\right)}{(\gamma + a) \left(e^{-\gamma(T-t)} - 1\right) + 2\gamma}$$

and

$$A(t, T) = \left[\frac{2\gamma \exp\left(a+\gamma\frac{(T-t)}{2}\right)}{(\gamma + a) \left(e^{-a\gamma(T-t)} - 1\right) + 2\gamma}\right]^{\frac{2\sigma b}{\sigma^2}}$$

with $\gamma = \sqrt{a^2 + 2\sigma^2}$. 
No-Arbitrage Models

Equilibrium models so far presented do not automatically fit today’s term structure of interest rates.

By choosing the parameters judiciously, they can be made to provide an approximate fit to many of the term structures that are encountered in practice.

But the fit is not usually an exact one and in some cases no reasonable fit can be found. Most traders find this unsatisfactory.

A no-arbitrage model is a model designed to be exactly consistent with today’s term structure of interest rates. The essential difference between an equilibrium and a no-arbitrage model is that today’s term structure of interest rates is an output, whereas a no-arbitrage model uses today’s term-structure of interest rates as an input.