Interest Rate Derivatives: Short Rates
Black’s Model

Consider a European call option on a variable whose value is $V$ - which does not have to be the price of a traded security. Define

$T$ : Time to maturity of the option

$F$ : Forward price of $V$ for a contract with maturity $T$

$F_0$ : Value of $F$ at time zero

$K$ : Strike price of the option

$P(t, T)$ : Price at time $t$ of a zero-coupon bond paying $1$ at time $T$

$V_T$ : Value of $V$ at time $T$

$\sigma$ : Volatility of $F$

We value the option by:

1. Assuming $\ln V_T$ is normal with mean $F_0$ and standard deviation $\sigma \sqrt{T}$

2. Discounting the expected payoff at the $T$-year rate (equivalent to multiplying the expected payoff by $P(0, T)$.)

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The payoff from the option at time $T$ is $\max\{V_T - K, 0\}$. The lognormal assumption for $V_T$ implies that the expected payoff is

$$E(V_T)N(d_1) - KN(d_2)$$

where $E(V_T)$ is the expected value of $V_T$ and

$$d_1 = \frac{\ln \frac{E(V_T)}{K} + \frac{\sigma^2}{2}T}{\sigma \sqrt{T}}$$
$$d_2 = \frac{\ln \frac{E(V_T)}{K} - \frac{\sigma^2}{2}T}{\sigma \sqrt{T}}$$

Because we are assuming that $E(V_T) = F_0$, the value of the option is

$$c = P(0, T) [F_0N(d_1) - KN(d_2)]$$

where

$$d_1 = \frac{\ln \frac{F_0}{K} + \frac{\sigma^2}{2}T}{\sigma \sqrt{T}}$$
$$d_2 = \frac{\ln \frac{F_0}{K} - \frac{\sigma^2}{2}T}{\sigma \sqrt{T}}$$
This is the Black model and, importantly, it does not assume geometric Brownian motion for the evolution of either \( V \) or \( F \). All that we require is that \( V_T \) be lognormal at time \( T \). The parameter \( \sigma \) is usually referred to as the volatility of \( F \) or the forward volatility of \( V \). Its only role is to define the standard deviation of \( \ln V_T \) by means of the relationship

\[
\text{Standard deviation of } \ln V_T = \sigma \sqrt{T}
\]

The volatility parameter does not necessarily say anything about the standard deviation of \( \ln V \) at times other than \( T \).

**Delayed Payoff:**

We can extend Black’s model to allow for the situation where the payoff is calculated from the value of the variable \( V \) at time \( T \), but the payoff is actually made at some later time \( T^* \). The expected payoff is discounted from time \( T^* \) instead of time \( T \) so that

\[
c = P(0, T^*) \left[ F_0 N(d_1) - KN(d_2) \right]
\]

where

\[
d_1 = \ln \frac{F_0}{K} + \frac{\sigma^2}{2} T
\]

\[
d_2 = \ln \frac{F_0}{K} - \frac{\sigma^2}{2} T
\]
Validity of Black’s Model:

Black’s model is appropriate when interest rates are assumed to be either constant or deterministic. In this case, the forward price of V equals its future price and $E(S_T) = F_0$ in a risk neutral world.

When interest rates are stochastic, there are two aspects of the derivation of the formulas that are open to question

1. Why do we set $E(V_T)$ equal to the forward price $F_0$ of V? This is not the same as the futures price.
2. Why do we ignore the fact that interest rates are stochastic when discounting?

These two assumptions offset each other. Black’s model has a sounder basis and wider applicability than first guessed.
Interest Rate Caps and Floors

Another interest rate option offered in the OTC market is an interest rate cap.

- Consider first a floating-rate note where the interest rate is reset periodically equal to LIBOR.
- The time between resets is known as the tenor. Suppose the tenor is 3 months. The interest rate on the note for the first 3 months is the initial 3-month LIBOR rate; the interest rate for the next 3 months is set equal to the 3-month LIBOR rate prevailing in the market at the 3-month point; and so forth.

An interest rate cap is designed to provide insurance against the rate of interest on the floating-rate note rising above a certain level, known as the cap rate.

- Suppose that the principal amount is $10 million, the tenor is 3 months, the life of the cap is 3 years, and the cap rate is 4%.
- The cap provides insurance against the interest on the floating rate note rising above 4%.
Assume that there are no day-count issues and there is exactly 0.25 year between each payment date.

- Suppose that on a particular reset date the 3-month LIBOR interest rate is 5%. The floating rate note would require

  \[0.25 \times 0.05 \times 10,000,000 = 125,000\]

  of interest to be paid 3 months later.
- With a 3-month LIBOR rate of 4% the interest payment would be

  \[0.25 \times 0.04 \times 10,000,000 = 100,000\]

- Therefore, the cap provides a payoff of $25,000.
- At each reset date during the life of the cap we observe LIBOR. If LIBOR is less than 4%, there is no payoff from the cap three months later.
- If LIBOR is greater than 4%, the payoff is one quarter of the excess applied to the principal of $10 million. Note that caps are usually defined so that the initial LIBOR rate, even if it is greater than the cap rate, does not lead to a payoff on the first reset date.
Cap as a Portfolio of Interest Rate Options:

Consider a cap with a total life of $T$, a principal of $L$, and a cap rate of $R_K$.

Suppose that the reset dates are $t_1, \ldots, t_n$ and define $t_{n+1} = T$. Define $R_k$ as the interest rate for the period between time $t_k$ and $t_{k+1}$ observed at time $t_k$. The cap leads to a payoff at time $t_{k+1}$ of

$$L \delta_k \max\{R_k - R_K, 0\}$$

where $\delta_k = t_{k+1} - t_k$. Both $R_k$ and $R_K$ are expressed with a compounding frequency equal to the frequency of resets.

Equation (1) is a call option on the LIBOR rate observed at time $t_k$ with the payoff occurring at time $t_{k+1}$. The cap is a portfolio of $n$ such options. LIBOR rates are observed at times $t_1, \ldots, t_n$ and the corresponding payoffs occur at times $t_2, t_3, \ldots, t_{n+1}$. The $n$ call options underlying the cap are known as caplets.
Cap as a Portfolio of Bond Options:

An interest rate cap can also be characterized as a portfolio of put options on zero-coupon bonds with payoffs on the puts occurring at the time they are calculated.

The payoff in (1) at time $t_{k+1}$ is equivalent to

$$\frac{L\delta_k}{1 + R_k\delta_k} \max\{R_k - R_K, 0\}$$

at time $t_k$. This reduces to

$$\max\{L - \frac{L(1 - R_K\delta_k)}{1 + R_k\delta_k}, 0\} \quad (2)$$

The expression

$$\frac{L(1 + R_K\delta_k)}{1 + R_k\delta_k}$$

is the value at time $t_k$ of a zero-coupon bond that pays off $L(1 + R_K\delta_k)$ at time $t_{k+1}$. The expression in (2) therefore the payoff from a put option with maturity $t_k$ on a zero-coupon bond with maturity $t_{k+1}$ when the face value of the bond is $L(1 + R_K\delta_k)$ and the strike price is $L$. It follows that an interest rate cap can be regarded as a portfolio of European put options on zero-coupon bonds.
Floors and Collars

Interest rate floors and interest rate collars (sometimes called floor-ceiling agreements) are defined analoguously to caps.

- A **floors** provides a payoff when the interest rate on the underlying floating-rate note falls below a certain rate.
- With the notation already introduced a floor provides a payoff at time $t_{k+1}$ of
  \[ L\delta_k \max\{R_K - R_k, 0\} \]

  Analogously to an interest rate cap, an interest rate floor is a portfolio of put options on interest rates or a portfolio of call options on zero-coupon bonds.

- Each of the individual options comprising a floor is known as a **floorlet**.
- A **collar** is an instrument designed to guarantee that the interest rate on the underlying floating rate note always lies between two levels. A collar is a combination of a long position in a cap and a short position in a floor. The cost of entering into the collar is then zero.
Valuation of Caps and Floors:

The caplet corresponding to the rate observed at time $t_k$ provides a payoff at time $t_{k+1}$ of

$$L \delta_k \max\{R_k - R_K, 0\}$$

If the rate $R_k$ is assumed to be lognormal with volatility $\sigma_k$, then our delayed payoff equation:

$$c = P(0, T^*) [F_0 N(d_1) - KN(d_2)]$$

in Black’s model implies the value of the caplet is

$$L \delta_k P(0, t_{k+1}) [F_k N(d_1) - R_K N(d_2)]$$  \hspace{1cm} (3)

where

$$d_1 = \frac{\ln \frac{F_k}{R_K} + \frac{\sigma_k^2 t_k}{2}}{\sigma_k \sqrt{t_k}}$$

$$d_2 = d_1 - \sigma_k \sqrt{t_k}$$

and $F_k$ is the forward rate for the period of time between $t_k$ and $t_{k+1}$. The corresponding floorlet is

$$L \delta_k P(0, t_{k+1}) [R_K N(-d_2) - F_k N(-d_1)]$$
**Example:** Consider a contract that caps the LIBOR interest rate on $10,000 at 8% per annum (with quarterly compounding) for 3 months starting in 1 year.

- This is a caplet and could be one element of a cap.
- Suppose that the LIBOR / swap zero curve is flat at 7% per annum with quarterly compounding and the volatility of the 3-month forward rate underlying the caplet is 20% per annum.
- The continuously compounded zero rate for all maturities is 6.9394%.
- We find $F_k = 0.07$, $\delta_k = 0.25$, $L = 10,000$, $R_K = 0.08$, $t_k = 1.0$, $t_{k+1} = 1.25$, $P(0, t_{k+1}) = e^{-0.069394 \times 1.25} = 0.9169$ and $\sigma_k = 0.20$. Also

$$d_1 = \frac{\ln \frac{0.07}{0.08} + 0.2^2 \times 1/2}{0.2 \times 1} = -0.5677 \quad d_2 = d_1 - 0.2 = -0.7677$$

so the caplet price is

$$0.25 \times 10,000 \times 0.9169 \left[0.07N(-0.5677) - 0.08N(-0.7677)\right] = 5.162$$
Swap options (swaptions) are options on interest rate swaps.

- The holder has the right to enter into a certain interest rate swap at a certain time in the future.
- Many large financial institutions that offer interest rate swap contracts also sell or buy swaptions.
- Example of a swap: Consider a company that knows that in 6 months it will enter into a 5-year floating-rate loan agreement and knows that it will wish to swap the floating interest payments for fixed interest rate.

Swaptions provide companies with a guarantee that the fixed rate of interest they will pay on a loan at some future time will not exceed come level.
Valuation of European Swap Options

The swap rate for a particular maturity at a particular time is the fixed rate that would be exchanged for LIBOR in a newly issued swap with that maturity.

- Suppose that the swap rate for an \( n \)-year swap starting at time \( T \) proves to be \( s_T \).
- By comparing the cash flows on a swap where the fixed rate is \( s_T \) to the cash flows on the swap where the fixed rate is \( s_K \), we see that the payoff from the swaption consists of a series of cash flows equal to
  \[
  \frac{L}{m} \max\{s_T - s_K, 0\}
  \]
- The cash flows are received \( m \) times per year for the \( n \) years of the life of the swap.
- Suppose that the swap payment dates are \( T_1, T_2, \ldots, T_{mn} \), measured in years from today. Here \( T_k \approx T + \frac{k}{m} \). Each cash flow is the payoff from a call option on \( s_T \) with strike price \( s_K \).
- The value of the cash flow received at time \( T_i \) is
  \[
  \frac{L}{m} P(0, T_i) \left[ s_0 N(d_1) - s_K N(d_2) \right]
  \]
  where
  \[
  d_1 = \frac{\ln \frac{s_0}{s_K} + \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}}
  \]
  \[
  d_2 = d_1 - \sigma \sqrt{T}
  \]
$s_0$ is the forward swap rate calculated at time zero and $\sigma$ is the volatility of the forward swap rate (so that $\sigma \sqrt{T}$ is the standard deviation of $\ln S_T$).

- The total value of the swaption is
  \[
  \sum_{i=1}^{mn} \frac{L}{m} P(0, T_i) \left[ s_0 N(d_1) - s_K N(d_2) \right]
  \]

- Defining $A$ as the value of a contract that pays $\frac{1}{m}$ at time $T_i$, $1 \leq i \leq mn$, the value of the swaption becomes
  \[
  LA \left[ s_0 N(d_1) - s_K N(d_2) \right]
  \]
  where
  \[
  A = \frac{1}{m} \sum_{i=1}^{mn} P(0, T_i)
  \]

- If the swaption gives the holder the right to receive a fixed rate of $s_K$ instead of paying it, the payoff from the swaption is
  \[
  \frac{L}{m} \max\{s_K - s_T, 0\}
  \]
  This put option on $s_T$. As before, the payoffs are received at time $T_i$, then we get the value of the swaption is
  \[
  LA \left[ s_K N(-d_2) - s_0 N(-d_1) \right]
  \]
**Example:** Suppose that the LIBOR yield curve is flat at 6% per annum with continuous compounding.

Consider a swaption that gives the holder the right to pay 6.2% in a 3-year swap starting in 5 years. The volatility of the forward swap rate is 20%. Payments are made semiannually and the principal is $100.

Then

\[
A = \frac{1}{2} \left[ e^{-0.06 \times 5.5} + e^{-0.06 \times 6} + e^{-0.06 \times 6.5} + e^{-0.06 \times 7} + e^{-0.06 \times 7.5} + e^{-0.06 \times 8} \right] = 2.0035
\]

The rate 6% per annum with continuous compounding translates into 6.09% with semiannual compounding.

It follows that \( s_0 = 0.0609, \ s_K = 0.062, \ T = 5, \) and \( \sigma = 0.2, \) so that

\[
d_1 = \frac{\ln \frac{0.0609}{0.062} + 0.2^2}{0.2 \sqrt{5}} \times 5 = 0.1836 \quad d_2 = d_1 - 0.2 \sqrt{5} = -0.2636
\]

The value of the swaption is

\[
100 \times 2.0035 \left[ 0.0609 \times N(0.1836) - 0.062 \times N(-0.2636) \right] = 2.07
\]

or $2.07.
Example

Suppose that the LIBOR yield curve is flat at 8% with annual compounding. A swaption gives the holder the right to receive 7.6% in a 5-year swap starting in 4 years. Payments are made annually. The volatility of the forward swap rate is 25% per annum and the principal is $1 million.

We use Black’s model to price the swaption.

The payoff from the swaption is a series of five cash flows equal to

$$\text{max}\{0.076 - s_T, 0\}$$

in millions of dollars where $s_T$ is the five-year swap rate in four years. The value of an annuity that provides $1 per year at the end of years 5, 6, 7, 8, and 9 is

$$\sum_{i=5}^{9} \frac{1}{1.08^i} = 2.9348$$
The value of the swaption in millions of dollars is therefore,

\[ 2.9348 \times [0.076N(-d_2) - 0.08N(-d_1)] \]

where

\[ d_1 = \frac{\ln \frac{0.08}{0.076} + \frac{0.25^2 \times 4}{2}}{0.25\sqrt{4}} = 0.3526 \]

and \( d_2 = d_1 - 0.25\sqrt{4} = -0.1474 \). The value of the swaption is

\[ 2.9348 \times [0.076N(0.1474) - 0.08N(-0.3526)] = 0.039554 \]

or $39,554.
Interest Rate Derivatives: Short Rate

So far we have modeled the pricing of interest rate options on the assumption that the probability distribution of an interest rate, a bond price, or some other variable at a future point in time is lognormal.

They are widely used for valuing caps, European bond options, and European swap options.

Problems: do not provide a description of how interest rates evolve through time. Not as useful for valuing interest rate derivatives such as American-style swap options, callable bonds, and structured notes.

Alternatives involve term structure model - describes all zero-coupon interest rates.
The short rate \( r \) at a time \( t \) is the rate that applies to an infinitesimally short period of time at time \( t \).

Called the **instantaneous short rate**. Bond prices, option prices, and other derivative prices depend only on the process followed by \( r \) in a risk-neutral world.

The process for \( r \) in the real world is irrelevant. The risk-neutral world we consider here will be the traditional risk-neutral world where, in a very short time period between \( t \) and \( t + \Delta t \), the investors earn on average \( r(t)\Delta t \).

All processes for \( r \) that we present will be processes in this risk-neutral world.

- The value at time \( t \) of an interest rate derivative that provides a payoff of \( f_T \) at time \( T \) is

\[
\hat{E} \left[ e^{-\bar{r}(T-t)} f_T \right]
\]

where \( \bar{r} \) is the average value of \( r \) in the time interval between \( t \) and \( T \), and \( \hat{E} \) denotes the expected value in the traditional risk-neutral world.
Let $P(t, T)$ as the price at time $t$ of a zero-coupon bond that pays off $1$ at time $T$. Then

$$P(t, T) = \hat{E} \left[ e^{-\bar{r}(T-t)} \right]$$

(6)

If $R(t, T)$ is the continuously compounded interest rate at time $t$ for a term of $T - t$ then

$$P(t, T) = e^{-R(t, T)(T-t)}$$

(7)

so that

$$R(t, T) = -\frac{1}{T-t} \ln P(t, T)$$

(8)

and so

$$R(t, T) = -\frac{1}{T-t} \ln \hat{E} \left[ e^{-\bar{r}(T-t)} \right]$$

(9)

This equation enables the term structure of interest rates at any given time to be obtained from the value of $r$ at that time and the risk-neutral process for $r$.

It shows that once we have fully defined the process for $r$, we have fully defined everything about the initial zero curve and its evolution through time.
Equilibrium Models

Equilibrium models usually start with assumptions about economic variables and derive a process for the short rate $r$. They then explore what the process for $r$ implies about bond prices and option prices.

In a one-factor equilibrium model, the process for $r$ involves only one source of uncertainty. Usually the risk-neutral process for the short rate is described by an Itô process of the form

$$dr = m(r)dt + s(r)dz$$

The instantaneous drift, $m$, and instantaneous standard deviation, $s$, are assumed to be functions of $r$, but are independent of time.

The assumption of a single factor is not as restrictive as it might appear. A one-factor model implies that all rates move in the same direction over any short time interval, but not that they all move by the same amount.

The shape of the zero curve can therefore change with the passage of time.
Consider three one-factor equilibrium models:

\[ m(r) = \mu r; \quad s(r) = \sigma r \]  
Rendleman-Bartter model

\[ m(r) = a(b - r); \quad s(r) = \sigma \]  
Vasicek model

\[ m(r) = a(b - r); \quad s(r) = \sigma \sqrt{r} \]  
Cox-Ingersoll-Ross model

**Rendleman-Bartter Model**: Here the risk-neutral process for \( r \) is

\[ dr = \mu r dt + \sigma r dz \]

where \( \mu \) and \( \sigma \) are constants. This means that \( r \) follows geometric Brownian motion. The process for \( r \) is of the same type as that assumed for a stock price. It can be represented using a binomial tree similar to the one used for stocks.

The assumption that the short-term interest rate behaves like a stock price is a natural starting point, but is less than ideal.

One important difference between interest rates and stock prices is that interest rates appear to be pulled back to some long-run average level over time. This phenomenon is known as **mean reversion**. When \( r \) is high, mean reversion tends to cause it to have a negative drift; when \( r \) is low, mean reversion tends to cause it to have a positive drift.
The Rendleman-Bartter model does not incorporate mean reversion.

There are compelling economic arguments in favor of mean reversion. When rates are high, the economy tends to low down, and there is low demand for funds from borrowers. As a result, rates decline. When rates are low, there tends to be a high demand for funds on the part of borrowers and rates tend to rise.

![Mean reversion diagram](image)
In the Vasicek Model the risk-neutral process for \( r \) is

\[
dr = a(b - r)dt + \sigma dz
\]

where \( a \), \( b \), and \( \sigma \) are constants. This model incorporates mean reversion. The short rate is pulled to a level \( b \) at rate \( a \). Superimposed upon this reversion is a normally distributed stochastic term \( \sigma dz \).

The expression for the price at time \( t \) of a zero-coupon bond that pays $1 at time \( T \) is

\[
P(t, T) = A(t, T)e^{-B(t,T)r(t)}
\]

where \( r(t) \) is the value of \( r \) at time \( t \),

\[
B(t, T) = \frac{1 - e^{-a(T-t)}}{a}
\]

and

\[
A(t, T) = \exp \left[ \frac{(B(t, T) - T + t) \left( a^2 b - \frac{\sigma^2}{2} \right)}{a^2} - \frac{\sigma^2 B(t, T)^2}{4a} \right]
\]
When $a = 0$, $B(t, T) = T - t$, and $A(t, T) = \exp\left[\frac{\sigma^2(T-t)^3}{6}\right]$.

From the fact that $R(t, T) = -\frac{1}{T-t} \ln P(t, T)$ then

$$R(t, T) = -\frac{1}{T-t} \ln A(t, T) + \frac{1}{T-t} B(t, T) r(t)$$

showing that the entire term structure can be determined as a function of $r(t)$ once $a, b, \sigma$ are chosen. Some possible shapes are:

**Figure 28.2** Possible shapes of term structure when Vasicek’s model is used.
Cox-Ingersoll-Ross Model

In Vasicek’s model the short-term interest rate \( r \) can become negative. In Cox-Ingersoll-Ross model, the interest rate is always positive and mean-reverting. Here

\[
dr = a(b - r)dt + \sigma \sqrt{r}dz
\]

This has the same mean-reverting properties as Vasicek, but the standard deviation is proportional to \( \sqrt{r} \). So as short-term interest rate increases, the standard deviation also increases.

Then

\[
P(t, T) = A(t, T)e^{-B(t,T)r(t)}
\]

where \( r(t) \) is the value of \( r \) at time \( t \),

\[
B(t, T) = \frac{2 \left( e^{-\gamma(T-t)} - 1 \right)}{(\gamma + a) \left( e^{-\gamma(T-t)} - 1 \right) + 2\gamma}
\]

and

\[
A(t, T) = \left[ \frac{2\gamma e^{(a+\gamma)(T-t)/2}}{(\gamma + a) \left( e^{-a\gamma(T-t)} - 1 \right) + 2\gamma} \right]^{\frac{2ab}{\sigma^2}}
\]

with \( \gamma = \sqrt{a^2 + 2\sigma^2} \).
No-Arbitrage Models

Equilibrium models so far presented do not automatically fit today’s term structure of interest rates.

By choosing the parameters judiciously, they can be made to provide an approximate fit to many of the term structures that are encountered in practice.

But the fit is not usually an exact one and in some cases no reasonable fit can be found. Most traders find this unsatisfactory.

A no-arbitrage model is a model designed to be exactly consistent with today’s term structure of interest rates. The essential difference between an equilibrium and a no-arbitrage model is that today’s term structure of interest rates is an output, whereas a no-arbitrage model uses today’s term-structure of interest rates as an input.
One no-arbitrage model consists of a binomial tree of bond prices with two parameters: short-rate standard deviation and the market price of risk of the short rate.

Then

\[ dr = \theta(t)dt + \sigma dz \]

where \( \sigma \), the instantaneous standard deviation of the short rate, is constant and \( \theta(t) \) is a function of time chosen to ensure that the model fits the initial term structure.

The variable \( \theta(t) \) defines the average direction that \( r \) moves at time \( t \). This is independent of the level \( r \). The market-price of risk is not used in the calculation of the price of derivatives.

The variable \( \theta(t) \) can be calculated analytically. It is

\[ \theta(t) = F_t(0,t) + \sigma^2 t \]

where the \( F(0,t) \) is the instantaneous forward rate for a maturity \( t \) as seen at time zero and the subscript \( t \) denotes a partial derivative with respect to \( t \).

As an approximation \( \theta = F_t(0,t) \). This means that the average direction that the short rate will be moving in the future is approximately equal to the slope of the instantaneous forward curve.
The Ho-Lee model looks like

![Diagram of the Ho-Lee model](image)

The slope of the forward curve defines the average direction that the short rate is moving at any given time. Superimposed on this slope is the normally distributed random outcome.

The expression for the price of a zero-coupon bond at time $t$ in terms of the short rate is

$$P(t, T) = A(t, T)e^{-r(t)(T-t)}$$

where

$$\ln A(t, T) = \ln \frac{P(0, T)}{P(0, t)} + (T - t)F(0, t) - \frac{1}{2} \sigma^2 t(T - t)^2$$
In these equations, time zero is today. Times \( t \) and \( T \) are general times in the future with \( T \geq t \).

The equations, define the price of a zero-coupon bond at a future time \( t \) in terms of the short rate at time \( t \) and the prices of bonds today. The latter can be calculated from today’s term structure.

**Hull-White Model:** Hull-White model extends the Vasicek model to provide exact fit to the initial term structure. One version would be

\[
dr = \left[\theta(t) - ar\right] dt + \sigma dz
\]

where \( a \) and \( \sigma \) are constants. This is Ho-Lee with a mean-reversion at rate \( a \). The \( \theta(t) \) can be calculated from the initial term structure:

\[
\theta(t) = F_t(0, t) + aF(0, t) + \frac{\sigma^2}{2a} \left( 1 - e^{-2at} \right)
\]

The last term in this equation is usually small. If we ignore it, then the drift of the process for \( r \) at time \( t \) is
Thus on average $r$ follows the slope of the initial instantaneous forward rate curve. When it deviates from that curve, it reverts back to it at rate $a$, i.e.

$$F_t(0, t) + a [F(0, t) - r]$$

Bond prices at time $t$ in the Hull-White model are

$$P(0, t) = A(t, T)e^{-B(t, T)r(t)}$$

where

$$B(t, T) = \frac{1 - e^{-a(T-t)}}{a}$$
Some of the models presented allow for zero-coupon bonds to be valued analytically.

For the Vasicek, Ho-Lee, and Hull-White, the price at time $t = 0$ of a call option that matures at time $T$ on a zero-coupon bond maturing at time $s$ is

$$LP(0, s)N(h) - KP(0, T)N(h - \sigma_P)$$

where $L$ is the principal of the bond, $K$ is the strike price, and

$$h = \frac{1}{\sigma_P} \ln \left( \frac{LP(0, s)}{P(0, T)K} \right) + \frac{\sigma_P}{2}$$

the price of a put option on the bond is

$$KP(0, T)N(-h + \sigma_P) - LP(0, s)N(-h)$$

In the case of the Vasicek and Hull-White moels

$$\sigma_P = \frac{\sigma}{a} \left[ 1 - e^{-a(s-T)} \right] \sqrt{\frac{1 - e^{-2aT}}{2a}}$$

and for Ho-Lee $\sigma_P = \sigma(s - T)\sqrt{T}$. 

Options, Futures, Derivatives / April 23, 2008
Options on Coupon-Bearing Bonds

In a one-factor model of \( r \), all zero-coupon bonds move up in price when \( r \) decreases and all zero-coupon bonds move down in price when \( r \) increases.

As a result a one-factor model allows us to express a European option on a coupon-bearing bond as the sum of European options on zero-coupon bonds:

1. Calculate \( r^* \), the critical value of \( r \) for which the price of the coupon-bearing bond equals the strike price of the option on the bond at option maturity.
2. Calculate the prices of options on the zero-coupon bonds that comprise the coupon-bearing bond. Set the strike price of each option equal to the value of the corresponding zero-coupon bond will have at time \( T \) when \( r = r^* \).
3. Set the price of the option on the coupon-bearing bond equal to the sum of the prices on the options on zero-coupon bonds calculated in step 2.
Volatility Structures

The models give rise to different volatility environments:

- Ho-Lee: the volatility of the 3-month forward rate is the same for all maturities
- Hull-White: the effect of mean-reversion is to cause the volatility of the 3-month forward rate to be declining function of maturity.

**Figure 28.5** Volatility of 3-month forward rate as a function of maturity for (a) the Ho-Lee model, (b) the Hull-White one-factor model, and (c) the Hull-White two-factor model (when parameters are chosen appropriately).
We can try to use our knowledge of derivatives to help value real objects, such as buildings, land, and equipment.

Often there are options embedded in these investment opportunities, and often difficult to price well.

**Capital Investment Appraisal**

Traditionally one values a potential capital investment project is known as net present value (NPV). The NPV of a project is the present value of its expected future incremental cash flows.

The discount rate used to calculate the present value is a risk-adjusted discount rate, chosen to reflect the risk of the project.

As the riskiness of the project increases, the discount rate also increases.

**Example** Consider an investment that costs $100 million and will last 5 years. The expected cash inflow in each year is estimated to be $25 million. If the risk-adjusted discount rate is 12% (with cont. compounding),
the net present value of the investment is

\[-100 + 25e^{-0.12\times1} + 25e^{-0.12\times2} + 25e^{-0.12\times3} + 25e^{-0.12\times4} + 25e^{-0.12\times5} = -11.53\]

• A negative NPV indicates that the project will reduce the value of the company to its shareholders and should not be undertaken.
• A positive NPV indicates that the project should be undertaken because it will increase the value of the company.

The risk-adjusted discount rate should be the return required by the company. This can be calculated in a number of ways.

One approach often used involves the capital asset pricing model:

1. Take a sample of companies whose main line of business is the same as that of the project being contemplated.
2. Calculate the betas of the companies and average them to obtain a proxy beta for the project.
3. Set the required rate of return equal to the risk-free rate plus the proxy beta times the excess return of the market portfolio over the risk-free rate.

There are difficulties using this approach, since there are usually embedded options in each project. In particular a company may consider abandoning a certain plant upgrade, etc. This will be discussed further next week.
Recall the market price of risk for variable $\theta$:

$$
\lambda = \frac{\mu - r}{\sigma}
$$

where $r$ is the risk-free rate, $\mu$ is the return on a traded security dependent only on $\theta$, and $\sigma$ is its volatility. We get the same market price of risk $\lambda$ regardless of the traded security chosen.

Suppose a real asset depends on several variables $\theta_i$. Let $m_i$ and $s_i$ be the expected growth rate and volatility of $\theta_i$ so that

$$
\frac{d\theta_i}{\theta_i} = m_i dt + s_i dz_i
$$

where $z_i$ is the Wiener process. Define $\lambda_i$ as the market price of risk of $\theta_i$. We can extend risk-neutral valuation to show that any asset dependent on the $\theta_i$ can be valued by:

1. Reducing the expected growth rate of each $\theta_i$ from $m_i$ to $m_i - \lambda_i s_i$
2. Discounting cash flows at the risk-free rate.
Final Exam: May 7th

Homework: Due April 30, 5PM.
Graded: 27.10, 27.11