Lecture 4.

The Greeks and Volatility Smiles
Delta of a Portfolio

The delta of a portfolio of options or other derivatives dependent on a single asset whose price is $S$ is

$$\frac{\partial \Pi}{\partial S}$$

where $\Pi$ is the value of the portfolio.

- The delta of the portfolio can be calculated from the deltas of the individual options in the portfolio. If a portfolio consists of a quantity $w_i$ of option, the delta of the portfolio is given by

$$\Delta = \sum_{i=1}^{n} w_i \Delta_i$$

where $\Delta_i$ is the delta of the $i$th option.

- The formula can be used to calculate the position in the underlying asset necessary to make the delta of the portfolio zero. When this position has been taken, the portfolio is referred to as being **delta neutral**.
Theta

The theta $\Theta$ of a portfolio of options is the rate of change of the value of the portfolio with respect to the passage of time with all else remaining the same.

Theta is sometimes referred as the **time decay** of the portfolio For a European call option on a non-dividend-paying stock, then

$$\Theta(\text{call}) = \frac{\partial c}{\partial t} = -\frac{S_0 N'(d_1) \sigma}{2\sqrt{T}} - rK e^{-rT} N(d_2)$$

where $d_1$ and $d_2$ are defined as before and

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

For a European put option on the stock

$$\Theta(\text{put}) = -\frac{S_0 N'(d_1) \sigma}{2\sqrt{T}} + rK e^{-rT} N(-d_2)$$
For a European call on a dividend paying stock with rate $q$ is

$$\Theta(\text{call}) = -\frac{S_0 N'(d_1)e^{-qT}\sigma}{2\sqrt{T}} - rK e^{-rT} N(d_2)$$

where

$$d_1 =$$

for a European put on a dividend paying stock with rate $q$ is

$$\Theta(\text{call}) = -\frac{S_0 N'(d_1)e^{-qT}\sigma}{2\sqrt{T}} + rK e^{-rT} N(-d_2)$$
Gamma

The *gamma*, $\Gamma$ of a portfolio of options on an underlying asset is the rate of change of the portfolio’s delta with respect to the price of the underlying asset.

It is the second partial derivative of the portfolio with respect to asset price:

$$\Gamma = \frac{\partial^2 \Pi}{\partial S^2}$$

If gamma is small, delta changes slowly, and adjustments to keep a portfolio delta neutral need to be made only relatively infrequently.

If gamma is large in absolute terms, delta is highly sensitive to the price of the underlying asset.
Making a Portfolio Gamma Neutral

- A position in the underlying asset itself or a forward contract on the underlying asset both have zero gamma and cannot be used to change the gamma of a portfolio.
- What is required is a position in an instrument (such as an option) that is not linearly dependent on the underlying asset.
- Suppose that a delta-neutral portfolio has a gamma equal to $\Gamma$, and a traded option has a gamma equal to $\Gamma_T$.
- If the number of traded options added to the portfolio is $w_T$, the gamma of the portfolio is $w_T\Gamma_T + \Gamma$.
- Thus the position in the traded option necessary to make the portfolio gamma neutral is $-\frac{\Gamma}{\Gamma_T}$.
- Including the traded option is likely to change the delta of the portfolio, so the position in the underlying asset cannot be changed continuously when delta hedging is used.
- Delta neutrality provides protection against larger movements in this stock price between hedge rebalancing.
**Example:** Suppose a portfolio is delta neutral and has a gamma of -3000. The delta and gamma of a particular traded call option are 0.62 and 1.50, respectively.

The portfolio can be made gamma neutral by including in the portfolio a long position of

\[
\frac{3000}{1.5} = 2000
\]

in the call option.

However, the delta of the portfolio will then change from zero to \(2000 \times 0.62 = 1240\). So a quantity 1240 of the underlying asset must be sold from the portfolio to keep it delta neutral.
Vega

So far have assumed that the volatility of the asset underlying a derivative is constant.

Volatilities change over time. In particular the value of a derivative is liable to change because of movements in volatility as well as because of changes in the asset price and the passage of time.

We set **vega** of a portfolio of derivatives, $\mathcal{V}$, is the rate of change of the value of the portfolio with respect to the volatility of the underlying asset:

$$\mathcal{V} = \frac{\partial \Pi}{\partial \sigma}$$

- If vega is high in absolute terms, then the portfolio’s value is very sensitive to small changes in volatility.
- If vega is low in absolute terms, then volatility changes have little impact on the value of the portfolio.
Calculating vega

For a European call or put on a non-dividend-paying stock, vega is given by

\[ \mathcal{V} = S_0 \sqrt{T} N'(d_1) \]

where \( d_1 = \frac{\ln \frac{S}{K} + (r + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}} \).

For a European call or put on a dividend-paying stock with yield \( q \), the vega is

\[ \mathcal{V} = S_0 \sqrt{T} N'(d_1) e^{-qT} \]

where \( d_1 = \frac{\ln \frac{S}{K} + (r - q + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}} \).

When the asset is a stock index, \( q \) is the dividend yield. When it is a currency contract, then set \( q \) to be the risk-free foreign rate \( r_f \). When it is a futures contract, \( S_0 = F_0 \) and \( q = r \).
• Confusing to compute the vega of a portfolio since we derived Black-Scholes by assuming constant volatility

• $\mathcal{V}$ neutrality protects from large changes in the volatility $\sigma$, whereas $\Gamma$ neutrality protects against large swings in the price of the underlying asset between hedge rebalancing.
Realities of Hedging

So it is possible to design a portfolio with zero delta, zero gamma, and zero vega.

Is this done in the "real world"?

Large portfolios usually work to maintain zero delta, but maintaining zero gamma and zero vega is much more difficult, since we need efficient access to large positions of nonlinear derivatives.

If the positions are too large, then even maintain zero delta is difficult. But moderately sized portfolios it is reasonable, since profits of trading cover the cost of daily rebalancing.
Portfolio Insurance

In general a portfolio manager wishes to acquire a put option to protect against large declines while achieving gains if the market appreciates.

One approach is to buy put options on a market index. Another approach is to create the put synthetically.

To create a synthetic put option, one maintains a position in the underlying asset so that the delta of the position is equal to the delta of the required option.

This can be more attractive than buying the put from the market:

- Options markets do not always have the liquidity to absorb trades that managers of large funds would like to have access to.
- Fund managers often require strike prices and exercise dates that are different from those available from the exchange-traded markets.
How to synthetically create the put?

The option can be created by trading the portfolio or by trading in index futures contracts.

- Consider the first approach - creating a put option by trading the portfolio. Recall the delta of a European put on the portfolio is

\[
\Delta = e^{-qT} \left[ N(d_1) - 1 \right]
\]  

(1)

where

\[
d_1 = \frac{\ln S_0 + (r - q + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}}
\]

with \( S_0 \) the value of the portfolio, \( K \) the strike price, \( r \) the risk-free rate, \( q \) the dividend yield on the portfolio, \( \sigma \) the volatility of the portfolio, and \( T \) the life of the option.

Assume that the volatility of the portfolio is \textbf{beta} times the volatility of a well-diversified market index.

Therefore, to create the put option synthetically, the fund manager should ensure that at any give time a proportion

\[
e^{-qT} \left[ N(d_1) - 1 \right]
\]

of the stocks in the original portfolio has been sold and the proceeds invested in riskless assets.
As the value of the original portfolio declines, the delta of the put given by (1) becomes more negative and the proportion of the original portfolio sold must be increased.

As the value of the original portfolio increases, the delta of the put becomes less negative and the proportion of the original portfolio sold must be decreased (and shares purchased)

- This strategy to create portfolio insurance entails dividing funds between the stock portfolio on which the insurance is required and riskless assets.
- As the value of the stock portfolio increases, riskless assets are sold and the position in the stock portfolio is increased.
- As the value of the stock portfolio decreases, the position in the stock portfolio is decreased and riskless assets are purchased.
- Insurance costs arise as the fact that selling occurs after a decline in the market and buying occurs after a rise in the market.
Stock Market Volatility

Note that the portfolio insurance created by synthetic means (selling portfolio at downswings and purchasing on upswings), can accentuate trends:

If a market declines, portfolio managers sell stock or sell index futures contracts. Both actions accentuate the decline. Sale of stock is liable to drive down market indices further.

This creates selling pressure on stocks - cause of 1987 stock market crash?

Whether portfolio insurance schemes affect volatility depends on how easily a market can absorb the trades that are generated by portfolio insurance.

If portfolio insurance trades are a small fraction of all trades, there is like no effect. As portfolio insurance becomes more popular, it is liable to have a destabilizing effect.
Volatility Smiles

How close are market prices to those predicted by Black-Scholes? Are Black-Scholes formulas used to price options?

Not entirely. Traders typically allow for volatility to depend on price strike price and time to maturity.

Plot of implied volatility of an option as a function of strike price is known as a volatility smile.
Implied Volatility

There are two ways to think about volatility:

- From price changes, we can compute the volatility via standard deviation.
- Another method is to consider data used in Black-Scholes:

\[
c = S_0 N(d_1) - K e^{-rT} N(d_2)
\]

with

\[
d_1 = \frac{\ln \frac{S_0}{K} + \left( r + \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}}
\]

\[
d_2 = \frac{\ln \frac{S_0}{K} + \left( r - \frac{\sigma^2}{2} \right) T}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T}
\]

If we know \( S_0, K, r, \) and \( c, \) then we can solve, implicitly, for \( \sigma. \) The resulting \( \sigma \) is called the \textbf{implied volatility}. 
We now consider our first volatility smile. This is a graph of volatility as a function of strike price. We assumed for Black-Scholes that this is a constant function...

On the other hand traders use the following volatility smile

- Volatility is relatively low for at-the-money options.
- Volatility is relatively high for the more in-the-money or out-of-the-money the strike price is.

**Figure 16.1** Volatility smile for foreign currency options.
The associated probability distribution should no longer be lognormal, since the crucial ingredient to $S$ being lognormal was

$$\frac{dS}{S} = \mu dt + \sigma \epsilon dz$$

Now $\sigma$ is a function....

The *implied distribution* turns out to be
The distribution with the same mean and same standard deviation has

- fat tails
- steeper

Why does this distribution hold? Consider deep out-of-the-money call option with high strike price $K_2$.

- The option pays off only if the exchange rate proves to be above $K_2$.
- The probability of this is higher for the implied probability distribution than for the lognormal distribution. Expect the implied distribution to give a relatively high price for this option.
- A relatively high price should lead to a relatively high implied volatility.

Next consider a deep in-the-money put option with a low strike price $K_1$.

- The option pays off only if the exchange rate proves to be below $K_1$.
- The probability of this is higher for implied probability distribution than for the lognormal distribution.
- Expect the implied distribution to give a relatively high price, and a relatively high implied volatility for this option too.
Empirical Tests of Volatility Smile

So the simile used by traders shows that they believe that the lognormal distribution understates the probability of extreme movements in exchange rates.

Consider the following data. We can measure the number of days in which the daily change of the exchange rate exceeds a certain number of standard deviations.

<table>
<thead>
<tr>
<th></th>
<th>Real world</th>
<th>Lognormal model</th>
</tr>
</thead>
<tbody>
<tr>
<td>&gt; 1 SD</td>
<td>25.04</td>
<td>31.73</td>
</tr>
<tr>
<td>&gt; 2 SD</td>
<td>5.27</td>
<td>4.55</td>
</tr>
<tr>
<td>&gt; 3 SD</td>
<td>1.34</td>
<td>0.27</td>
</tr>
<tr>
<td>&gt; 4 SD</td>
<td>0.29</td>
<td>0.01</td>
</tr>
<tr>
<td>&gt; 5 SD</td>
<td>0.08</td>
<td>0.00</td>
</tr>
<tr>
<td>&gt; 6 SD</td>
<td>0.03</td>
<td>0.00</td>
</tr>
</tbody>
</table>

We can see there are much fatter tails in the real world - hence more likely to have a large movements.
Why?

Why is there a **smile** in the foreign currency option? We need two conditions to hold for the lognormal distribution to hold:

- Volatility of the asset is constant
- Price of the asset changes smoothly with no jumps

**Neither** of these two assumptions hold for an exchange rate.

Volatility of an exchange rate is not constant and exchange rates frequently jump. Both of these tend to increase the likelihood of extreme events.

The percentage impact of a nonconstant volatility on prices becomes more pronounced as the maturity of the option is increased, but the volatility smile created by nonconstant volatility usually becomes less pronounced as the maturity of the option increases.

**Business Snapshot** Traders in the mid 80’s understood that there are heavier tails and purchased cheap put /call options and waited. These options occurred with greater frequency than lognormal and were too cheap (from Black-Scholes pricing). These traders made a lot of money! By late 80’s the volatility smile was introduced.
Before the crash of 1987, stocks were generally assumed to follow the lognormal distribution.

After the crash, a volatility smile for equity options was introduced by Rubinstein and Jackwerth-Rubinstein.

The volatility smile or **volatility skew**, has the form of a downward sloping parabola.
• Volatility to price a **low-strike-price** option (deep-out-of-the-money put or deep-in-the-money call) is significantly higher than that used to price a **high-strike-price** option (deep-in-the-money put or deep-out-of-the-money call).

• The volatility smile for equity options corresponds to the implied probability distribution given by below:

![Diagram of implied and lognormal distributions](Image)  
compared to the corresponding lognormal distribution.
Why consistent?

- Consider a deep-out-of-the-money call option with a strike price of $K_2$. This has a lower price when the implied distribution is used than when the lognormal distribution is used.
- This is because the option pays off only if the stock price proves to be above $K_2$, and the probability of this is lower for the implied probability distribution than for the lognormal distribution.
- Thus expect the implied distribution to give a relatively low price for the option. A low price leads to a relatively low implied volatility which is what is observed.

Consider now a deep-out-of-the-money put option with strike price $K_1$.

- The option pays off only when the stock price is below $K_1$. The probability of this is higher for implied probability distribution.
- Expect the implied distribution to give a relatively higher price, and a relatively high price implies higher implied volatility.
Why the smile?

Reasons for the equity volatility smile

- **Fear of a crash.** Traders are concerned about the possibility of a crash, so they price the option accordingly.
- **Leverage.** As a company’s equity declines in value, the equity becomes more risky and its volatility increases. As a company’s equity increases in value, the equity becomes less risky and its volatility decreases.
Computing the Probability Distributions

Recall that the European call option on an asset with strike price $K$ and maturity $K$ is given by

$$c = e^{rT} \int_{S_T=K}^{\infty} (S_T - K)g(S_T) dS_T$$

where $r$ is the constant interest rate, $S_T$ is the asset price at time $T$, and $g$ is the risk-neutral probability density function of $S_T$.

Differentiating with respect to $K$ yields

$$\frac{\partial c}{\partial K} = -e^{rT} \int_{S_T=K}^{\infty} g(S_T) dS_T$$

and again

$$\frac{\partial^2 c}{\partial K^2} = e^{-rT} g(K)$$

Thus we get

$$g(K) = e^{rT} \frac{\partial^2 c}{\partial K^2} \approx e^{rT} \frac{c(K - \delta) - 2c(K) + c(K + \delta)}{\delta^2}$$

This implies that we can estimate the probability curve from the volatility smile curve.
Traders also consider the volatility term structure when pricing options.

In other words the volatility used to price an at-the-money option depends on the maturity of the option.

- Volatility tends to be an increasing function of maturity when short-dated volatilities are historically low, since there is expectation that volatility will increase.
- Volatility tends to be a decreasing function of maturity when short-dated volatilities are historically high, since there is expectation that volatility will decrease.

**Volatility surfaces** combine volatility smiles with the volatility term structure to tabulate the volatilities appropriate for pricing an option with any strike price and any maturity.

Consider an example

<table>
<thead>
<tr>
<th>Strike Price</th>
<th>0.90</th>
<th>0.95</th>
<th>1.00</th>
<th>1.05</th>
<th>1.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 month</td>
<td>14.2</td>
<td>13.0</td>
<td>12.0</td>
<td>13.1</td>
<td>14.5</td>
</tr>
<tr>
<td>3 month</td>
<td>14.0</td>
<td>13.0</td>
<td>12.0</td>
<td>13.1</td>
<td>14.2</td>
</tr>
<tr>
<td>6 month</td>
<td>14.1</td>
<td>13.3</td>
<td>12.5</td>
<td>13.4</td>
<td>14.3</td>
</tr>
<tr>
<td>1 year</td>
<td>14.7</td>
<td>14.0</td>
<td>13.5</td>
<td>14.0</td>
<td>14.8</td>
</tr>
<tr>
<td>2 year</td>
<td>15.0</td>
<td>14.4</td>
<td>14.0</td>
<td>14.5</td>
<td>15.1</td>
</tr>
<tr>
<td>5 year</td>
<td>14.8</td>
<td>14.6</td>
<td>14.4</td>
<td>14.7</td>
<td>15.0</td>
</tr>
</tbody>
</table>
Some of the implied probability entries are computed using market data and Black-Scholes.

The rest are found via simple linear interpolation.

When a new option has to be valued, traders look up the appropriate volatility in the table.

**Example:** Consider the value of a 9-month option with a strike price of 1.05. The trader would interpolate between 13.4 and 14.0 in the table to obtain a volatility of

\[
\frac{1}{2}(14.0 + 13.4) = 13.7\%
\]

The volatility of 13.7% would then be used in the Black-Scholes formula or binominal tree.

- The shape of the volatility smile depends on the option maturity. The smile tends to become less pronounced as the option maturity increases.
Role of the Model

How important is the pricing model if traders are prepared to use a different volatility for every option?

We can think of Black-Scholes as an interpolation tool used by traders to ensure that an option is priced consistently with the market prices of other actively traded options.

If traders used a different model, then volatility surfaces and the shape of the smile would change, but the dollar prices found in the market should not change appreciably.
Consider now an unusual situation which can greatly affect the shape of a volatility smile.

- Suppose that a pharmaceutical stock is currently at $50 and an announcement on a pending lawsuit against it is expected in a few days. The news is expected to send the stock either up by $8 or down by $8.
- The probability distribution of the stock price in one month should consist of a mixture of two lognormal distributions, the first corresponding to favorable news, the second to unfavorable news.

Figure 16.5 Effect of a single large jump. The solid line is the true distribution; the dashed line is the lognormal distribution.
The solid line show the mixtures of the two lognormal distributions, whereas the dashed line is the single lognormal distribution.

- The true probability distribution is bimodal. We can see this as an example of the binomial tree method:

![Figure 16.6 Change in stock price in 1 month.](image)

Suppose that the stock price is currently $50 and that it is known that in 1 month the price will be either $42 or $58. Suppose the risk-free rate is 12%. The option can be valued using binomial model. Here

\[ u = \frac{58}{50} = 1.16 \quad d = 0.84 \quad a = e^{rT} = 1.0101 \]

so

\[ p = \frac{a - d}{u - d} = \frac{1.0101 - 0.84}{1.16 - 0.84} = 0.5314 \]
Now we compute for different strike prices the value of the European call or European put.

- Consider the European call with a strike price of 44. Then from binomial tree operation:

\[
c = e^{-rT}[pf_u + (1 - p)f_d]
\]

\[
= e^{-0.12 \times \frac{1}{12}} \left[ 0.5314 \times (58 - 44) + (1 - 0.5314) \times 0 \right]
\]

\[
= 7.37
\]

Then we solve for the volatility in Black-Scholes and get \( \sigma = 58.8\% \).

- Consider the European call with a strike price of 48. Then from binomial tree operation:

\[
c = e^{-rT}[pf_u + (1 - p)f_d]
\]

\[
= e^{-0.12 \times \frac{1}{12}} \left[ 0.5314 \times (58 - 48) + (1 - 0.5314) \times 0 \right]
\]

\[
= 5.26
\]

Then we solve for the volatility in Black-Scholes and get \( \sigma = 69.5\% \).
Continue with other strike prices and get the implied volatilities, which yields the chart:

<table>
<thead>
<tr>
<th>Strike price</th>
<th>Call price</th>
<th>Put price</th>
<th>Implied volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td>42</td>
<td>8.42</td>
<td>0.00</td>
<td>0.0</td>
</tr>
<tr>
<td>44</td>
<td>7.37</td>
<td>0.93</td>
<td>58.8</td>
</tr>
<tr>
<td>46</td>
<td>6.31</td>
<td>1.86</td>
<td>66.6</td>
</tr>
<tr>
<td>48</td>
<td>5.26</td>
<td>2.78</td>
<td>69.5</td>
</tr>
<tr>
<td>50</td>
<td>4.21</td>
<td>3.71</td>
<td>69.2</td>
</tr>
<tr>
<td>52</td>
<td>3.16</td>
<td>4.64</td>
<td>66.1</td>
</tr>
<tr>
<td>54</td>
<td>2.10</td>
<td>5.57</td>
<td>60.0</td>
</tr>
<tr>
<td>56</td>
<td>1.05</td>
<td>6.50</td>
<td>49.0</td>
</tr>
<tr>
<td>58</td>
<td>0.00</td>
<td>7.42</td>
<td>0.0</td>
</tr>
</tbody>
</table>
Graphing the implied volatility as a function of Strike Price yields a volatility frown.

Therefore, nonstandard volatility smiles can result from nonstandard expected behavior of a stock process.
Summary

- Black-Scholes model and its extensions assume that the probability distribution of the underlying asset at any given future time is lognormal.
- This assumption is not made by traders.
- Traders assume a probability distribution of an equity price has a heavier left tail and a less heavy right tail than the lognormal distribution.
- Traders also assume that the probability distribution of an exchange rate has a heavier right tail and heavier left tail than the lognormal distribution.
- Traders use volatility smiles to allow for nonlognormality.
- The volatility smile defines the relationship between the implied volatility of an option and its strike price.
- For equity options, the volatility smile tends to be downward sloping.
  - Out-of-the-money puts and in-the-money calls tend to have high implied volatility.
  - Out-of-the-money calls and in-the-money puts tend to have low implied volatility.
- Foreign currency options, the volatility smile is $U$-shaped. Both out-of-money and in-the-money options have higher implied volatilities than at-the-money options.
- Often traders use volatility term structure. The implied volatility of an option depends on the duration of the option.
- When volatility smiles and volatility term structures are used together, we get a volatility surface.
- This defines volatility as a function of both strike price and time to maturity.
Basic Numerical Procedures

We cover a couple of numerical methods that are used to price derivatives.

This is useful when there is no closed formula for the price, such as for American options.

- Generalized Binomial Trees
- Monte Carlo Methods for Black-Scholes
- Finite Difference Methods
Binomial Trees

Black-Scholes theory provides exact formulas for the pricing of European options under ideal situations.

American options do not have such a nice representation. Binomial trees are very useful for such derivatives:

Recall the Binomial Tree setup:

- Assume that in a short period of time $\Delta t$ the stock either rises to $Su$ or drops to $Sd$.
- Thus $u > 1$ and $d < 1$. The probability of the up movement is $p$ and the probability of a down movement is $1 - p$.
- Assume that the world is risk neutral:
  - Assume that the expected return from all traded assets is the risk-free interest rate
  - Value payoffs from the derivative by calculating their expected values and discounting at the risk-free interest rate.
• The values $p, u, d$ must give correct values for the mean and variance of asset price changes during the time interval $\Delta t$.

• Since we assume the risk-neutral world hypothesis holds, the expected return from the asset is the risk-free interest rate, $r$.

• Suppose that the asset provides a yield of $q$, then the expected return of the capital gains must be $r - q$. Therefore, the expected value of the asset price at the end of the time interval of length $\Delta t$ becomes

$$Se^{(r-q)\Delta t}$$

where $S$ is the value of the stock at the start of the time period.

Therefore, we find the expected value of capital gains increase is

$$Se^{(r-q)\Delta t} = pSu + (1 - p)Sd$$

or

$$e^{(r-q)\Delta t} = pu + (1 - p)d \quad (2)$$

independent of the stock price. Here we used the Mean Growth to determine a relationship between $p, u, d$. 

We now use the volatility. The variance of a variable $\text{var } Q = E(Q^2) - E(Q)^2$.

The variance over a time interval $\Delta t$ then $\sigma^2 \Delta t$. We compute

$$\sigma^2 \Delta t = pu^2 + (1 - p)d^2 - e^{2(r-q)\Delta t}$$

From (2) we find $p = \frac{a-d}{u-d}$ where $a = e^{(r-q)\Delta t}$, we see

$$\sigma^2 \Delta t = e^{(r-q)\Delta t} (u + d) - ud - e^{2(r-q)\Delta t} \quad (3)$$

We get two conditions on $u, d, p$ from (2) and (3)

Finally we impose a simplifying condition $d = \frac{1}{u}$. Ignoring higher-order terms in $\Delta t$ we get

$$p = \frac{a-d}{u-d} \quad (4)$$

$$u = e^{\sigma \sqrt{\Delta t}} \quad (5)$$

$$d = e^{-\sigma \sqrt{\Delta t}} \quad (6)$$

$$a = e^{(r-q)\Delta t} \quad (7)$$

$a$ is the growth factor.
• For a one step tree we evaluate

\[ f = e^{-r\Delta t} [pf_u + (1 - p)f_d] \]

where \( e^{-r\Delta t} \) is the discounting factor on the time-step.

• Recursively, we check at the end time \( f_{N,j} = \max\{K - S_0u^jd^{N-j}, 0\} \). We then evaluate the worth recursively.

• More after an example:
Expressing Binomial Trees Algebraically

- Recursively, we check at the end time \( f_{N,j} = \max\{K - S_0u^j d^{N-j}, 0\} \). We then evaluate the worth recursively.
- At some node \((i, j)\) there is a probability \(p\) of moving from \((i, j)\) at time \(i\Delta t\) to \((i + 1, j + 1)\) at time \((i + 1)\Delta t\), and a probability \((1 - p)\) of moving from \((i + 1, j - 1)\) at time \((i + 1)\Delta t\).
- If no early exercise then the risk-neutral value is
  \[
  f_{i,j} = e^{-r\Delta t} [pf_{i+1,j+1} + (1 - p)f_{i+1,j-1}]
  \]
  for \(0 \leq 1 \leq N - 1\) and \(0 \leq j \leq i\).
- If early exercise is taken into account then we get
  \[
  f_{i,j} = \max\{K - S_0u^j d^{i-j}, e^{-r\Delta t} [pf_{i+1,j+1} + (1 - p)f_{i+1,j-1}]\}
  \]
  In the limit as \(\Delta t \rightarrow 0\), we get the true price for an American put option.
Next Time

We

• study how the calculate the Greek Letters using a Binomial Tree
• take into account discrete dividend payments.
• consider \textit{time-dependent} parameters.
Homework

Homework: Due Feb. 13, 5PM.
Graded: 15.25, 15.26, 17.25