Lecture 7.

Finite Difference Methods

Solutions

Problem: 15.25

A financial institution has the following portfolio of over-the-counter options on sterling

Type	Position	Delta of option	Gamma of option	Vega of option
Call	-1000	0.50	2.2	1.8
Call	-500	0.8	0.6	0.2
Put	-2000	-0.40	1.3	0.7
Call	-500	0.70	1.8	1.4

A traded option is available with delta of 0.6, a gamma of 1.5, and a vega of 0.8.

(a) What position in the traded option and in sterling would make the portfolio both gamma neutral and delta neutral?

(b) What position in the traded option and in sterling would make the portfolio both vega neutral and delta neutral?

Solution

The delta of the portfolio is

$$-1000 \times 0.50 - 500 \times 0.80 - 2000 \times (-0.40) - 500 \times 0.70 = -450$$

The gamma of the portfolio is

$$-1000 \times 2.2 - 500 \times 0.6 - 2000 \times 1.3 - 500 \times 1.8 = -6000$$

The vega of the portfolio is

$$-1000 \times 1.8 - 500 \times 0.2 - 2000 \times 0.7 - 500 \times 1.4 = -4000$$

(a) A long position in 4000 traded options will give a gamma-neutral portfolio since the long position has a gamma of $4000 \times 1.5 = 6000$. The delta of the whole portfolio, including the traded options, is

$$4000 \times 0.6 - 450 = 1950$$

Thus, in addition to the 4000 traded options, a short position in £1950 is necessary so that the portfolio is both gamma and delta neutral.

(b) A long position in 5000 traded options will give a vega-neutral portfolio since the long position has a vega of $5000 \times 0.8 = 4000$. The delta of the whole portfolio, including the traded options, is

$$5000 \times 0.6 - 450 = 2550$$

Thus in addition to the 5000 traded options, a short position in £2550 is necessary so that the portfolio is both vega and delta neutral.

Solutions

Problem: 15.26

Consider again the situation in 15.25. Suppose that the second traded option with a delta of 0.1, gamma of 0.5 and vega of 0.6 is available. How could the portfolio be made delta, gamma, and vega neutral?

Solution

Let w_1 be the position in the first traded option and w_2 be the position in the second traded option. We require

$$6000 = 1.5w_1 + 0.5w_2$$

$$4000 = 0.8w_1 + 0.6w_2$$

We can solve this easily to get $w_1 = 3200$ and $w_2 = 2400$. The whole portfolio then has a delta of

$$-450 + 3200 \times 0.6 + 2400 \times 0.1 = 1710$$

Therefore, the whole portfolio can be delta, gamma, vega neutral by taking a long position in 3200 of the first traded option, a long position in 2400 of the second traded option, and a short position of £1710.

Solutions

Problem: 17.25

An American put option to sell a Swiss franc for dollars has a strike price of \$0.80 and a time to maturity of 1 year. The volatility of the Swiss franc is 10% , the dollar interest rate is 6% , the Swiss franc interest rate is 3% , and the current exchange rate is 0.81. Use a three-time-step tree to value the option. Estimate the delta of the option from your tree.

Solution

The value of the option is 0.0207, and its delta can be estimated as

$$\frac{0.006221 - 0.041153}{0.858142 - 0.764559} = -0.404$$

Monte Carlo and Derivatives

We can use Monte Carlo to offer a risk-neutral valution by computing sample paths. Consider a derivative dependent on a single market variable S that provides a payoff at time T.

1. Sample a random path for S in a risk-neutral world.
2. Calculate the payoff from the derivative
3. Repeat steps 1 and 2 to get many sample values of the payoff from the derivative in a risk-neutral world.
4. Calculate the mean of the sample payoffs to get an estimate of the expected payoff in a risk-neutral world.
5. Discount the expected payoff at the risk-free rate to get an estimate of the value of the derivative

Example of Stock Process

Consider a process underlying the market variable is our stock process

$$dS = \mu S dt + \sigma S dz \tag{1}$$

where dz is the Wiener process, μ is the expected return in a risk-neutral world, and σ is the volatility.

To simulate the path of S we divide up the life of the stock process into N time-steps of length Δt. Then the discrete, approximate equation to (1) is

$$S(t + \Delta t) - S(t) = \Delta S = \mu S \Delta t + \sigma S \epsilon \sqrt{\Delta t} \tag{2}$$

where $S(t)$ is the value of S at time t.

Here ϵ is a random sample from a normal distribution with mean zero and standard deviation of 1.0.

The method generates a path from the initial value of S.

In practice, one usually simulates the process of $\ln S$ instead of S. From Ito's lemma then process then satisifes

$$d \ln S = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dz$$

thus

$$\ln S(t + \Delta t) - \ln S(t) = \left(\mu - \frac{\sigma^2}{2}\right) \Delta t + \sigma \epsilon \sqrt{\Delta t}$$

Therefore,

$$S(t + \Delta t) = S(t) \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) \Delta t + \sigma \epsilon \sqrt{\Delta t}\right] \tag{3}$$

In fact the result it true of all T, since it is a generalized Wiener process. Thus

$$\ln S(T) - \ln S(0) = \left(\mu - \frac{\sigma^2}{2}\right) T + \sigma \epsilon \sqrt{T}$$

Therefore,

$$S(T) = S(0) \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) T + \sigma \epsilon \sqrt{T}\right] \tag{4}$$

This can be used to check Black-Scholes.

Checking Black-Scholes

One can numerically check the veracity of the Black-Scholes formula. How?

We are given constants S_0, K, r, σ, T that we can use in the Black-Scholes formula.

- Compute the stock process via a Monte Carlo method:

$$S(T) = S(0)\exp\left[\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\epsilon\sqrt{T}\right]$$

  by choosing a sample from the standard normal distribution.
- Given $S(T)$ we evaluate the option value as

$$e^{-rT}\max\{S(T) - K, 0\}$$

- We repeat the procedure a number of trials and average the value.

Number of Trials

- Accuracy of the result given by Monte Carlo simulation depends on the number of trails.
- Usually one calculates the standard deviation and the mean of the discounted payoffs given by the simulation trials.
- Denote μ and ω to be the mean and standard deviation, and we assume μ is the price of the derivative at the end of the simulated trial.
- The standard error of the estimate of the price of the derivative is given by

$$\frac{\omega}{\sqrt{M}}$$

  where M is the number of trials.
- A 95% confidence interval for the price f of a derivative is given by

$$\mu - 1.96\frac{\omega}{\sqrt{M}} < f < \mu + 1.96\frac{\omega}{\sqrt{M}}$$

Applications

- Monte Carlo simulation are usually more efficient than other methods when there are many stochastic variables.
- The Monte Carlo simulation tends to be increase linearly with the number of of variables, whereas other methods may be exponential in the number of variables.
- Monte Carlo simulation also provides a standard error for the estimates that it makes.
- Can handle more complex stochastic processes.

Greek letters

In order to compute the Greek letters from a Monte Carlo simulation, one needs to compute the partial derivative of f with respect to a derivative. We consider the approximate derivative of f with respect to x.

- First compute the Monte Carlo simulation in the usual way to calculate $\hat{f}$ with a fixed value of x.
- Second compute the value of the derivative $\hat{f}^*$ with a new $x + \Delta x$.
- Third compute
$$\frac{\hat{f}^* - \hat{f}}{\Delta x}$$
- In order to minimize the standard error of the estimate, the number of intervals N, the number of random streams, and the number of trials M should be the same for calculating both $\hat{f}$ and $\hat{f}^*$.

Sampling through a Tree

Instead of implementing Monte Carlo simulation by randomly sampling from the stochastic process for the underlying variable, we can use an N-step binomial tree and sample from the 2^N paths that are possible.

- Suppose we have a binomial tree here the probability of an up-movement is 0.6. The procedure for sampling a random path through the tree is as follows.
- At each node, we sample a random number between 0 and 1. If the number is less than 0.4, we take the down path.
- Once we have a complete path from the initial node to the end of the tree, we can calculate a payoff.
- This completes a first trial. Similar procedure is used to complete more trials. The mean of the payoffs is discounted at the risk-free rate to get an estimate of the value of the derivative.

Finite Difference Schemes

Finite difference methods are useful for solving partial differential equations. The differential equation is converted into a set of **difference equations** that are solved iteratively.

Consider how we might value an American put option on a stock paying a dividend yield of q. The differential equation that the option must satisfy is the associated Black-Scholes equation

$$\frac{\partial f}{\partial t} + (r - q)\, S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial S^2} = r f$$

- Suppose the life of the option is T. Divide this into N equally spaced intervals of length $\Delta t = \frac{T}{N}$. A total of $N + 1$ times are therefore considered

$$0, \Delta t, 2\Delta t, \ldots, T$$

- Suppose that S_{max} is a stock price sufficiently high that (if reached) the put has virtually no value. Define $\Delta S = \frac{S_{max}}{M}$ and consider a total of $M + 1$ equally spaced stock price:

$$0, \Delta S, 2\Delta S, \ldots, S_{max}$$

  Choose S_{max} so that a $k\Delta S$ is the current stock price.

- The time points and stock price points define a grid consisting of $(N+1) \times (M+1)$ points. The point (i, j) on the grid is the point that corresponds to time $i\Delta t$ and stock price $j\Delta S$.
- Use the discrete variable $f_{i,j}$ to denote the value of the option on the grid point (i, j).

We can now define the discrete derivatives. In particular we have the **forward difference approximation**:

$$\frac{\partial f}{\partial S} \approx \frac{f_{i,j+1} - f_{i,j}}{\Delta S}$$

the **backward difference approximation**:

$$\frac{\partial f}{\partial S} \approx \frac{f_{i,j} - f_{i,j-1}}{\Delta S}$$

and the **symmetric difference approximation**:

$$\frac{\partial f}{\partial S} \approx \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S}$$

For the time derivative we use a forward difference scheme:

$$\frac{\partial f}{\partial t} \approx \frac{f_{i+1,j} - f_{i,j}}{\Delta t}$$

and the second derivative of f with respect to S we get

$$\frac{\partial^2 f}{\partial S^2} \approx \frac{1}{\Delta S}\left[\frac{f_{i,j+1} - f_{i,j}}{\Delta S} - \frac{f_{i,j} - f_{i,j-1}}{\Delta S}\right] = \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{\left(\Delta S\right)^2}$$

We can now write down the finite difference scheme for the Black-Scholes, using $S = j\Delta S$,

$$\begin{aligned}\frac{f_{i+1,j} - f_{i,j}}{\Delta t} &+ (r-q)\, j\Delta S \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S} \\ &+ \frac{1}{2}\sigma^2 j^2 \left(\Delta S\right)^2 \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{\left(\Delta S\right)^2} = r f_{i,j}\end{aligned}$$

We rearrange the terms to get

$$\begin{aligned} f_{i+1,j} = & \left(\frac{1}{2}(r-q)j\Delta t - \frac{1}{2}\sigma^2 j^2 \Delta t\right) f_{i,j-1} \\ & + \left(1 + \sigma^2 j^2 \Delta t + r\Delta t\right) f_{i,j} \\ & + \left(-\frac{1}{2}(r-q)j\Delta t - \frac{1}{2}\sigma^2 j^2 \Delta t\right) f_{i,j+1} \\ = & \, a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1} \end{aligned}$$

We now choose boundary conditions for our problem.

Next the value of the put at time T is $\max\{K - S_T, 0\}$, where S_T is the stock price at time T.

$$f_{N,j} = \max\{K - j\Delta S, 0\} \qquad j = 0, \ldots, M$$

The value of the put option when the stock price is zero is K. Hence

$$f_{i,0} = K \qquad i = 0, \ldots, N$$

The value of the put option when the stock price is S_{max} is zero. Hence

$$f_{i,M} = 0 \qquad i = 0, \ldots, N$$

Finite Difference Cont.

We now solve for the rest of the $f_{i,j}$'s. We know $f_{N,j}$ then our equation yields equations

$$a_j f_{N-1,j-1} + b_j f_{N-1,j} + c_j f_{N-1,j+1} = f_{N,j}$$

for $j = 1, \ldots, M-1$. The right hand sides are known from the boundary condition:

$$f_{N-1,0} = K \qquad f_{N-1,M} = 0$$

- Therefore, we have $M-1$ **linear** equations for $M-1$ unknowns. This can be solved easily to get
$$f_{N-1,1}, f_{N-1,2}, \ldots, f_{N-1,M-1}$$
- We now check whether $f_{N-1,j}$ is optimal. If $f_{N-1,j} < K - j\Delta S$ then we should exercise early and $f_{N-1,j}$ is reassigned the value $K - j\Delta S$.
- Once $T - \Delta t$ has been evaluated at points $(N-1, j)$, we move to the points on the grid referring to $T - 2\Delta t$.
- Finally, we get the grid points $f_{0,1}, f_{0,2}, \ldots, f_{0,M-1}$. We choose the point that is

Explicit Finite Difference Method

The implicit finite difference scheme is very robust, and as Δt and $\Delta S \to 0$ then the solution goes to the solution of the Black-Scholes.

On the other hand implicit finite difference requires the solution of a set of equations at each fixed time.

We can do a simpler method that doesn't require solving the system of equations at each time step.

- Assume that $\frac{\partial f}{\partial S}$ and $\frac{\partial^2 f}{\partial S^2}$ is the same at (i, j) as at $(i+1, j)$. Then

$$\frac{\partial f}{\partial S} \approx \frac{f_{i+1,j+1} - f_{i+1,j-1}}{2\Delta S}$$

  and

$$\frac{\partial^2 f}{\partial S^2} \approx \frac{f_{i+1,j+1} - 2f_{i+1,j} + f_{i+1,j-1}}{(\Delta S)^2}$$

This yields a finite difference scheme

$$
\begin{aligned}
f_{i,j} &= \frac{1}{1+r\Delta t}\left(-\frac{1}{2}(r-q)j\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t\right) f_{i+1,j-1} \\
&\quad + \frac{1}{1+r\Delta t}\left(1-\sigma^2 j^2 \Delta t\right) f_{i,j} \\
&\quad + \frac{1}{1+r\Delta t}\left(\frac{1}{2}(r-q)j\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t\right) f_{i+1,j+1} \\
&= a_j^* f_{i+1,j-1} + b_j^* f_{i+1,j} + c_j^* f_{i+1,j+1}
\end{aligned}
$$

Since we know the information at the previous time, we can directly compute the value of $f_{i,j}$ without solving a system of equations.

Change of Variables Improvement

We can improve the efficiency of the finite difference methods by using $\ln S$ rather than S as the underlying variable. Setting $Z = \ln S$ then Black-Scholes becomes

$$\frac{\partial f}{\partial t} + \left(r - q - \frac{\sigma^2}{2}\right)\frac{\partial f}{\partial Z} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial Z^2} = rf$$

We discretize in equal Z steps, rather than for S steps. The difference equation for the implicit method becomes

$$\frac{f_{i+1,j} - f_{i,j}}{\Delta t} + \left(r - q - \frac{\sigma^2}{2}\right)\frac{f_{i,j+1} - f_{i,j-1}}{2\Delta Z} + \frac{1}{2}\sigma^2\frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\Delta Z)^2} = rf_{i,j}$$

So:

We get the finite difference scheme

$$\alpha_j f_{i,j-1} + \beta_j f_{i,j} + \gamma_j f_{i,j+1} = f_{i+1,j}$$

where

$$\alpha_j = \frac{\Delta t}{2\Delta Z}\left(r - q - \frac{\sigma^2}{2}\right) - \frac{\Delta t}{2\left(\Delta Z\right)^2}\sigma^2$$

$$\beta_j = 1 + \frac{\Delta t}{\left(\Delta Z\right)^2}\sigma^2 + r\Delta t$$

$$\gamma_j = -\frac{\Delta t}{2\Delta Z}\left(r - q - \frac{\sigma^2}{2}\right) - \frac{\Delta t}{2\left(\Delta Z\right)^2}\sigma^2$$

The explicit finite difference scheme

$$\alpha_j^* f_{i+1,j-1} + \beta_j^* f_{i+1,j} + \gamma_j^* f_{i+1,j+1} = f_{i,j}$$

where

$$\alpha_j^* = \frac{1}{1+r\Delta t}\left[-\frac{\Delta t}{2\Delta Z}\left(r-q-\frac{\sigma^2}{2}\right)+\frac{\Delta t}{2\left(\Delta Z\right)^2}\sigma^2\right]$$

$$\beta_j^* = \frac{1}{1+r\Delta t}\left[1-\frac{\Delta t}{\left(\Delta Z\right)^2}\sigma^2\right]$$

$$\gamma_j^* = \frac{1}{1+r\Delta t}\left[\frac{\Delta t}{2\Delta Z}\left(r-q-\frac{\sigma^2}{2}\right)+\frac{\Delta t}{2\left(\Delta Z\right)^2}\sigma^2\right]$$

The change of variables approach has the property that α_j, β_j, γ_j as well as α_j^*, β_j^*, γ_j^*.

Most efficient if

$$\Delta Z = \sigma\sqrt{3\Delta t}$$

Relationship to Trinomial Tree Approaches

Explicit finite difference method is equivalent to the trinomial tree approach. In the expression for a_j^*, b_j^*, c_j^*, we can interpret as

- $-\frac{1}{2}(r-q)j\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t$: Probability of the stock price decreasing from $j\Delta S$ to $(j-1)\Delta S$ in time Δt
- $1 - \sigma^2 j^2 \Delta t$: Probability of the stock price staying constant at $j\Delta S$ in time Δt
- $\frac{1}{2}(r-q)j\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t$: Probability of the stock price increasing from $j\Delta S$ to $(j+1)\Delta S$ in time Δt

Note that the sum of the three probabilities is 1. The probabilities

- give the expected increase in the stock price in time Δt as $(r-q)j\Delta S\Delta t = (r-q)S\Delta t$.
- give the variance of the change in the stock price in time Δt as

$$\sigma^2 j^2 \Delta S^2 \Delta t = \sigma^2 S^2 \Delta t$$

- The value of f at time $i\Delta t$ is calculated as the expected value of f at time $(i+1)\Delta t$ in the risk-neutral world discounted at the risk-free rate.

For the explicit version of the finite difference method to work well, the three probabilities

$$-\frac{1}{2}\left(r-q\right)j\Delta t+\frac{1}{2}\sigma^2 j^2\Delta t$$

$$1-\sigma^2 j^2\Delta t$$

$$\frac{1}{2}(r-q)j\Delta t+\frac{1}{2}\sigma^2 j^2\Delta t$$

should be all positive. If the probabilities are negative, then the method doesn't make sense. When the change of variables formula is used $Z=\ln S$ then we get probabilities for the explicit difference scheme

$$-\frac{\Delta t}{2\Delta Z}\left(r-q-\frac{\sigma^2}{2}\right)+\frac{\Delta t}{2(\Delta Z)^2}\sigma^2$$

$$1-\frac{\Delta t}{(\Delta Z)^2}\sigma^2$$

$$\frac{\Delta t}{2\Delta Z}\left(r-q-\frac{\sigma^2}{2}\right)+\frac{\Delta t}{2(\Delta Z)^2}\sigma^2$$

Choosing $\Delta Z = \sigma\sqrt{3\Delta t}$ then the probabilities become

$$-\sqrt{\frac{\Delta t}{12\sigma^2}}\left(r - q - \frac{\sigma^2}{2}\right) + \frac{1}{6}$$

$$\frac{2}{3}$$

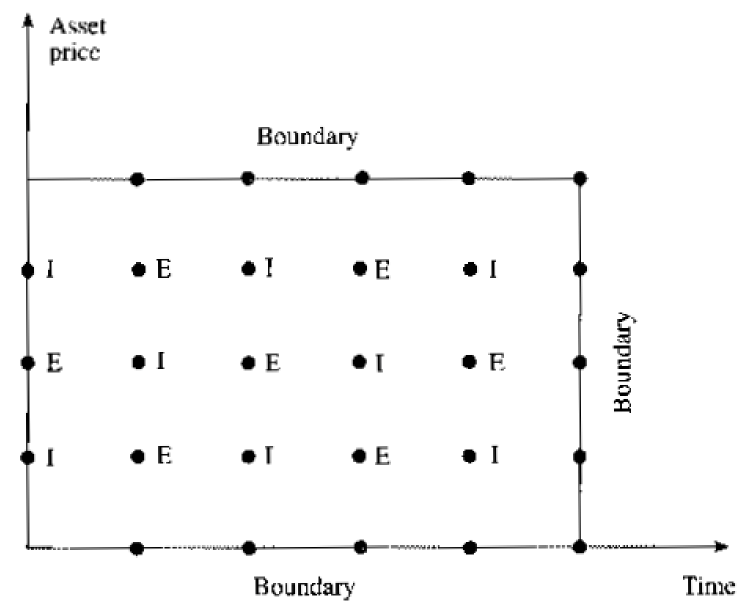
$$\sqrt{\frac{\Delta t}{12\sigma^2}}\left(r - q - \frac{\sigma^2}{2}\right) + \frac{1}{6}$$

These are exactly the probabilities found in the trinomial trees.

Other Finite Difference Methods

There are other methods that integrate some the features of implicit and explicit schemes.

- **Hopscotch method** alternates between explicit and implicit calculations as one moves from node to node.



This allows one to deal with the implicit nodes without having to solve a simultaneous system of equations.

- **Crank-Nicolson scheme** is an average of the explicit and implicit method.
  The implicit method yields:

$$f_{i,j} = a_j f_{i-1,j-1} + b_j f_{i-1,j} + c_j f_{i-1,j+1}$$

The explicit method yields:

$$f_{i-1,j} = a_j^* f_{i,j-1} + b_j^* f_{i,j} + c_j^* f_{i,j+1}$$

Then, C-N is

$$f_{i,j} + f_{i-1,j} = a_j f_{i-1,j-1} + b_j f_{i-1,j} + c_j f_{i-1,j+1} + a_j^* f_{i,j-1} + b_j^* f_{i,j} + c_j^* f_{i,j+1}$$

Setting

$$g_{i,j} = f_{i,j} - a_j^* f_{i,j-1} - b_j^* f_{i,j} - c_j^* f_{i,j+1}$$

then

$$g_{i,j} = a_j f_{i-1,j-1} + b_j f_{i-1,j} + c_j f_{i-1,j+1} - f_{i-1,j}$$

which is similar to the implicit method, but has a faster rate of convergence.

Notes:

- Finite difference schemes useful for computing American and European options prices but not as useful for computing more exotic options that depend on past history of the underlying variables (say an *Asian option*).
- Finite difference can also compute options prices when several state variables are present.
- Not difficult to compute Δ, Γ, Θ using the difference quotients to approximate the derivatives. $\mathcal{V}$ is computed by computing the prices, changing $\sigma \mapsto \sigma + \Delta\sigma$ and recomputing the price.

Summary

Three numerical methods presented to compute options prices:

- **Binomial Trees**
  - Assume that in each short interval of time Δt, a stock price either moves up by a multiplicative amount u or down by a multiplicative amount d.
  - The constants u and d and the associated probabilities are chosen so that the change in the stock price has the correct mean and standard deviation in a risk-neutral world
  - Derivative prices are calculated by starting at the end of the tree and working backwards.
  - For an American option, the value at a node is the great of
    (a) the value if it is exercised immediately
    (b) the discounted expected value if it is held for a further period of time Δt.
  - Work well for American options. Less well for options that depend on the history of the option.
- **Monte Carlo Methods**
  - Monte Carlo Simulation involves using random numbers to sample many different paths that the variables underlying the derivative could follow in a risk-neutral world.
  - For each path, the payoff is calculated and discounted at the risk-free interest rate.
  - The arithmetic average of the discounted payoffs is the estimated value of the derivative.
  - Work well for European options and other options that depend on the history of the option. Relatively more efficient when the number of underlying variables is increased. Less well for American options.

- **Finite Difference Schemes**
  - Finite difference methods solve the underlying differential equation by converting it to a difference equation.
  - Similar to tree methods in that the computations work back from the maturity time to the initial time.
  - Explicit method is functionally the same as using a trinomial tree.
  - Implicit method is more complicated but convergence is assured.
  - Work well for American options. Less well for options that depend on the history of the option.

Value at Risk

We now look for a quantity that gives a measure of the total risk of a portfolio

Value at Risk (VaR) - attempts to provide a single number that summarizes the total risk in a portfolio of financial assets.

This is widely used by corporate treasurers and fund managers as well as by financial institutions. Central bank regulators use VaR in determining the capital the bank is required to keep to reflect the market risks it is bearing.

- The value-at-risk measure, we are interested is of the form

  We are X percent certain that we will not lose more than V dollars in the next N days.
- The variable V is the VaR of the portfolio
- The VaR is a function of two parameters - the time horizon N and the confidence level X. It measure the loss level over N days that we are X certain will not be exceeded.
- Bank regulators require that banks calculate VaR with $N = 10$ and $X = 99$.

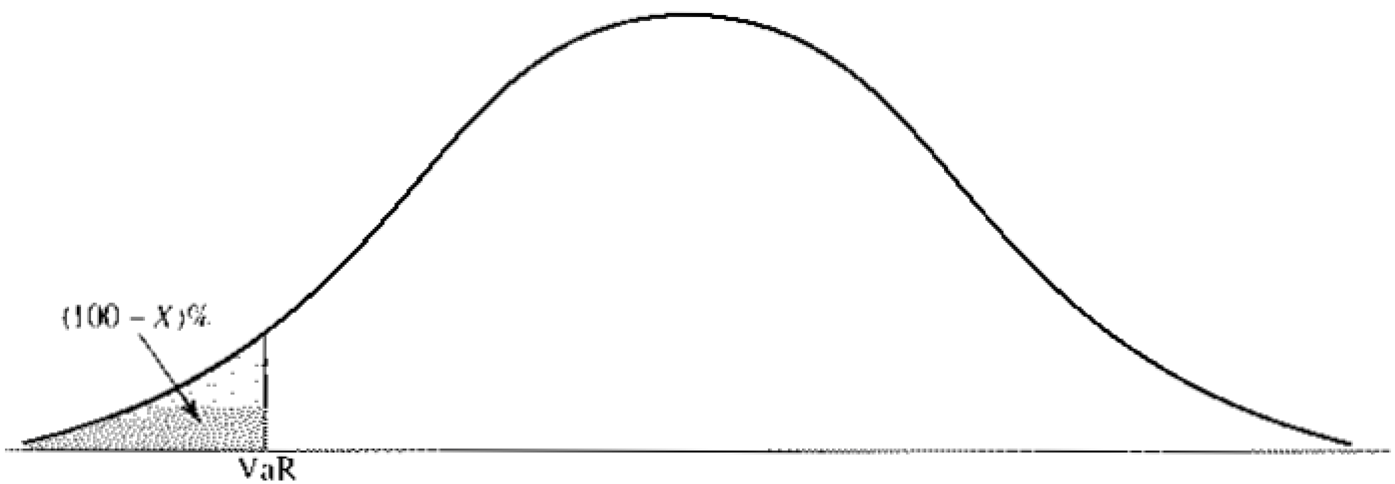
VaR

- VaR is attractive since it simply asks

  How bad can things get?

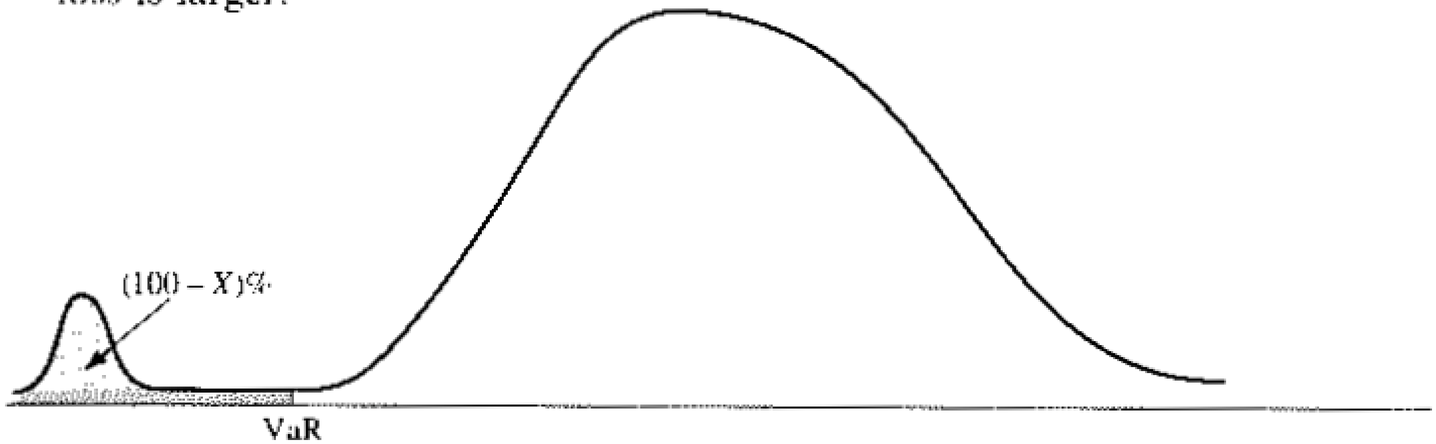
- When the value of the portfolio is normally distributed we see VaR looks like:

Figure 18.1 Calculation of VaR from the probability distribution of the change in the portfolio value; confidence level is $X\%$.



- There is a problem if the portfolio is not normally distributed. Consider a case where there is a larger probability of a very large down movement and less otherwise:

Figure 18.2 Alternative situation to Figure 18.1. VaR is the same, but the potential loss is larger.



- The two graphs have the same VaR, but the second is riskier, since there is a larger probability of a very large loss.

Some traders may look for a different measure of total risk, **Conditional VaR**.

- C-VaR asks

  If things do get bad, how much can we expect to lose?
- C-VaR is the expected loss during an N-day period conditional that we are in the $100 - X\%$ left tail of the distribution.
- VaR is a most popular measure than other such risk measures.

The Time Horizon

VaR has two parameters - the N-day time horizon, and the X confidence interval.

- In practice $N = 1$, since there is usually not enough data for a longer period.
- Usually one assumes
$$N - \text{day VaR} = 1 - \text{day VaR} \times \sqrt{N}$$
- Since bank's are required to have capital at least three times the 10-day 99% VaR, then it is required to have
$$3 \times \sqrt{10} = 9.49$$
times the 1-day 99% VaR.
- We discuss several ways to estimate VaR.
  - Historical Simulations
  - Model-Building
  - Linear & Quadratic Model
  - Monte Carlo Simulation

Historical Simulations

Historical simulation is one approach of estimating VaR.

- Use past data in a direct way as a guide to what might happen in the future.
- Suppose that we wish to calculate the 99% confidence level with a 1-day horizon using 500 days of data.
  1. Identify the market variables affection the portfolio (typically **exchange rates, equity prices, interest rates,** etc).
  2. Collect data on the movements in these markets variables over the most recent 500 days.
  3. We have 500 alternative scenarios for what can happen between today and tomorrow
  4. Scenario 1 is where the percentage changes in the values of all variables are the same as they were on the first day for which we have collected data.
  5. Scenario 2 is where the percentage changes in the values of all variables are the same as they were on the second day for which we have collected data.
  6. Etc...
  7. For each scenario we calculate the dollar change in the value of the portfolio between today and tomorrow.
  8. This yields a probability distribution for daily changes in the value of our portfolio.
  9. Example:

Consider 500 observations on 500 consecutive days of n market variables:

Day	Market Var. 1	Market Var. 2	$\cdots$	Market Var. n
0	20.33	0.1132	$\cdots$	65.37
1	20.78	0.1159	$\cdots$	64.91
2	21.44	0.1162	$\cdots$	65.02
3	20.97	0.1184	$\cdots$	64.90
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
498	25.72	0.1312	$\cdots$	62.22
499	25.75	0.1323	$\cdots$	61.99
500	25.85	0.1343	$\cdots$	62.10

The next table shows the values of the market variables tomorrow if their percentage changes between today and tomorrow are the same as they were between Day $i-1$ and Day i.

Day	Market Var. 1	Market Var. 2	$\cdots$	Market Var. n	Port. Val. millions	Chg in Val. millions
0	26.42	0.1375	$\cdots$	61.66	23.71	0.21
1	26.67	0.1346	$\cdots$	62.21	23.12	-0.38
2	25.28	0.1368	$\cdots$	61.99	22.94	-0.56
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$
499	25.88	0.1354	$\cdots$	61.87	23.63	0.13
500	25.95	0.1363	$\cdots$	62.21	22.87	-0.63

For example: $26.42 = 25.95 \times \frac{20.78}{20.33}$.

- Define v_i to be the value of the market variable on Day i and suppose that today is Day m. The ith scenario assumes that the value of the market variable tomorrow will be

$$v_m \frac{v_i}{v_{i-1}}$$

- We assume the value of the portfolio is \$23.5 million. This leads to the numbers in the final column for the change in the value between today and tomorrow for all different scenarios. For scenario 2 the change in value is $+\$210,000$, for scenario 2 it is $+\$380,000$, and so on.
- We are interested in the 1-percentile point of the distribution of changes in the portfolio value.
- Because there are 500 scenarios we can estimate the as the fifth worst number in the final column in the table.
- Alternatively we can use **Extreme Value Theory** which smooths the numbers in the left tail of the distribution in an attempt to obtain a more accurate estimate of the 1% point of the distribution.
- Each day of the VaR estimate in the example would be updated using the most recent 500 days of data.
- Consider what happens on Day 501. We find out new values for all the market variables and we are able to calculate a new value for our portfolio. We then go through the procedure to calculate a new VaR. We use the market variables from Day 1 to Day 501 (leaving Day 0 from the list).

Model-Building Approach

A major approach outside of the historical approach is to use model approach.

Daily Volatilities

- Usually we measure volatilities in years. Model-building approach to VaR, we measure time in days and the volatility of an asset is usually quoted as "volatility per day"
- Define σ_{year} to be the volatility per year of a certain asset and σ_{day} as the equivalent volatility per day of the asset. Assuming 252 trading days per year, we use

$$\sigma_{year} = \sigma_{day}\sqrt{252} = \sigma_{day}\sqrt{N}$$

or

$$\sigma_{day} = \frac{\sigma_{year}}{\sqrt{252}} \approx 0.063\sigma_{year}$$

- σ_{day} is approximately equal to the standard deviation of the percentage change in the asset price in one day. For purposes of calculating VaR we assume exact equality. We define the daily volatility of an asset price as equal to the standard deviation of the percentage change in one day.
- We will apply this to different assets next time.