Value at Risk (VaR)
We now look for a quantity that gives a measure of the total risk of a portfolio

**Value at Risk** (VaR) - attempts to provide a single number that summarizes the total risk in a portfolio of financial assets.

This is widely used by corporate treasurers and fund managers as well as by financial institutions. Central bank regulators use VaR in determining the capital the bank is required to keep to reflect the market risks it is bearing.

- The value-at-risk measure, we are interested is of the form
  
  We are $X$ percent certain that we will not lose more than $V$ dollars in the next $N$ days.

- The variable $V$ is the VaR of the portfolio
- The VaR is a function of two parameters - the time horizon $N$ and the confidence level $X$. It measure the loss level over $N$ days that we are $X$ certain will not be exceeded.
- Bank regulators require that banks calculate VaR with $N = 10$ and $X = 99$. 
• VaR is attractive since it simply asks
  How bad can things get?
• When the value of the portfolio is normally distributed we see VaR looks like:

\[
\text{Figure 18.1} \quad \text{Calculation of VaR from the probability distribution of the change in the portfolio value; confidence level is } X\%.
\]

• There is a problem if the portfolio is not normally distributed. Consider a case where there is a larger probability of a very large down movement and less otherwise:
The two graphs have the same VaR, but the second is riskier, since there is a larger probability of a very large loss.

Some traders may look for a different measure of total risk, **Conditional VaR**.

- C-VaR asks
  
  If things do get bad, how much can we expect to lose?

- C-VaR is the expected loss during an $N$-day period conditional that we are in the $100 - X\%$ left tail of the distribution.

- VaR is a most popular measure than other such risk measures.
The Time Horizon

VaR has two parameters - the $N$-day time horizon, and the $X$ confidence interval.

- In practice $N = 1$, since there is usually not enough data for a longer period.
- Usually one assumes
  \[ N \text{ - day VaR} = 1 \text{ - day VaR} \times \sqrt{N} \]
- Since bank’s are required to have capital at least three times the 10-day 99% VaR, then it is required to have
  \[ 3 \times \sqrt{10} = 9.49 \]
  times the 1-day 99% VaR.
- We discuss several ways to estimate VaR.
  - Historical Simulations
  - Model-Building
  - Linear & Quadratic Model
  - Monte Carlo Simulation
Historical Simulations

Historical simulation is one approach of estimating VaR.

- Use past data in a direct way as a guide to what might happen in the future.
- Suppose that we wish to calculate the 99% confidence level with a 1-day horizon using 500 days of data.
  1. Identify the market variables affecting the portfolio (typically exchange rates, equity prices, interest rates, etc).
  2. Collect data on the movements in these market variables over the most recent 500 days.
  3. We have 500 alternative scenarios for what can happen between today and tomorrow.
  4. Scenario 1 is where the percentage changes in the values of all variables are the same as they were on the first day for which we have collected data.
  5. Scenario 2 is where the percentage changes in the values of all variables are the same as they were on the second day for which we have collected data.
  6. Etc...
  7. For each scenario we calculate the dollar change in the value of the portfolio between today and tomorrow.
  8. This yields a probability distribution for daily changes in the value of our portfolio.
  9. Example:
Consider 500 observations on 500 consecutive days of $n$ market variables:

<table>
<thead>
<tr>
<th>Day</th>
<th>Market Var. 1</th>
<th>Market Var. 2</th>
<th>⋮</th>
<th>Market Var. n</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>20.33</td>
<td>0.1132</td>
<td>⋮</td>
<td>65.37</td>
</tr>
<tr>
<td>1</td>
<td>20.78</td>
<td>0.1159</td>
<td>⋮</td>
<td>64.91</td>
</tr>
<tr>
<td>2</td>
<td>21.44</td>
<td>0.1162</td>
<td>⋮</td>
<td>65.02</td>
</tr>
<tr>
<td>3</td>
<td>20.97</td>
<td>0.1184</td>
<td>⋮</td>
<td>64.90</td>
</tr>
<tr>
<td>⋮</td>
<td>⋮</td>
<td>⋮</td>
<td>⋮</td>
<td>⋮</td>
</tr>
<tr>
<td>498</td>
<td>25.72</td>
<td>0.1312</td>
<td>⋮</td>
<td>62.22</td>
</tr>
<tr>
<td>499</td>
<td>25.75</td>
<td>0.1323</td>
<td>⋮</td>
<td>61.99</td>
</tr>
<tr>
<td>500</td>
<td>25.85</td>
<td>0.1343</td>
<td>⋮</td>
<td>62.10</td>
</tr>
</tbody>
</table>

The next table shows the values of the market variables tomorrow if their percentage changes between today and tomorrow are the same as they were between Day $i - 1$ and Day $i$.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>26.42</td>
<td>0.1375</td>
<td>⋮</td>
<td>61.66</td>
<td>23.71</td>
<td>0.21</td>
</tr>
<tr>
<td>1</td>
<td>26.67</td>
<td>0.1346</td>
<td>⋮</td>
<td>62.21</td>
<td>23.12</td>
<td>-0.38</td>
</tr>
<tr>
<td>2</td>
<td>25.28</td>
<td>0.1368</td>
<td>⋮</td>
<td>61.99</td>
<td>22.94</td>
<td>-0.56</td>
</tr>
<tr>
<td>⋮</td>
<td>⋮</td>
<td>⋮</td>
<td>⋮</td>
<td>⋮</td>
<td>⋮</td>
<td>⋮</td>
</tr>
<tr>
<td>499</td>
<td>25.88</td>
<td>0.1354</td>
<td>⋮</td>
<td>61.87</td>
<td>23.63</td>
<td>0.13</td>
</tr>
<tr>
<td>500</td>
<td>25.95</td>
<td>0.1363</td>
<td>⋮</td>
<td>62.21</td>
<td>22.87</td>
<td>-0.63</td>
</tr>
</tbody>
</table>

For example: $26.42 = 25.95 \times \frac{20.78}{20.33}$. 
• Define \( v_i \) to be the value of the market variable on Day \( i \) and suppose that today is Day \( m \). The \( i \)th scenario assumes that the value of the market variable tomorrow will be

\[
v_{m} \frac{v_i}{v_{i-1}}
\]

• We assume the value of the portfolio is $23.5 million. This leads to the numbers in the final column for the change in the value between today and tomorrow for all different scenarios. For scenario 2 the change in value is +$210,000, for scenario 2 it is +$380,000, and so on.

• We are interested in the 1-percentile point of the distribution of changes in the portfolio value.

• Because there are 500 scenarios we can estimate the as the fifth worst number in the final column in the table.

• Alternatively we can use **Extreme Value Theory** which smooths the numbers in the left tail of the distribution in an attempt to obtain a more accurate estimate of the 1% point of the distribution.

• Each day of the VaR estimate in the example would be updated using the most recent 500 days of data.

• Consider what happens on Day 501. We find out new values for all the market variables and we are able to calculate a new value for our portfolio. We then go through the procedure to calculate a new VaR. We use the market variables from Day 1 to Day 501 (leaving Day 0 from the list).
A major approach outside of the historical approach is to use model approach.

**Daily Volatilities**

- Usually we measure volatilities in years. Model-building approach to VaR, we measure time in days and the volatility of an asset is usually quoted as "volatility per day"
- Define $\sigma_{year}$ to be the volatility per year of a certain asset and $\sigma_{day}$ as the equivalent volatility per day of the asset. Assuming 252 trading days per year, we use
  
  $$\sigma_{year} = \sigma_{day} \sqrt{252} = \sigma_{day} \sqrt{N}$$

  or
  
  $$\sigma_{day} = \frac{\sigma_{year}}{\sqrt{252}} \approx 0.063 \sigma_{year}$$

- $\sigma_{day}$ is approximately equal to the standard deviation of the percentage change in the asset price in one day. For purposes of calculating VaR we assume exact equality. We define the daily volatility of an asset price as equal to the standard deviation of the percentage change in one day.
- We will apply this to different assets next time.
Single-Asset Case

We consider the case of VaR calculated using the model-building approach in a simple situation.

- Assume that the portfolio consists $10 million shares of Microsoft. Suppose that \( N = 10 \) and \( X = 99 \). We wish to consider the loss level over 10 days that we are 99% confident will not be exceeded.
- Consider 1-day time horizon first and extrapolate out.
- Assume that the volatility of Microsoft is 2% per day. The size of the position of $10 million implies that the standard deviation of daily changes in the value of the position is \( 0.02 \times 10,000,000 = $200,000. \)
- Expect that the change in the market variable is constant over the time period being considered is zero.
  - Not a bad assumption. If Microsoft’s growth rate is 20% per year then over one day, the expected return is \( 0.20/252 = 0.08\% \).
  - On the other hand the standard deviation of the return is 2%.
  - Over a 10-day period, the expected return is \( 0.08 \times 10 = 0.8\% \), whereas the standard deviation of the return is \( 2\sqrt{10} = 6.3\% \).
- The value of the portfolio has a standard deviation over a 1-day period of $200,000 and a mean of zero.
• Assume that the change is normally distributed then

\[ N(-2.33) = 0.01 \]

Therefore, there is a 1% probability that a normally distributed variable will decrease in value by more than 2.33 standard deviations.

• The 1-day 99% VaR for our portfolio consisting of a $10 million position in Microsoft is therefore

\[ 2.33 \times 200,000 = \$466,000 \]

• The \( N \)-day VaR is simple to compute:

\[ 466,000 \times \sqrt{10} = \$1,473,621 \]

**Example:** Consider a portfolio consisting of a $5 million position in AT&T, and suppose the daily volatility of AT&T is 1% (corresponds to 16% per year). A similar calculation shows that the standard deviation of the change in the value of the portfolio in 1 day is

\[ 5,000,000 \times 0.01 = 50,000 \]

Assuming the change is normally distributed, the 1-day 99% VaR is

\[ 50,000 \times 2.33 = \$116,500 \]

and the 10-day 99% VaR is

\[ 116,500 \times \sqrt{10} = \$368,405 \]
Two-Asset Case

Consider a portfolio consisting of both $10 million of Microsoft shares and $5 million of AT&T shares. Suppose that the returns on the two shares have a bivariate normal distribution and a correlation of 0.3.

A standard result in statistics says that if two variables $X$ and $Y$ have standard deviations equal to $\sigma_X$ and $\sigma_Y$ with the coefficients of correlation between them equal to $\rho$, the standard deviation of $X + Y$ is given by

$$\sigma_{X+Y} = \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y}$$

To apply the result, we set $X$ equal to the change in the value of the position in Microsoft over a 1-day period and $Y$ equal to the change in the value of the position in AT&T over a 1-day period, so that

$$\sigma_X = 200,000 \text{ and } \sigma_Y = 50,000$$

The standard deviation of the change in the value of the portfolio consisting of both stocks over a 1-day period is therefore

$$\sqrt{200,000^2 + 50,000^2 + 2 \times 0.3 \times 200,000 \times 50,000} = 220,227$$

The mean is assumed to be zero. The change is normally distributed, so the 1-day 99% VaR if therefore

$$220,227 \times 2.33 = 513,129$$

The 10-day 99% VaR is $\sqrt{10}$ times this or $1,622,657$. 
Benefits of Diversification

In the example we have worked on we have

1. The 10-day 99% VaR for the portfolio of Microsoft shares is $1,473,621.
2. The 10-day 99% VaR for the portfolio of AT&T shares is $368,405.
3. The 10-day 99% VaR for the portfolio of both Microsoft and AT&T shares is $1,622,657.

The amount

\[(1,473,621 + 368,405) - 1,622,657 = 219,369\]

represents the benefits of diversification. If Microsoft and AT&T were perfectly correlated, the VaR for the portfolio of both Microsoft and AT&T would equal the VaR for the Microsoft portfolio plus the VaR for the AT&T portfolio.

Less than perfect correlation leads to some of the risk being "diversified away".
To calculate the standard deviation of $\Delta P$, we define $\sigma_i$ as the daily volatility of the $i$th asset and $\rho_{ij}$ as the coefficient of correlation between returns on asset $i$ and asset $j$. This means that $\sigma_i$ is the standard deviation of $\Delta x_i$ and $\rho_{ij}$ is the coefficient of correlation between $\Delta_i$ and $\Delta x_j$.

The variance of $\Delta P$, denoted $\sigma_P^2$ is given by

$$\sigma_P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \alpha_i \alpha_j \sigma_i \sigma_j$$

or

$$\sigma_P^2 = \sum_{i=1}^{n} \alpha_i^2 \sigma_i^2 + 2 \sum_{i=1}^{n} \sum_{j<i} \rho_{ij} \alpha_i \alpha_j \sigma_i \sigma_j$$  \hspace{1cm} (1)$$

The standard deviation of the change over $N$ days is $\sigma_P \sqrt{N}$, and the 99% VaR for an $N$-day time horizon is $2.33\sigma_P \sqrt{N}$.  


Example: From our previous example, $\sigma_1 = 0.02$, $\sigma_2 = 0.01$, and $\rho_{12} = 0.3$. As noted $\alpha_1 = 10$ and $\alpha_2 = 5$ so that

$$\sigma_P^2 = 10^2 \times 0.02^2 + 5^2 \times 0.01^2 + 2 \times 10 \times 5 \times 0.3 \times 0.02 \times 0.01 = 0.0485$$

and $\sigma_P = 0.220$. This is the standard deviation of the change in the portfolio value per day in millions of dollars. The 10-day 99% VaR is $2.33 \times 0.220 \times \sqrt{10} = $1.623 million. (as seen earlier).
Handling Interest Rates

It is not easy to define a separate market variable for every single bond price or interest rate to which a company is exposed.

- One possibility is to assume that only parallel shifts in the yield curve occur.
- Then it is necessary to define only one market variable: the size of the parallel shift. The changes in the value of the bond portfolio can then be calculated using the duration relationship

\[ \Delta P = -DP \Delta y \]

where \( P \) is the value of the portfolio, \( \Delta P \) is the change in \( P \) in one day, \( D \) is the modified duration of the portfolio, and \( \Delta y \) is the parallel shift in 1 day.

- This approach does not usually give enough accuracy. The procedure usually followed is to choose as market variables the prices of zero-coupon bonds with standard maturities: 1 month, 3 month, 1 year, 2 years, 5 years, 7 years, 10 years, and 30 years.
- For the purposes of calculating VaR, the cash flows from instruments in the portfolio are mapped into cash flows occurring on the standard maturity dates.
**Example**: Consider a $1 million position in a Treasury bond lasting 1.2 years that pays a coupon of 6% semiannually. Coupons are paid in 0.2, 0.7, and 1.2 years, and the principal is paid in 1.2 years.

- This bond is, in the first case, regarded as a $30,000 position in 0.2 year zero-coupon bond plus a $30,000 position in a 0.7 year zero-coupon bond plus a $1.03 million position in a 1.2 year zero-coupon bond.
- The position in the 0.2 year bond is then replaced by an equivalent position in 1-month and 3-month zero-coupon bonds; the position in the 0.7-year bond is replaced by an equivalent position in 6-month and 1-year zero-coupon bonds; and the position in the 1.2-year bond is replaced by an equivalent position in 1-year and 2-year zero-coupon bonds.
- The result is that the position in the 1.2-year coupon-bearing bond is for VaR purposes regarded as a position in zero-coupon bonds having maturities of 1 month, 3 months, 6 months, 1 year, and 2 years.
- This is known as **cash-flow mapping**
Consider an example of a portfolio consisting of a long position in a single Treasury bond with a principal of $1 million maturing in 0.8 years. Suppose that the bond provides a coupon of 10% per annum payable semiannually. This means that the bond provides coupon payments of $50,000 in 0.3 years and 0.8 years.

It provides a principal payment of $1 million in 0.8 years. The Treasury bond can therefore be regarded as a position in a 0.3-year zero-coupon bond with a principal of $50,000 and a position in a 0.8-year zero-coupon bond with a principal of $1,050,000.

The position in the 0.3-year zero-coupon bond is mapped into an equivalent position in 3-month and 6-month zero-coupon bonds. The position in the 0.8-year zero-coupon bond is mapped into an equivalent position in 6-month and 1-year zero-coupon bonds. The result is that the position in the 0.8-year coupon-bearing bond is, for VaR purposes, regarded as a position in zero-coupon bonds having maturities of 3-months, 6-months, and 1 year.
**Mapping Procedure:** Consider the $1,050,000 that will be received in 0.8 years. We suppose that zero rates, daily bond price volatilities, and correlations between bond returns are:

<table>
<thead>
<tr>
<th>Maturity:</th>
<th>3-month</th>
<th>6-month</th>
<th>1-year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero rate</td>
<td>5.50</td>
<td>6.00</td>
<td>7.00</td>
</tr>
<tr>
<td>Bond price volatility percent per day</td>
<td>0.06</td>
<td>0.10</td>
<td>0.20</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Correlation between daily returns</th>
<th>3-mo. bond</th>
<th>6-mo. bond</th>
<th>1-yr bond</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-month bond</td>
<td>1.0</td>
<td>0.9</td>
<td>0.6</td>
</tr>
<tr>
<td>6-month bond</td>
<td>0.9</td>
<td>1.0</td>
<td>0.7</td>
</tr>
<tr>
<td>1-year bond</td>
<td>0.6</td>
<td>0.7</td>
<td>1.0</td>
</tr>
</tbody>
</table>

1. Interpolate between 6-month rate of 6% and the 1-year rate of 7% to obtain a 0.8-year rate of 6.6%. Annual compounding is assumed for all rates. The present value of $1,050,000 cash flow to be received in 0.8 years is

\[
\frac{1,050,000}{1.066^{0.8}} = 997,662
\]

2. Interpolate between the 0.1% volatility for the 6-month bond and the 0.2% volatility for the 1-year bond to get a 0.16% volatility for the 0.8-year bond.

3. Suppose we allocate \( \alpha \) of the present value to the 6-month bond and \( 1 - \alpha \) of the present value to the 1-year bond. Using

\[
\sigma_P^2 = \sum_{i=1}^{n} \alpha_i^2 \sigma_i^2 + 2 \sum_{i=1}^{n} \sum_{j < i} \rho_{ij} \alpha_i \alpha_j \sigma_i \sigma_j
\]
and matching variances, we obtain

\[ 0.0016^2 = 0.001^2 \alpha^2 + 0.002^2 (1 - \alpha)^2 + 2 \times 0.7 \times 0.001 \times 0.002 \alpha (1 - \alpha) \]

This is a quadratic equation that can be solved in the usual way to give \( \alpha = 0.320337 \). This means that

\[ 32.0337\% \]

of the value should be allocated to a 6-month zero-coupon bond and 67.9663% of the value should be allocated to a 1-year zero-coupon bond. The 0.8-year bond worth $997.662 is therefore replaced by a 6-month bond worth

\[ 997,662 \times 0.320337 = \$319,589 \]

and a 1-year bond worth

\[ 997,662 \times 0.679663 = \$678,074 \]

This cash-flow mapping scheme has the advantage that it preserves both the value and the variance of the cash flow. It can be shown that the weights assigned to the two adjacent zero-coupon bonds are always positive.
For the $50,000 cash flow received at time 0.3 years, we can carry out similar calculations. It turns out that the present value of the cash flow is $49,189. It can be mapped into a position worth $37,397 in a 3-month bond and a position worth $11,793 in a 6-month bond. We see:

<table>
<thead>
<tr>
<th>Position in 3-month bond($)</th>
<th>$50,000 received in 0.3 years</th>
<th>$1,050,000 received in 0.8 years</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Position in 6-month bond($)</td>
<td>37,397</td>
<td>319,589</td>
<td>331,382</td>
</tr>
<tr>
<td>Position in 1-year bond($)</td>
<td>11,793</td>
<td>678,074</td>
<td>678,074</td>
</tr>
</tbody>
</table>

Using the volatilities and correlations in the previous table, gives the variance of the change in the price of the 0.8-year bond with $n = 3$, $\alpha_1 = 37,397$, $\alpha_2 = 331,382$, $\alpha_3 = 678,074$, $\alpha_4 = 678,074$, $\sigma_1 = 0.0006$, $\sigma_2 = 0.0001$, $\sigma_3 = 0.0002$, and $\rho_{12} = 0.9$, $\rho_{13} = 0.6$, $\rho_{23} = 0.7$.

This variance is 2,628,518. The standard deviation of the change in the price of the bond is therefore

$$\sqrt{2,628,518} = 1,621.3$$

Because we are assuming that the bond is the only instrument in the portfolio, the 10-day 99% VaR is

$$1621.3 \times \sqrt{10} \times 2.33 = 11,946$$

or about $11,950.
Consider how a linear model can be used when there are options. Consider first a portfolio consisting of options on a single stock whose current price is $S$.

Suppose that the delta of the position is $\delta$. Since $\delta$ is the rate of change of the value of the portfolio with $S$, it is approximately true that

$$\delta = \frac{\Delta P}{\Delta S}$$

or

$$\Delta P = \delta \Delta S$$

where $\Delta S$ is the dollar change in the stock price in 1 day and $\Delta P$ is the dollar change in the portfolio in 1 day.

We define $\Delta x$ as the percentage change in the stock price in 1 day, so that

$$\Delta x = \frac{\Delta S}{S}$$

It follows that an approximate relationship between $\Delta P$ and $\Delta x$ is

$$\Delta P = S \delta \Delta x$$

When we have several underlying market variables that includes options, we can derive an approximate linear relationship between $\Delta P$ and $\Delta x$ similarly.
The relationship is

$$\Delta P = \sum_{i=1}^{n} S_i \delta_i \Delta x_i$$

where $S_i$ is the value of the $i$th market variable and $\delta_i$ is the delta of the portfolio with respect to the $i$th market variable. This yields

$$\Delta P = \sum_{i=1}^{n} \alpha_i \Delta x_i$$

with $\alpha_i = S_i \delta_i$.

**Example:** A portfolio consists of options on Microsoft and AT&T. The options on Microsoft have a delta of 1000, and the option on AT&T have a delta of 20,000. The Microsoft share price is $120, and the AT&T share price is $30. We find that approximately

$$\Delta P = 120 \times 1000 \times \Delta x_1 + 30 \times 20,000 \times \Delta x_2$$

or

$$\Delta P = 120,000 \Delta x_1 + 600,000 \Delta x_2$$

where $\Delta x_1$ and $\Delta x_2$ are the returns from Microsoft and AT&T in 1 day and $\Delta P$ is the resultant change in the value of the portfolio.
Assuming that the daily volatility of Microsoft is 2% and the daily volatility of AT&T is 1% and the correlation between the daily changes is 0.3, the standard deviation of $\Delta P$ is

$$\sqrt{(120 \times 0.02)^2 + (600 \times 0.01)^2 + 2 \times 120 \times 0.02 \times 600 \times 0.01 \times 0.3} = 7099$$

Since $N(-1.65) = 0.05$, the 5-day 95% VaR is $1.65 \times \sqrt{5} \times 7099 = $26,193.
Quadratic Model

The linear model for a portfolio with options is an approximation that does not take into account the $\Gamma$ of the options.

- $\Gamma$ measures the curvature of the relationship between the price of the option and the price of the underlying asset.
- The VaR for a portfolio is critically dependent on the left tail of the probability distribution of the portfolio value.
- When confidence level used is 99%, the VaR is the value in the left tail below which there is 1% of the distribution.
- A positive $\Gamma > 0$ tends to have a less heavy left tail than the normal distribution. If the distribution is normal, we will tend to calculate a VaR that is too high.
- A negative $\Gamma < 0$ tends to have a more heavy left tail than the normal distribution. If the distribution is normal, we will tend to calculate a VaR that is too low.

Figure 18.3  Probability distribution for value of portfolio: (a) positive gamma; (b) negative gamma.
• To calculate more precisely the VaR, we will need to take into account both delta and gamma to measure both $\Delta P$ and $\Delta x$.

Consider a portfolio dependent on a single asset whose price is $S$.

• Suppose $\delta$ and $\gamma$ are the delta and gamma of the portfolio. From the appendix, the equation

$$\Delta P = \delta \Delta S + \frac{1}{2} \gamma (\Delta S)^2$$

which is a more precise measure of the relationship between $\Delta S$ and $\Delta P$.

• Setting

$$\Delta x = \frac{\Delta S}{S}$$

then we find

$$\Delta P = S \delta \Delta x + \frac{1}{2} S^2 \gamma (\Delta x)^2$$
For a portfolio with \( n \) underlying market variables, with each instrument in the portfolio being dependent on only one of the market variables, we find

\[
\Delta P = \sum_{i=1}^{n} S_i \delta_i \Delta x_i + \sum_{i=1}^{n} \frac{1}{2} S_i^2 \gamma_i (\Delta x_i)^2
\]

where \( S_i \) is the value of the \( i \)th market variable, and \( \delta_i \) and \( \gamma_i \) are the delta and gamma of the portfolio with respect to the \( i \)th market variable.

When individual instruments in the portfolio may be dependent on more than one market variable, we get

\[
\Delta P = \sum_{i=1}^{n} S_i \delta_i \Delta x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{2} S_i S_j \gamma_{ij} \Delta x_i \Delta x_j
\]

where \( \gamma_{ij} \) is the "cross gamma" defined as

\[
\gamma_{ij} = \frac{\partial^2 P}{\partial S_i \partial S_j}
\]
Monte Carlo Simulation

1. Value the portfolio today in the usual way using the current values of market variables.
2. Sample once from the multivariate normal probability distribution of the $\Delta x_i$
3. Use the values of the $\Delta x_i$ that are sampled to determine the value of each market variable at the end of one day.
4. Revalue the portfolio at the end of the day in the usual way
5. Subtract the value calculated in Step 1 from the value in Step 4 to determine a sample $\Delta P$.
6. Repeat Steps 2 to 5 many times to build up a probability distribution for $\Delta P$

The VaR is calculated as the appropriate percentile of the probability distribution of $\Delta P$.

**Example:** Suppose that we calculate 5000 different sample values of $\Delta P$ in the above approach. The 1-day 99% VaR is the value of $\Delta P$ for the 50th worst outcome; the 1-day VaR 95% is the value of the $\Delta P$ for the 250th worst outcome, etc. The $N$-day VaR is usually assumed to be the 1-day VaR multiplied by $\sqrt{N}$.

One deficiency is that Monte Carlo simulation tends to be slow because the company’s complete portfolio has to be revalued many times.
Comparison of Approaches

Two methods

- **Historical Simulation**: historical data determines the joint probability distribution of the market variables. Advantage - avoids the need for cash-flow mapping. Disadvantages - computationally expensive and does not easily allow volatility updating schemes.

- **Model Building**: Advantages - results can be produced very quickly and can be used in conjunction with volatility updating schemes. Disadvantages - assumes that the market variables have a multivariate normal distribution (daily changes are usually not normally distributed) and gives poor results for small delta portfolios.
Many corporations carry out stress testing of their portfolios.

- Stress testing involves estimating how the portfolio would have performed under some of the most extreme market moes seen in the last 10 to 20 years.
- For example to test extreme movements in US equity prices, a company might set the percentage changes in all market variables equal to those on Oct. 19, 1987 (the S&P 500 moved by 22.3 standard deviations) or the variables equal to those on January 8, 1988 (the S&P 500 moved by 6.8 standard deviations).
- The scenarios used in stress testing are also sometimes generated by senior management.
- One technique sometimes used is to ask senior management to meet periodically and "brainstorm" to develop extreme scenarios that might occur given the current economic environment and global uncertainties.
- Stress testing can be considered as a way of taking into account extreme events that do occur from time to time but that are virtually impossible according to the probability distributions assumed for market variables.
- A 5-standard-deviation daily move in a market variable is one such extreme event. Under the assumption of a normal distribution, it happens about once every 7000 years, but in practice it is not uncommon to see a 5-standard-deviation daily move once or twice every 10 years.
- **Back testing**: It involves testing how well the VaR estimates would have performed in the past.
- Suppose that we are calculating a 1-day 99% VaR. Back testing would involve looking at how often the loss in a day exceeded in the 99% VaR that would have been calculated for that day. If this happened on about 1% of the days, we can feel reasonably comfortable with the methodology for calculating VaR. If it happened on say 7% of days, the methodology is suspect.
Summary

- VaR calculations are aimed at making a statement of the form:
  We are $X\%$ certain that we will not lose more than $V$ dollars in the next $N$ days.
  The variable $V$ is the VaR, $X\%$ is the confidence level, and $N$ days is the time horizon.

- One approach is to calculate VaR via historical simulation. This involves creating a database consisting of the daily movements in all market variables over a period of time.
  1. The first simulation trial assumes that the percentage changes in each market variable are the same as those on the first day covered by the database.
  2. The second simulation trial assumes that the percentage changes are the same as those on the second day, and so on.
  3. The change in the portfolio value, $\Delta P$ is calculated for each simulation trial, and the VaR is calculated as the appropriate percentile of the probability distribution of $\Delta P$.

- An alternative approach is via model-building. This is straightforward if two assumptions are made:
  1. The change in the value of the portfolio $\Delta P$ is linearly dependent on percentage changes in market variables.
  2. The percentage changes in market variables are multivariate normally distributed.

- The probability distribution of $\Delta P$ is then normal, and there are analytic formulas for relating the standard deviation of $\Delta P$ to the volatilities and correlations of the underlying market variables. The VaR can be calculated from well-known properties of the normal distribution.
When a portfolio includes options $\Delta P$ is not linearly related to the percentage changes in market variables. From knowledge of the gamma of the portfolio, we can derive an approximate quadratic relationship between $\Delta P$ and percentage changes in market variables. Monte Carlo simulation can then be used to estimate VaR.
Estimating Volatilities and Correlations

We discuss how to use historical data to extract estimates on the current and future levels of volatilities and correlations.

**Estimating Volatility:** Define $\sigma_n$ as the volatility of a market variable on day $n$, as estimated at the end of day $n - 1$. The square of the volatility, $\sigma^2_n$, on day $n$ is the **variance rate**. We described the standard approach to estimating $\sigma_n$ from historical data.

Suppose that the value of the market variable at the end of day $i$ is $S_i$. The variable $u_i$ is defined as the continuously compounded return during day $i$

$$u_i = \ln \frac{S_i}{S_{i-1}}$$

An unbiased estimate of the variance rate per day, $\sigma^2_n$ using the most recent $m$ observations on the $u_i$ is

$$\sigma^2_n = \frac{1}{m - 1} \sum_{i=1}^{m} (u_{n-i} - \bar{u})^2$$

where $\bar{u}$ is the mean of the $u_i$'s:

$$\bar{u} = \frac{1}{m} \sum_{i=1}^{m} u_{m-i}$$
For the purposes of monitoring daily volatility, the formula is usually changed in a number of ways

1. $u_i$ is defined as the percentage change in the market variable between the end of the day $i - 1$ and the end of day $i$ so that

$$u_i = \frac{S_i - S_{i-1}}{S_{i-1}}$$  \hspace{1cm} (3)

2. $\bar{u}$ is assumed to be zero

3. $m - 1$ is replaced by $m$

These three changes make very little difference to the estimates that are calculated, but they allow us to simplify the formula for the variance rate to

$$\sigma^2_n = \frac{1}{m} \sum_{i=1}^{m} u_{n-i}^2$$  \hspace{1cm} (4)

where $u_i$ is given by (3).
Weighting Schemes

The sigma given by (4) gives equal weight to

\[ u_{n-1}^2, u_{n-2}^2, \ldots, u_{n-m}^2 \]

Our objective is to estimate the current level of volatility \( \sigma_n \). Therefore, it makes sense to give more weight to recent data. One such model is

\[
\sigma_n^2 = \sum_{i=1}^{m} \alpha_i u_{n-i}^2
\]  

(5)

The variable \( \alpha_i \) is the amount of weight given to the observation \( i \) days ago. The \( \alpha \)'s are positive.

If we choose them so that \( \alpha_i < \alpha_j \) when \( i > j \), less weight is given to older observations. The weights must sum to unity, so we have

\[
\sum_{i=1}^{m} \alpha_i = 1
\]
An extension of the idea, called \textbf{ARCH}(m) in equation (5) is to assume that there is a long-run average variance rate and that this should be given some weight. This leads to the model that takes the form

\[
\sigma_n^2 = \gamma V_L + \sum_{i=1}^{m} \alpha_i u_{n-i}^2
\]  

(6)

where \(V_L\) is the long-run variance rate and \(\gamma\) is the weight assigned to \(V_L\). Because the weights must sum to unity, we have

\[
\gamma + \sum_{i=1}^{m} \alpha_i = 1
\]

Define \(\omega = \gamma V_L\), the model equation becomes

\[
\sigma_n^2 = \omega + \sum_{i=1}^{m} \alpha_i u_{n-i}^2
\]  

(7)
The Exponentially Weighted Moving Average Model or EWMA is a particular case of (5) where the weights $\alpha_i$ decrease exponentially as we move back through time.

Specifically $\alpha_{i+1} = \lambda \alpha_i$, where $\lambda$ is a constant between 0 and 1.

The formula is

$$\sigma_n^2 = \lambda \sigma_{n-1}^2 + (1 - \lambda) u_{n-1}^2 \quad (8)$$

The estimate $\sigma_n$ of the volatility of day $n$ is calculated from $\sigma_{n-1}$ and $u_{n-1}$.

We can see why this corresponds to exponentially decreasing weights.

$$\sigma_n^2 = \lambda \left[ \lambda \sigma_{n-2}^2 + (1 - \lambda) u_{n-2}^2 \right] + (1 - \lambda) u_{n-1}^2$$

$$= (1 - \lambda) (u_{n-1}^2 + \lambda u_{n-2}^2) + \lambda^2 \sigma_{n-2}^2$$

Substitute for $\sigma_{n-2}^2$ yields

$$\sigma_n^2 = (1 - \lambda) (u_{n-1}^2 + \lambda u_{n-2}^2 + \lambda^2 u_{n-3}^2) + \lambda^3 \sigma_{n-3}^2$$
Continuing yields

\[
\sigma_n^2 = (1 - \lambda) \sum_{i=1}^{m} \lambda^{i-1} u_{n-i}^2 + \lambda^m \sigma_{n-m}^2
\]  

(9)

For large \( m \), the term \( \lambda^m \sigma_{n-m}^2 \) is sufficiently small to be ignored so that this is the same as an equation with \( \alpha_i = (1 - \lambda) \lambda^{i-1} \). The weights for the \( u_i \) decline at rate \( \lambda \) as we move back through time. Each weight is \( \lambda \) times the previous weight.
Homework: Due Feb. 27, 5PM.
Graded:

- Prove the probability formulas for the trinomial trees. Hint, use the arguments from our binomial pricing.
- Program an explicit finite difference method to compute the price of
  - Price an American put option with $S_0 = 20, K = 21, \sigma = 30\%, r = 7\%, T = 0.5$.
  - Check using 3-step binary tree.
- Problem 18.17