Math 8680 Homework Fall 2005

1. Problem #2, p. 22 in Björner-Brenti.

2. Problem #4, p. 57 in Björner-Brenti.

3. For $w \in S_n$, let $l^*(w)$ be the minimum number k of transpositions whose product is $w = t_1 t_2 \cdots t_k$. So if w itself is any transposition, $l^*(w) = 1$. Show that

$$\sum_{w \in S_n} t^{l^*(w)} = (1+t)(1+2t)\cdots(1+(n-1)t).$$

4. Let $l_J(w)$ be the number of left cosets xW_J fixed by w under left multiplication.
(a) Prove (see §1.16 of Humphreys)

$$\text{(ALT)} \qquad \sum_{J \subset S} (-1)^{|J|} l_J(w) = det(w).$$

(b) In type A_{n-1}, note that $l_J(w)$ is the character of a permutation representation of the symmetric group S_n. Moreover, $det(w) = sign(w)$ is a character, so (ALT) is a character identity. Which Schur function identity is equivalent to (ALT)?

5. In this problem you will prove that if (W, S) is a finite Coxeter group of rank n whose basic invariants have degrees $d_1, \cdots, d_n$, then

$$\text{(Length GF)} \qquad W(q) = \sum_{w \in W} q^{l(w)} = \prod_{i=1}^{n} \frac{1-q^{d_i}}{1-q}.$$

Recall that we have already shown the recurrence relation

$$\sum_{J \subset S} \frac{(-1)^{|J|}}{W_J(q)} = \frac{q^N}{W(q)}, \qquad N = \#\text{ of positive roots.}$$

So to prove (Length GF), it is enough to show that the same recurrence holds for the right side

$$(5) \qquad \sum_{J \subset S} \frac{(-1)^{|J|}}{\overline{W}_J(q)} = \frac{q^N}{\overline{W}(q)}.$$

where

$$\overline{W}(q) = \prod_{i=1}^{n} \frac{1-q^{d_i}}{1-q}.$$

To define $\overline{W}_J(q)$ in (5), let W_J act on the vector space V_J of dimension $|J|$ spanned by the roots corresponding to $|J|$, with invariants $d_1(J), \cdots, d_{|J|}(J)$. Then

$$\overline{W}_J(q) = \prod_{i=1}^{|J|} \frac{1-q^{d_i(J)}}{1-q}$$

Rewrite (5) as

(5a)
$$\sum_{J\subset S}\frac{(-1)^{|J|}}{(1-q)^n\overline{W}_J(q)}=\frac{q^N}{(1-q)^n\overline{W}(q)}.$$

(a) Let Alt_k be the vector space of homogeneous alternating polynomials of degree k (that is $wp(x) = sign(w)p(x),\ deg(p) = k$). Show that the right side of (5a) is $\sum_{k=0}^{\infty} dim(Alt_k)t^k$. (Possible hints: Any alternating polynomial must be divisible by

$$\prod_{\alpha\in\Phi^+} H_\alpha(x), \text{what's left?}$$

or try Molien's theorem with $\lambda = det(w)$, a 1-dimensional irreducible representation of W.)

(b) Let $\mathbb{C}^{W_J}[x_1, x_2, \cdots, x_n]$ be the ring of W_J-invariants as W_J acts on V, whose k^{th} homogeneous part is $\mathbb{C}^{W_J}[x_1, x_2, \cdots, x_n](k)$. Show that

$$1/[(1-q)^n\overline{W}_J(q)] = \sum_{k=0}^{\infty} dim(\mathbb{C}^{W_J}[x_1, x_2, \cdots, x_n](k))q^k.$$

(c) Conclude that (5a) holds if, and only of,

$$\sum_{J\subset S}(-1)^{|J|}dim(\mathbb{C}^{W_J}[x_1, x_2, \cdots, x_n](k)) = dim(Alt_k).$$

(d) Use (ALT) to verify the equality in (c). (Possible hint: Check that

$$D = \frac{1}{|W|}\sum_{w\in W}(det\ w)w$$

is an idempotent, and that the trace of D on polynomials of degree k is $dim(Alt_k)$. For the left side of (c), you need to use (ALT) and the Frobenius reciprocity theorem for induced characters, after realizing $l_J(w) = 1_{W_J}^W(w)$ is an induced character.)