Math 8680 Homework #1a

1. In the HRR formula

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k}\frac{d}{dx}(...)$$

show that the kth term is $O(exp(\pi\sqrt{2n}/k\sqrt{3}))$, and conclude that $p(n)$ is asymptotic to

$$\frac{1}{4n\sqrt{3}}e^{\pi\sqrt{2n/3}}.$$

Solution: Fix k. Since $A_k(n)$ is a sum of $\phi(k)$ terms of modulus 1, we have $|A_k(n)| \leq \phi(k) < k$, independent of n. (In class I stated that $|A_k(n)| \leq 2k^{3/4}$.) So we need to check the n dependence in the derivative. The main term in the derivative is up to a constant, $cosh(\pi/k\sqrt{2(n-1/24)/3})/(n-1/24)$, which gives the claimed O term. If we explicitly work out the $k=1$ term using $A_1(n) = 1$ we get the stated first term.

2. Verify that the modular transformation

$$\sum_{n=-\infty}^{\infty} e^{sn-tn^2} = \sqrt{\pi/t} \sum_{n=-\infty}^{\infty} e^{(s+2\pi in)^2/4t}$$

does imply $\eta(-1/\tau) = \sqrt{\tau/i}\ \eta(\tau)$.

Solution: We have

$$\eta(\tau) = e^{\pi i\tau/12} \prod_{n=1}^{\infty}(1 - e^{2\pi i\tau n}).$$

So using the Euler pentagonal number theorem we have

$$\eta(\tau) = e^{\pi i\tau/12} \sum_{n=-\infty}^{\infty} e^{\pi in+\pi i\tau n(3n-1)}.$$

Choose $s = \pi i(1-\tau)$ and $t = -3\pi i\tau$ in the modular transformation to obtain

$$\begin{aligned}
\eta(\tau) =& e^{\pi i\tau/12}\sqrt{i/3\tau} \sum_{n=-\infty}^{\infty} e^{-(\pi i(1-\tau)+2\pi in)^2/12\pi i\tau} \\
=& e^{\pi i\tau/12}\sqrt{i/3\tau} \sum_{n=-\infty}^{\infty} e^{\pi i/6-\pi i/12\tau-\pi i\tau/12+\pi in/3-\pi in/3\tau-\pi in^2/3\tau} \\
=& e^{-\pi i/12\tau+\pi i/6}\sqrt{i/3\tau} \sum_{n=-\infty}^{\infty} w^{n^2}(we^{\pi i/3})^n, \text{ where } w = e^{-\pi i/3\tau} \\
=& e^{-\pi i/12\tau+\pi i/6}\sqrt{i/3\tau}(w^2, w^2e^{\pi i/3}, e^{-\pi i/3}; w^2)_\infty, \text{(by the JTP identity)} \\
=& e^{-\pi i/12\tau+\pi i/6}(1-e^{-\pi i/3})\sqrt{i/3\tau} \prod_{n=1}^{\infty}(1-w^{6n}) \\
=& e^{-\pi i/12\tau}\sqrt{i/\tau} \prod_{n=1}^{\infty}(1-e^{-2\pi in/\tau}) \\
=& \sqrt{i/\tau}\quad \eta(-1/\tau).
\end{aligned}$$

3. True or False? (Amaze your friends.)

$$\sum_{n=-\infty}^{\infty} e^{-n^2/\pi} = \pi.$$

Which decimal place is off?

Solution: False. If $s = 0$, $t = 1/\pi$ in the modular transformation, we have

$$\sum_{n=-\infty}^{\infty} e^{-n^2/\pi} = \pi \sum_{n=-\infty}^{\infty} e^{-n^2\pi^3}$$

So the left side is gretaer than π, but not by much. The right side converges very rapidly, if $R = e^{-\pi^3}$, the error is at most $2R/(1-R)$ which is less than 10^{-13}. Actually the error is very close to $2R$.