Math 8680 Homework #1b

1. Verify that $Pf(XAX^T) = det(X)Pf(A)$ for any skew-symmetric matrix A and a square matrix X.

Solution: $Pf(XAX^T)^2 = det(XAX^T) = det(X)^2 det(A) = det(X)^2 Pf(A)^2$. Since each side is a polynomial function of the entries of the matrices A and X, $Pf(XAX^T)/(det(X)Pf(A))$ must be ± 1. Then choosing $X = I$ yields the plus sign. Another proof is to verify the identity if X is an elementary matrix (from elementary row operations).

2. Suppose that G is a bipartite graph with vertex set $V = V_1 \cup V_2$, $V_1 \cap V_2 = \varnothing$, $|V_1| = |V_2|$. Order the vertices so that the incidence matrix A of G may be written

$$A = \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix}$$

Relate $Pf(A)$ and $Haf(A)$ to well-known functions of B. What does the Pfaffian-Hafnian method beome?

Solution: $Pf(A) = (-1)^{\binom{n}{2}} det(B)$, $Haf(A) = Perm(B)$. In this case the matchings are just permutations. We must check that the parity of the number of crossings is equal to the parity of the number of inversions$+\binom{n}{2}$. We do this by verifying this it works initially, then is preserved under adjacent switches.

The identity permutation corresponds to the matching $(1, n+1)(2, n+2)\cdots(n, 2n)$ which has $\binom{n}{2}$ crossings.

Next suppose that we switch two adjacent points in a permutation, so that the number of inversions changes by one. The same is true for the corresponding matching.

3. Verify that the sign chosen in

$$Pf\left(\frac{x_i - x_j}{1 - x_i x_j}\right)_{1\le i,j\le 2n} = \prod_{1\le i<j\le 2n} \frac{x_i - x_j}{1 - x_i x_j}$$

is correct.

Solution: We must show that the sign in

$$\sum_{m\in M_{2n}} (-1)^{c(m)} \prod_{i<j,\{i,j\}\in m} (x_i - x_j) \prod_{s<t,\{s,t\}\notin m} (1 - x_s x_t) = \Delta(x_1, \cdots, x_{2n})$$

is correct. Divide both sides by $x_1 - x_2$, and let $x_1, x_2 \to 1$,

$$\lim_{x_1,x_2\to 1} \frac{\Delta(x_1,\cdots,x_{2n})}{x_1 - x_2} = \Delta(x_3,\cdots,x_{2n}) \prod_{i=3}^{2n} (1 - x_i)^2.$$

For the left side, if 12 is an edge of m, the only crossings occur after 2, so we have

$$\prod_{i=3}^{2n}(1 - x_i)^2 \sum_{m'\in M_{2n-2}} (-1)^{c(m')} \prod_{i<j,\{i,j\}\in m'} (x_i - x_j) \prod_{s<t,\{s,t\}\notin m'} (1 - x_s x_t)$$

$$= \prod_{i=3}^{2n}(1 - x_i)^2 \Delta(x_3, \cdots, x_{2n})$$

by induction on n.

If 12 is not an edge of m, suppose that the edges are $1p$ and $2q$, and pair this match to the match which has edges $1q$ and $2p$. Since

$$\begin{aligned}&(x_1-x_p)(x_2-x_q)(1-x_1x_q)(1-x_2x_p)-(x_1-x_q)(x_2-x_p)(1-x_1x_p)(1-x_2x_q)\\ =&(x_1-x_2)(x_p-x_q)(1-x_1x_2)(1-x_px_q),\end{aligned}$$

when we divide by x_1-x_2 and then put $x_1=x_2=1$, the remaining terms are zero.

4. Let $A_{ij}=1$ for all $i<j$. Prove that $Pf(A)=1$

Solution: Suppose that 1 is connected to i, then the remaining edges form a matching on $2n-2$ points. If i is even, then there must be an even number of edges crossing with the $1i$ edge, so the sign is the same. If i is odd, then there must be an odd number of edges crossing with the $1i$ edge, so the sign is switched. There are n even integers from 2 to $2n$, and $n-1$ odd integers in the same interval, so by induction we obtain $Pf(A)=n-(n-1)=1$.

5. Suppose that A is an upper triangular matrix whose ij entry is a function of two variables, $A_{ij}(x_i,x_j)$, with $A_{ij}(z,z)=0$. Prove that $Pf(A_{ij})_{1\le i,j\le 2n}=0$ if we put $x_i=x_{i+1}$ for any $1\le i\le 2n-1$. Conclude that

$$Pf\left(\frac{x_i-x_j}{x_i}\right)_{1\le i,j\le 2n}=\prod_{i=1}^{2n-1}\frac{x_i-x_{i+1}}{x_i}.$$

Solution: We show that if $x_i=x_{i+1}$, then $Pf(A)=0$, since the terms in the Pffafian cancel each other. Let's pair matchings by switching the two edges connected to i and $i+1$, any matching with the edge $i,i+1$ is left unpaired. Since $x_i=x_{i+1}$, the unapired terms contribute $A(x_i,x_i)=0$. The remaining terms cancel because $A(x_j,x_i)A(x_{i+1},x_k)=A(x_j,x_{i+1})A(x_i,x_k)$, and we introduce (or delete) exactly one new crossing when the edges to i and $i=1$ are switched.

In the example this proves that x_i-x_{i+1} is a factor for $1\le i\le 2n-1$. Since the Pfaffian is a homogeneous rational function of degree 0, with poles at $x_1,\cdots,x_{2n-1}$, it must be a multiple of the claimed product. But the constant term of the right side product is one, which is exactly the constant term in the Pfaffian when all of the edge weights are 1 (see problem 4).

6. (This problem is too hard, but may be fun for you to think about. We will prove this before the semester ends.) Let

$$a_n(q)=\sum_{m\in M_{2n}}q^{crossing(m)}$$

be the generating function for the number of crossings of all complete matchings of $[2n]$. For example

$$a_1(q)=1,\quad a_2(q)=2+q,\quad a_3(q)=5+6q+3q^2+q^3.$$

Show that

$$a_n(q)=\frac{\sum_{k=-n}^{n}\binom{2n}{n+k}(-1)^k q^{\binom{k}{2}}}{(1-q)^n}.$$

What is the constant term of $a_n(q)$? What is $a_n(-1)$?

Solution: The constant term is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, and $a_n(-1) = 1$ by problem 4. For the rest of the problem we need to know that these are moments for q-Hermite polynomials.

7. Show that

$$det\left((x_i - y_j)\prod_{p=1,\neq j}^{n}(x_i + y_p)\right)_{1\le i,j\le n} = 2^{n-1}\Delta(x)\Delta(y)(x_1x_2\cdots x_n + (-1)^n y_1y_2\cdots y_n)$$

where $\Delta(x)$ is the Vandermonde determinant, and conclude that

$$Pf\left(\frac{x_i - x_j}{x_i + x_j}\right)_{1\le i,j\le 2n} = \prod_{1\le i<j\le 2n}\frac{x_i - x_j}{x_i + x_j}.$$

Solution: In fact

$$det\left((x_i - ty_j)\prod_{p=1,\neq j}^{n}(x_i + y_p)\right)_{1\le i,j\le n} = (1+t)^{n-1}\Delta(x)\Delta(y)(x_1x_2\cdots x_n + (-1)^n ty_1y_2\cdots y_n)$$

But if we put $t = 1$ we have

$$\begin{aligned}
&Pf^2\left(\frac{x_i - x_j}{x_i + x_j}\right)_{1\le i,j\le 2n} = det\left(\frac{x_i - x_j}{x_i + x_j}\right)_{1\le i,j\le 2n}\\
&= \prod_{i,j=1}^{2n}(x_i + x_j)^{-1} det\left((x_i - x_j)\prod_{p=1,\neq j}^{n}(x_i + x_p)\right)_{1\le i,j\le 2n}\\
&= 2^{2n}x_1x_2\cdots x_{2n}\Delta(\vec{x})\Delta(\vec{x})\prod_{i,j=1}^{2n}(x_i + x_j)^{-1} \quad \text{(by choosing } y_j = x_j\text{)}\\
&= \Delta(\vec{x})^2\prod_{1<i<j\le 2n}(x_i + x_j)^{-2}.
\end{aligned}$$

To prove the determinant (for $t = 1$), note that it is an anitsymmetric function of $\vec{x}$ and $\vec{y}$, just by permuting rows and columns, so we obtain the factors $\Delta(x)\Delta(y)$. The remaining part must be a symmetric polynomial (separately in $\vec{x}$ and $\vec{y}$) which is homogeneous of degree n. If we put $y_1 = 0$, we see that the first row has a factor x_1, so $x_1\cdots x_n$ divides the determaint if $y_1 = 0$. If $x_1 = 0$ the first row is the constant $-y_1\cdots y_n$, thus the remaining factor is $cx_1x_2\cdots x_n + dy_1y_2\cdots y_n$ for some constants c and d. Moreover since the determinant is unchanged if $y_i \to -x_i$, $x_i \to -y_i$, we have $d = (-1)^n c$.

So it remains to do the $y_1 = 0$ case. In this case the ij entry is a polynomial of degree $n-1$ in x_i,

$$(x_i - y_j)\prod_{p=2,p\neq j}^{n}(x_i + y_j), \quad j \ge 2$$

If we subtract the 1st column

$$\prod_{p=2,p\neq j}^{n}(x_i + y_j).$$

from the jth column, the ij entry becomes a polynomial in x_i of degree $n-2$,

$$-2y_j \prod_{p=2,p\neq j}^{n} (x_i + y_j).$$

Now use the 2nd column to reduce the degree in x_i to $n-3$ of columns past two, the new ij entry for $j \geq 3$ is

$$-2y_j(y_2 - y_j) \prod_{p=3,p\neq j}^{n} (x_i + y_j).$$

We continue in this way, reducing the degree in x_i of the column past 3 by one, until the the ij entry has degree $n-j$ in x_i

$$2\prod_{p=1}^{j-1}(y_p - y_j) \prod_{p=j+1}^{n} (x_i + y_j), \quad j \geq 2$$

Now we can use the Vandermonde determinant to complete the evaluaiton.

For Pfaffian we need the corect sign. As in problem 3 this is to check that the sign in

$$\sum_{m\in M_{2n}} (-1)^{c(m)} \prod_{i<j,\{i,j\}\in m} (x_i - x_j) \prod_{s<t,\{s,t\}\notin m} (x_s + x_t) = \Delta(x_1, \cdots, x_{2n})$$

is correct. We show that the coefficient of $x_1^{2n-1}x_2^{2n-2}\cdots x_{2n}^0$ in this homogeneous polynomial of degree $\binom{2n}{2}$ is 1. We must always choose the term x_i and the term x_s to obtain this term, thus the coefficient is 1, by problem 4.