Math 8680 Homework #2

1. Suppose that an OPS $\{p_n(x)\}$ exists for the linear functional L. Prove that if all of the odd moments are 0, then $p_n(-x) = (-1)^n p_n(x)$. Prove that

$$p_n(x) = (x - c_n)p_{n-1}(x) - \lambda_n p_{n-2}(x),$$

where $c_n = 0$.

Solution: We show that $(-1)^n p_n(-x)$ is another monic OPS for L, so by uniqueness we have $(-1)^n p_n(-x) = p_n(x)$. This clearly implies that $c_n = 0$. By replacing x by $-x$ in the 3-term recurrence relation for $p_n(x)$ we have a 3-term recurrence relation for $(-1)^n p_n(-x)$. It remains to show that that these polynomials are orthogonal to 1. However,

$$L(p_n(-x)) = L(p_n(x))$$

because all odd moments of L are zero. By orthogonality of the $p_n(x)$, we have $L(p_n(x)) = 0$, so $L(p_n(-x)) = 0$.

2. Recall that the Charlier polynomials are defined by

$$C_n^a(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{x}{k} k!(-a)^{n-k}$$

Show that

$$\text{(2a)} \qquad \sum_{n=0}^{\infty} C_n^a(x)\frac{t^n}{n!} = exp(-at)(1+t)^x.$$

The measure for $C_n^a(x)$ is purely discrete with jumps of $e^{-a}a^x/x!$ at $x = 0, 1, \cdots$. Show that the normalized moments are given by

$$\text{(2b)} \qquad \mu_n = \sum_{k=1}^{n} S(n,k)a^k,$$

where $S(n,k)$ is the Stirling number of the 2nd kind. Find the exponential generating function for

$$f(n_1, \cdots, n_k) = L(C_{n_1}^a(x)C_{n_2}^a(x)\cdots C_{n_k}^a(x))$$

and give a combinatorial interpretation for $f(n_1, \cdots, n_k)$ similar to the matching interpretation for the Hermite polynomials. Also, find a combinatorial interpretation for $C_n^a(x)$ which immediately implies (2a) from the exponential formula.

Solution: From (2a) we have

$$\begin{aligned} exp(-at)(1+t)^x &= \sum_{j=0}^{\infty} \frac{(-at)^j}{j!} \sum_{k=0}^{\infty} \frac{(-x)_k(-t)^k}{k!} \\ &= \sum_{n=0}^{\infty} t^n/n! \sum_{k=0}^{n} \binom{n}{k}\binom{x}{k}(-a)^{n-k}. \end{aligned}$$

We have

$$
\begin{aligned}
L((x)_n) =& e^{-a}\sum_{x=0}^{\infty}\frac{a^x}{x!}(x)_n = e^{-a}\sum_{x=n}^{\infty}\frac{a^x}{(x-n)!} \\
=& a^n.
\end{aligned}
$$

Thus

$$
\mu_n = L(x^n) = L\left(\sum_{k=1}^{n} S(n,k)(x)_k\right) = \sum_{k=1}^{n} S(n,k)a^k.
$$

From (2a) we have

$$
\begin{aligned}
\sum_{n_1,\cdots,n_k\geq 0} f(n_1,\cdots,n_k)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_k^{n_k}}{n_k!} =& L\left(\prod_{i=1}^{k} e^{-at_i}(1+t_i)^x\right) \\
=& e^{-ae_1(t_1,\cdots,t_k)}e^{-a}e^{a\prod_{i=1}^{k}(1+t_i)} \\
=& e^{a\sum_{i=2}^{k} e_i(t_1,\ldots,t_k)}.
\end{aligned}
$$

So by the exponential formula, $f(n_1,\cdots,n_k)$ is the generating function for inhomogeneous decompositions of $K_{n_1+\cdots+n_k}$ into complete subgraphs $K_2,\cdots,K_k$, where a counts the number of connected components.

Let's renormalize to consider

$$
\sum_{k=0}^{n}\binom{n}{k}(x)_k a^{n-k} = \sum_{A\subset[n],\pi:A\to A} wt(\pi)a^{n-|A|},
$$

where π is a permutation on A and $wt(\pi) = x^{\#cycles(\pi)}$. The connected components of such a π are cycles ($(n-1)!$ of them), with weigth x, which give a contribuion of

$$
exp(\sum_{n\geq 1} xt^n(n-1)!/n!) = (1-t)^{-x},
$$

and isolated points outside of A, with weight a, so e^{at}. With our rescaling this gives the generating function (2a).

3. State and prove a theorem that basically says that the eigenvalues of a symmetric tridiagonal matrix are the zeros of an orthogonal polynomial, and that the eigenvectors are given by orthogonal polynomials evaluated at these zeros

Solution: Consider the symmetric tridiagonal matrix M_n whose diagonal entries are A_i, and whose off diagonal entries are B_i, $1 \leq i \leq n$. Let $p_n(x) = det(M_n - xI)$ be the characteristic polynomial of M_n. By expanding the determinant along the last row we have

$$
p_n(x) = (A_n - x)p_{n-1}(x) - B_{n-1}^2 p_{n-2}(x).
$$

Thus $(-1)^n p_n(-x)$ is a monic OPS for a positive definite linear function L (assuming B_i is a non-zero real for all i), and the eigenvalues are the zeros of $p(x)$.

Let
$$p_n(x) = (-1)^n q_n(x) B_1 \cdots B_n.$$
Then the 3-term recurrence relation for $q_n(x)$ is

(3a) $$xq_n(x) = B_n q_n(x) + A_n q_{n-1}(x) + B_{n-1} q_{n-2}(x).$$

If x_{nk}, $1 \le k \le n$ are the n real simple zeros of $p_n(x)$, define the eigenvector $v_k = (q_0(x_{nk}), \cdots, q_{n-1}(x_{nk}))$, so that $M_n(v_k) = x_{nk} v_k$ by (3a).

4. Given a positive definite linear functional L, and a real number a which is not a zero of any orthogonal polynomnial $p_n(x)$ for L, let $L^*(p(x)) = L(p(x)(x-a))$. Show that the monic OPS for L^* is
$$p_n^*(x) = \frac{1}{(x-a)p_n(a)}\big(p_{n+1}(x)p_n(a) - p_{n+1}(a)p_n(x)\big).$$

Solution: It is clear that $p_n^*(x)$ is a monic polynomial in x of degree n. To check orthogonality, we use
$$\begin{aligned} L^*(p_n^*(x)x^j) =& L\left(\frac{1}{(x-a)p_n(a)}\big(p_{n+1}(x)p_n(a) - p_{n+1}(a)p_n(x)\big)(x-a)x^j\right) = \\ =& \frac{1}{p_n(a)} L\big((p_{n+1}(x)p_n(a) - p_{n+1}(a)p_n(x))x^j\big) \\ =& \frac{1}{p_n(a)}\big(p_n(a)L(p_{n+1}(x)x^j) - p_{n+1}(a)L(p_n(x)x^j)\big). \end{aligned}$$
The last term is zero for $j < n$ by L-orthogonality.

5. Give the details that Gaussian quadrature is unique: if there exists real numbers $A'_{nk} > 0, 1 \le k \le n$, and $x'_{n1} < x'_{n2} < \cdots < x'_{nn}$ such that
$$L(p(x)) = \sum_{k=1}^{2n} A'_{nk} p(x'_{nk})$$
for all real polynomials $p(x)$ of degree $\le 2n-1$, then $A'_{nk} = A_{nk}$ and $x'_{nk} = x_{nk}$ for all k.

Solution: Suppose that some x'_{nk} is not one of the x_{ni}, $1 \le i \le n$. Define a polynomial $p(x)$ of degree $2n-1$ by
$$p(x) = \prod_{i=1}^{n}(x - x_{ni}) \prod_{i \ne k}(x - x'_{ni}).$$
Then Gaussian quadrature gives
$$L(p(x)) = 0,$$
while the alternative form above gives
$$L(p(x)) = A'_{nk} p(x'_{nk}) \ne 0.$$
This is a contradiction, so the locations x_{nk} are uniquely determined. In class we proved a formula for A_{nk} in terms of the locations, so it is uniquely determined also.