Math 8680 Homework #3

1. Prove that the eigenspace decomposition of a distance transitive graph yields inequivalent, irreducible representations of the automorphism group.

Solution: Let G be such a graph with n vertices. Let $A = Aut(G)$ and B be the stablizer subgroup of a fixed vertex v of the graph. By distance transitivity, the B-orbits on the set of vertices of G are precisely the spheres of radius i, $0 \le i \le d$. Now choose any linear transformation on the complex vector space spanned by the vertices. By distance transitivity, any such linear transformation is a linear combination of the adjacency matrices A_i, $0 \le i \le d$. So the algebra generated by A is the commuting algebra of the representation. It is clearly commutative, so the representation is multiplicity free.

2. The Krawtchouk polynomials $\hat{K}_k(x, 1/2, N)$, $0 \le k \le N$, may be defined by

$$\hat{K}_k(x, 1/2, N) = \sum_{j=0}^{k} \binom{x}{j} \binom{N-x}{k-j} (-1)^{k-j}.$$

(a) Show that if x is an integer between 0 and N, then

$$\text{(GF)} \qquad \sum_{k=0}^{N} \hat{K}_k(x, 1/2, N) t^k = (1+t)^x (1-t)^{N-x}.$$

and conclude that $\hat{K}_k(x, 1/2, N) = (-1)^k \hat{K}_k(N-x, 1/2, N)$.

(b) Show that if $K_k(x, 1/2, N) = \hat{K}_k(x, 1/2, N)/\binom{N}{k}(-1)^k$, then for integer values of x between 0 and N we have

$$K_k(x, 1/2, N) = K_x(k, 1/2, N).$$

(c) Show that

$$\frac{1}{2^N} \sum_{k,j=0}^{N} t_1^k t_2^j \sum_{x=0}^{N} \binom{N}{x} \hat{K}_k(x, 1/2, N) \hat{K}_j(x, 1/2, N) = (1 + t_1 t_2)^N,$$

and conclude that $\hat{K}_k(x, 1/2, N)$, $0 \le k \le N$ is a finite set of orthogonal polynomials. What is the L^2-norm?

(d) Find the 3-term recurrence relation for $\hat{K}_k(x, 1/2, N)$.

(e) Let G be the graph of the N-dimensional cube. Find the constants α_i, β_i, and γ_i such that

$$A_1 A_i = \alpha_i A_{i+1} + \beta_i A_i + \gamma_i A_{i-1}.$$

Find the eigenvalues λ_i, $0 \le i \le N$, and the polynomials $p_i(\lambda_j)$. Write out the orthogonality relation, and identify $dim\ V_i$, and the size if a sphere of radius i, v_i.

Solution: If we find the coefficient of t^k in (GF) we get

$$\sum_{j=0}^{k} \binom{x}{j} \binom{N-x}{k-j} (-1)^{k-j} = \hat{K}_k(x, 1/2, N).$$

Since replacing x by $N-x$ just replaces t by $-t$, we therefore have $\hat{K}_k(x,1/2,N) = (-1)^k \hat{K}_k(N-x,1/2,N)$.

As a hypergeometric series we have

$$K_k(x,1/2,N) = \frac{\binom{N-x}{k}}{\binom{N}{k}} \, {}_2F_1(-x,-k;N-x-k+1;-1) = \, {}_2F_1(-x,-k;-N;2)$$

from the terminating ${}_2F_1$ transformation

$$(B)_k \, {}_2F_1(-k,A;B;x) = (B-A)_k \, {}_2F_1(-k,A;1-k+A-B;1-x).$$

This is clearly symmetric in x and k.

The summation claim is evident from (GF) and the binomial theorem,

$$\frac{1}{2^N}\left((1+t_1)(1+t_2)+(1-t_1)(1-t_2)\right)^N = (1+t_1t_2)^N.$$

Since this is a function of t_1t_2, we have orthogonality. This also gives $||p_k||^2 = \binom{N}{k}$.

The leading term of $\hat{K}_k(x,1/2,N)$ is $2^k/k!$, so

$$\text{(RR)} \qquad (k+1)\hat{K}_{k+1}(x) = (2x-b_k)\hat{K}_k(x) - \lambda_k \hat{K}_{k-1}(x).$$

Since $\hat{K}_k(0,1/2,N) = (-1)^k \hat{K}_k(N,1/2,N) = \binom{N}{k}(-1)^k$ we can substitute $x=0,N$ and find two equations for b_k and λ_k. The result is

$$b_k = N, \quad \lambda_k = N-k+1.$$

To compute α_i, we fix a pair of points (x,y), $d(x,y)=i+1$, and count the number of z such that $d(x,z)=i$, $d(z,y)=1$. It is $\alpha_i = i+1$. Similarly, $\beta_i = 0$, $\gamma_i = N-i+1$, so that

$$AP_i(A) = (i+1)P_{i+1}(A) + (N-i+1)P_{i-1}(A).$$

So by (RR) we have the eigenvalues $\lambda_j = N-2j$, and

$$P_i(\lambda_j) = (-1)^i \hat{K}_i(j,1/2,N).$$

We already wrote out the orthogonality relation, which implies $dim(V_j) = \binom{N}{j}$.

3. What are the orthogonal polynomials whose moment sequence is $\mu_n = M_n$, the Motzkin numbers?

Solution: We put $b_k = \lambda_k = 1$. Since $b_k = 1$ is a constant, we are just translating the $b_k = 0$ case, which was $\hat{U}_k(x-1)$.

4. Let $p_n(x) = \sum_{k=0}^n p_{nk}x^k$ be a monic OPS. Define μ_{nk} to be the generating function of all Motzkin paths from $(0,0)$ to (n,k). Thus $\mu_{n0} = \mu_n$, the moment sequence for $p_n(x)$. Show that $x^n = \sum_{k=0}^n \mu_{n,k}p_k(x)$, and reinterpret this statement as an inverse of a triangular matrix.

Solution: Let $x^n = \sum_{k=0}^n A_{n,k}p_k(x)$, then $L(x^np_k(x)) = A_{n,k}L(p_k^2) = \lambda_1\cdots\lambda_k A_{n,k}$. By our main theorem, this is the generating function for Motzkin paths from $(0,0)$ to (n,k). Clearly the $p_{n,k}$ matrix is inverse to the $\mu_{n,k}$ matrix.

5. Use the generalized Lloyd's theorem and the sphere packing condition to prove that there are no non-trivial perfect 2-codes in the N-dimensional cube.

Solution: This takes a few details. Note that for $N=5$ there is such a trivial code just $\{00000, 11111\}$. The generalized Lloyd polynomial is

$$\begin{aligned}\hat{K}_0(j,1/2,N)-&\hat{K}_1(j,1/2,N)+\hat{K}_2(j,1/2,N)\\ &=(2+N+N^2-4j-4Nj+4j^2)/2=L(j),\end{aligned} \tag{Lloyd}$$

while the sphere packing condition is

$$1+N+\binom{N}{2}|2^N, \text{ so } \quad 1+N+\binom{N}{2}=2^f, \quad 2L(j)=(2^{f+1}-4j-4Nj+4j^2).$$

For $N=5$ the Lloyd polynomial is $2(j-2)(j-4)$, which does have integer roots, and the sphere packing condition is $16|32$ which is correct. We know that $L(j)$ must have two distinct roots a and b, which must lie between 2 to $N-1$,

$$2L(j)=4(j-a)(j-b)=(2^{f+1}-4j-4Nj+4j^2)$$

This gives

$$ab=2^{f-1}, \quad a+b=N+1.$$

So both $a=2^A$ and $b=2^B$ are powers of 2, where $1\leq A<B$. Since $2^f-(N+1)=\binom{N}{2}$ we have

$$2^{A+B+1}-2^A-2^B=(2^A+2^B-1)(2^{A-1}+2^{B-1}-1).$$

By reducing mod 2 we see that $A=1$, so

$$2^{B+2}-2-2^B=(2^B+1)2^{B-1}.$$

which implies that $2=2^{B-1} \mod 2^{B-1}$, or $B=2$.

6. Let H be the infinite matrix whose ij entry is μ_{i+j}. If $\vec{a}=(a_1<\cdots<a_k)$ denotes k distinct rows of H, and $\vec{b}=(b_1<\cdots<b_k)$ denotes k distinct columns of H, let $H(\vec{a},\vec{b})$ denote the value of the corresponding $k\times k$ minor of H. Prove that

$$H(\vec{a},\vec{b})=\sum_{(\omega_1,\cdots,\omega_k,\sigma)} sgn(\sigma)\prod_{i=1}^{k} wt(\omega_i) \tag{6a}$$

is the generating function for all tuples $(\omega_1,\cdots,\omega_k,\sigma)$, where $\sigma\in S_k$, and ω_i is a Motzkin path from $(-a_i,0)$ to $(b_{\sigma_i},0)$. Conclude that you may take only the tuples of paths in (6a) which have no pairwise intersections at lattice points. Evaluate this determinant if $\vec{a}=\vec{b}=(0,1,\cdots,k-1)$, and explain how we previously derived this result.

Solution: Since the weights on the steps of a Motzkin path only depend upon the y-coordinate we can shift Motzkin paths to the left and retain the same weight. So (6a) is the definition of the determinant. If two paths intersect at a lattice point, we can do tail swapping to define a weight preserving, sign reversing involution to reduce to tuples with no lattice point intersection. For $\vec{a} = \vec{b} = (0, 1, \cdots, k-1)$, the only non-intersecting paths available go straight up and then down, the weight is $(\lambda_1)(\lambda_1\lambda_2)\cdots(\lambda_1\cdots\lambda_{k-1})$.

7. The discrete q-Hermite polynomials (when renormalized) have the monic 3-term recurrence relation

$$p_{n+1}(x) = xp_n(x) - q^{n-1}[n]_q p_{n-1}(x).$$

Find an matching interpretation for the polynomials $p_n(x)$ and for the moment sequence μ_n.

Solution: Let

$$p_n(x) = \sum_{matching\ m\ of K_n} x^{\#fixedpoints(m)}(-1)^{\#edges(m)}q^{stat(m)}$$

where

$$stat(m) = \sum_{edges\ e} stat(e),$$

where if $e = (a, b)$, $a < b$,

$$stat(e) = b - 1 + |\{c : a < c < b, m(c) < b\}|.$$

Also

$$\mu_n = \sum_{complete\ matches\ m\ of K_n} q^{stat2(m)},$$

where

$$stat2(m) = \#crossings(m) + \sum_{edges\ e} stat2(e),$$

where if $e = (a, b)$, $a < b$,

$$stat2(e) = |\{edges\ f : f = (c, d), c < b < d\}|.$$