# Complete monotonicity for inverse powers of some combinatorially defined polynomials

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**Reference:** A.D. Scott and A.D. Sokal, Complete monotonicity for inverse powers of some combinatorially defined polynomials, `arXiv:1301.2449`, to appear in *Acta Mathematica*.

## A class of positivity problems

**Given:**

- Polynomial $P$ (univariate or multivariate) with real coefficients, and $P(0) > 0$.
- Real number $\beta > 0$.

**Question:** Does $P^{-\beta}$ have all nonnegative Taylor coefficients?

**Example** (Friedrichs + Lewy, late 1920s):

$$P(y_1, y_2, y_3) = (1 - y_1)(1 - y_2) + (1 - y_1)(1 - y_3) + (1 - y_2)(1 - y_3)$$

and $\beta = 1$. Szegő (1933) solved a generalization of this problem:

$$P_n(y_1, \ldots, y_n) = \sum_{i=1}^{n} \prod_{j \neq i} (1 - y_j)$$

Then $P_n^{-\beta}$ has nonnegative Taylor coefficients whenever $\beta \geq 1/2$. Szegő's proof was surprisingly indirect (exploits identities for Bessel functions).

Alternate proofs:

- Kaluza (1933): Elementary but intricate (only for $n = 3$, $\beta = 1$).
- Askey and Gasper (1972): Jacobi polynomials instead of Bessel functions.
- Straub (2008): Elementary and simple (but only for $n = 3, 4$, $\beta = 1$).

The Lewy–Askey problem

Consider

$$
\begin{aligned}
P(y_1, y_2, y_3, y_4) \; = \;& \\
& (1-y_1)(1-y_2) + (1-y_1)(1-y_3) + (1-y_1)(1-y_4) + \\
& (1-y_2)(1-y_3) + (1-y_2)(1-y_4) + (1-y_3)(1-y_4)
\end{aligned}
$$

**Question:** Does the rational function $P(y_1, y_2, y_3, y_4)^{-1}$ have nonnegative Taylor coefficients?

Askey (1975): This problem "has caused me many hours of frustration":

> "So far the most powerful method of treating problems of this type is to translate them into another problem involving special functions and then use the results and methods which have been developed for the last two hundred years to solve the special function problem. So far I have been unable to make a reduction in [Lewy's problem] and so have no place to start."

But Dick adds that "it is possible to solve some problems without using special functions, so others should not give up on [Lewy's problem]."

**Theorem** (Scott–Sokal): $P(y_1, y_2, y_3, y_4)^{-\beta}$ has nonnegative Taylor coefficients for all $\beta \geq 1$.

Our methods are completely elementary; we don't use special functions.

This is a corollary of a much more general result ...

## A combinatorial point of view

Askey and Gasper (1972):

> "There should be a combinatorial interpretation of these results."
>
> "This might suggest new methods."

**Definition:** For a connected graph $G = (V, E)$, we define the spanning-tree generating polynomial

$$T_G(\mathbf{x}) = \sum_{T \in \mathcal{T}(G)} \prod_{e \in T} x_e$$

where $\mathbf{x} = \{x_e\}_{e \in E}$ are indeterminates indexed by the edges of $G$, and $\mathcal{T}(G)$ denotes the family of edge sets of spanning trees in $G$.

**Observation:** The Szegő polynomial

$$P_n(y_1, \ldots, y_n) = \sum_{i=1}^{n} \prod_{j \neq i} (1 - y_j)$$

is the spanning-tree polynomial $T_G(\mathbf{x})$ for the $n$-cycle $G = C_n$, after the change of variables $x_i = 1 - y_i$.

**Question:** Might Szegő's result extend to the spanning-tree polynomials of some wider class of graphs?

**Answer:** Yes!

In fact, we can generalize the change of variables $x_i = 1 - y_i$ to $x_i = c_i - y_i$ for constants $c_i > 0$ that are not necessarily equal. (This turns out to be important.)

# A result for series-parallel graphs

**Definition:** A connected graph $G = (V, E)$ is called *series-parallel* if it can be obtained from a tree by a finite sequence of series and parallel extensions of edges.

**Theorem**: Let $G = (V, E)$ be a connected series-parallel graph, and let $T_G(\mathbf{x})$ be its spanning-tree polynomial. Then, for all $\beta \geq 1/2$ and all choices of strictly positive constants $\mathbf{c} = \{c_e\}_{e \in E}$, the function $T_G(\mathbf{c} - \mathbf{y})^{-\beta}$ has nonnegative Taylor coefficients in the variables $\mathbf{y}$.

Conversely, if $G$ is a connected graph and there exists $\beta \in (0, 1) \setminus \{\frac{1}{2}\}$ such that $T_G(\mathbf{c} - \mathbf{y})^{-\beta}$ has nonnegative Taylor coefficients (in the variables $\mathbf{y}$) for all $\mathbf{c} > \mathbf{0}$, then $G$ is series-parallel.

Proof of direct half is completely elementary.

Proof of converse half relies on a deep result from harmonic analysis on Euclidean Jordan algebras (Gindikin 1975). More on this later.

## A rephrasing in terms of complete monotonicity

**Definition:** A $C^\infty$ function $f(x_1, \ldots, x_n)$ defined on $(0, \infty)^n$ is called *completely monotone* if its partial derivatives of all orders alternate in sign, i.e.

$$(-1)^k \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}} \;\geq\; 0$$

for all $x \in (0, \infty)^n$, all $k \geq 0$ and all choices of indices $i_1, \ldots, i_k$.

**Theorem** (rephrased): Let $G = (V, E)$ be a connected series-parallel graph, and let $T_G(\mathbf{x})$ be its spanning-tree polynomial. Then $T_G^{-\beta}$ is completely monotone on $(0, \infty)^E$ for all $\beta \geq 1/2$.

Conversely, if $G = (V, E)$ is a connected graph and there exists $\beta \in (0, 1) \setminus \{\frac{1}{2}\}$ such that $T_G^{-\beta}$ is completely monotone on $(0, \infty)^E$, then $G$ is series-parallel.

Allowing arbitrary constants $\mathbf{c} > \mathbf{0}$ allows the result to be formulated in terms of complete monotonicity, and leads to a characterization that is both necessary and sufficient.

Szegő's result (or rather, its generalization to arbitrary $\mathbf{c}$) extends to series-parallel graphs and no farther.

## But this is not the end . . .

- We can go far beyond series-parallel graphs if we relax our demands about the set of $\beta$ for which $T_G^{-\beta}$ has nonnegative coefficients.
- Key here is Kirchhoff's *matrix-tree theorem*, which expresses spanning-tree polynomials as determinants.

A Szegő-like result holds for very general determinantal polynomials:

**Theorem**: Let $A_1, \ldots, A_n$ $(n \geq 1)$ be $m \times m$ real or complex matrices or hermitian quaternionic matrices, and let us form the polynomial

$$P(x_1, \ldots, x_n) \;=\; \det\left( \sum_{i=1}^{n} x_i A_i \right)$$

in the variables $\mathbf{x} = (x_1, \ldots, x_n)$. [In the quaternionic case, det denotes the Moore determinant.] Assume further that there exists a linear combination of $A_1, \ldots, A_n$ that has rank $m$ (so that $P \not\equiv 0$).

(a) If $A_1, \ldots, A_n$ are real symmetric positive-semidefinite matrices, then $P^{-\beta}$ is completely monotone on $(0, \infty)^n$ for $\beta = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and for all real $\beta \geq (m-1)/2$.

(b) If $A_1, \ldots, A_n$ are complex hermitian positive-semidefinite matrices, then $P^{-\beta}$ is completely monotone on $(0, \infty)^n$ for $\beta = 0, 1, 2, 3, \ldots$ and for all real $\beta \geq m-1$.

(c) If $A_1, \ldots, A_n$ are quaternionic hermitian positive-semidefinite matrices, then $P^{-\beta}$ is completely monotone on $(0, \infty)^n$ for $\beta = 0, 2, 4, 6, \ldots$ and for all real $\beta \geq 2m-2$.

Proof is completely elementary.

## A more general approach

To understand better these curious conditions on $\beta$, take a slightly more general perspective:

**Definition:** Let $C$ be an open convex cone in a finite-dimensional real vector space $V$. Then a $C^\infty$ function $f\colon\, C \to \mathbb{R}$ is called *completely monotone* if for all $k \geq 0$, all choices of vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k \in C$, and all $x \in C$, we have

$$(-1)^k D_{\mathbf{u}_1} \cdots D_{\mathbf{u}_k} f(x) \;\geq\; 0$$

where $D_{\mathbf{u}}$ denotes a directional derivative.

We then have the following result that "explains" the previous one:

**Theorem**:

(a) Let $V$ be the real vector space $\mathrm{Sym}(m, \mathbb{R})$ of real symmetric $m \times m$ matrices, and let $C \subset V$ be the cone $\Pi_m(\mathbb{R})$ of positive-definite matrices. Then the map $A \mapsto (\det A)^{-\beta}$ is completely monotone on $C$ *if and only if* $\beta \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\} \cup [(m-1)/2, \infty)$.

Indeed, if $\beta \notin \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\} \cup [(m-1)/2, \infty)$, then the map $A \mapsto (\det A)^{-\beta}$ is not completely monotone on any nonempty open convex subcone $C' \subseteq C$.

(b,c) Analogous things for complex hermitian and quaternionic hermitian matrices.

Proof of direct half is completely elementary.

Proof of converse half again relies on the deep result from harmonic analysis on Euclidean Jordan algebras.

## Unified formulation in terms of Euclidean Jordan algebras

**Theorem:** Let $V$ be a simple Euclidean Jordan algebra of dimension $n$ and rank $r$, with $n = r + \frac{d}{2}r(r-1)$. Let $\Omega \subset V$ be the positive cone, and let $\Delta\colon V \to \mathbb{R}$ be the Jordan determinant. Then the map $x \mapsto \Delta(x)^{-\beta}$ is completely monotone on $\Omega$ if and only if $\beta \in \{0, \frac{d}{2}, \ldots, (r-1)\frac{d}{2}\}$ or $\beta > (r-1)\frac{d}{2}$.

Indeed, if $\beta \notin \{0, \frac{d}{2}, \ldots, (r-1)\frac{d}{2}\} \cup ((r-1)\frac{d}{2}, \infty)$, then the map $x \mapsto \Delta(x)^{-\beta}$ is not completely monotone on any nonempty open convex subcone $\Omega' \subseteq \Omega$.

Proof of "if" is completely elementary.

Proof of "only if" relies (once again) on a deep result from harmonic analysis on Euclidean Jordan algebras: the characterization of parameters for which the Riesz distribution is a positive measure (Gindikin 1975; but see also Casalis and Letac 1994, Sokal 2011 for elementary proofs).

Indeed, this theorem is essentially *equivalent* to the characterization of parameters for which the Riesz distribution is a positive measure.

The set of values of $\beta$ described here is known as the *Gindikin–Wallach set* and arises in a number of contexts in representation theory.

## Some applications

**Corollary 1:** Let $G = (V, E)$ be a connected graph with $p$ vertices, and let $T_G(\mathbf{x})$ be its spanning-tree polynomial. Then $T_G^{-\beta}$ is completely monotone on $(0, \infty)^E$ for $\beta = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and for all real $\beta \geq (p-2)/2$.

- Proof uses matrix-tree theorem to write $T_G(\mathbf{x})$ as determinantal polynomial involving real symmetric matrices.
- Claimed set of $\beta$ is *not* best possible in general. (Cf. series-parallel graphs.)
- Open problem: Determine exact set of allowable $\beta$ for each graph $G$.

**Second application:** Represent elementary symmetric polynomial

$$E_{2,4}(x_1, x_2, x_3, x_4) \;=\; x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$$

as determinantal polynomial with complex hermitian matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad A_4 = \begin{pmatrix} 1 & e^{-i\pi/3} \\ e^{i\pi/3} & 1 \end{pmatrix}$$

We then have:

**Corollary 2**: The function $E_{2,4}^{-\beta}$ is completely monotone on $(0, \infty)^4$ if and only if $\beta = 0$ or $\beta \geq 1$.

In particular, the Lewy–Askey function

$$\left( \sum_{1 \leq i < j \leq 4} (1 - y_i)(1 - y_j) \right)^{-\beta}$$

has nonnegative Taylor coefficients for all $\beta \geq 1$.
[And same holds if $1 - y_i$ is replaced by $c_i - y_i$ for arbitrary $c_i > 0$.]

## General formulation in terms of matroids

- Let $M$ be a matroid with ground set $E$.
- Let $\mathcal{B}(M)$ be its set of bases.
- Then the *basis generating polynomial* of $M$ is

$$B_M(\mathbf{x}) \;=\; \sum_{S \in \mathcal{B}(M)} \prod_{e \in S} x_e$$

  where $\mathbf{x} = \{x_e\}_{e \in E}$ are indeterminates.
- **Examples:**
  - Graphic matroid $M(G)$: Then $B_M(\mathbf{x}) =$ spanning-tree polynomial $T_G(\mathbf{x})$.
  - Uniform matroid $U_{r,n}$: Then $B_M =$ elementary symmetric polynomial $E_{r,n}$.

**Corollary:** Let $M$ be a matroid of rank $r$ on the ground set $E$, and let $B_M(\mathbf{x})$ be its basis generating polynomial.

(a) If $M$ is a regular [= real-unimodular] matroid, then $B_M^{-\beta}$ is completely monotone on $(0,\infty)^E$ for $\beta = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and for all real $\beta \ge (r-1)/2$. (This holds in particular if $M$ is a graphic or cographic matroid, i.e. for the spanning-tree or complementary-spanning-tree polynomial of a connected graph.)

(b) If $M$ is a complex-unimodular matroid [= sixth-root-of-unity matroid], then $B_M^{-\beta}$ is completely monotone on $(0,\infty)^E$ for $\beta = 0, 1, 2, 3, \ldots$ and for all real $\beta \ge r-1$.

[There would also be a result for quaternionic-unimodular matroids if we could understand better what they are!]

## Quadratic forms

**Observation:** $E_{2,3}$ (Szegő) and $E_{2,4}$ (Lewy–Askey) are quadratic forms in the variables $\mathbf{x}$.

**Question:** Can we say something about more general quadratic forms?

**Theorem:** Let $V$ be a finite-dimensional real vector space, let $B$ be a symmetric bilinear form on $V$ having inertia $(n_+, n_-, n_0)$, and define the quadratic form $Q(x) = B(x,x)$. Let $C \subset V$ be a nonempty open convex cone with the property that $Q(x) > 0$ for all $x \in C$. Then $n_+ \geq 1$, and moreover:

(a) If $n_+ = 1$ and $n_- = 0$, then $Q^{-\beta}$ is completely monotone on $C$ for all $\beta \geq 0$. For all other values of $\beta$, $Q^{-\beta}$ is not completely monotone on any nonempty open convex subcone $C' \subseteq C$.

(b) If $n_+ = 1$ and $n_- \geq 1$, then $Q^{-\beta}$ is completely monotone on $C$ for $\beta = 0$ and for all $\beta \geq (n_- - 1)/2$. For all other values of $\beta$, $Q^{-\beta}$ is not completely monotone on any nonempty open convex subcone $C' \subseteq C$.

(c) If $n_+ > 1$, then $Q^{-\beta}$ is not completely monotone on any nonempty open convex subcone $C' \subseteq C$ for any $\beta \neq 0$.

Here proofs of *both* sufficiency and necessity are completely elementary.

**Corollary:** The function $E_{2,n}^{-\beta}$ is completely monotone on $(0,\infty)^n$ if and only if $\beta = 0$ or $\beta \geq (n-2)/2$.

(Provides an alternate proof for Szegő and Lewy–Askey problems.)

## Proof of direct half of theorem for real symmetric matrices

**Step 1.** Let $A$ be a real symmetric positive-definite $m \times m$ matrix. We then have the Gaussian integral

$$(\det A)^{-1/2} \;=\; \int\limits_{\mathbb{R}^m} \exp(-\mathbf{x}^{\mathrm{T}} A \mathbf{x}) \prod_{j=1}^{m} \frac{dx_j}{\sqrt{\pi}}$$

where $\mathbf{x} = (x_1, \ldots, x_m)$. Differentiating under the integral sign shows that the $k$-fold directional derivative of $(\det A)^{-1/2}$ in directions $B_1, \ldots, B_k \in \Pi_m(\mathbb{R})$ has sign $(-1)^k$, because each derivative brings down a factor $-\mathbf{x}^{\mathrm{T}} B_i \mathbf{x} \le 0$. This proves complete monotonicity for $\beta = 1/2$.

**Step 2.** Any product of completely monotone functions is completely monotone. So we get complete monotonicity for $\beta = k/2$ for $k = 1, 2, 3, \ldots$ .

**Step 3.** For real $\beta > (m-1)/2$, use the integral representation

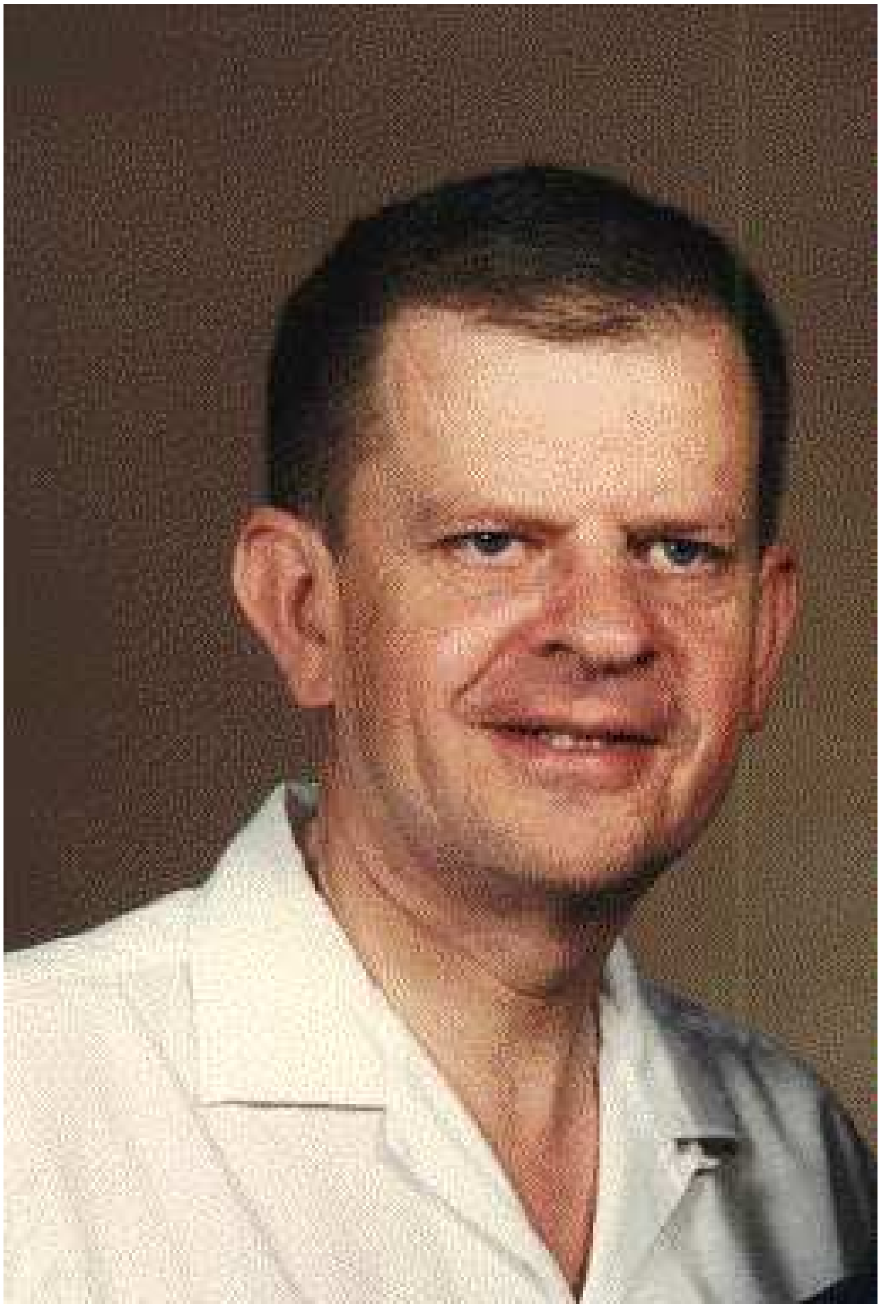
$$(\det A)^{-\beta} \;=\; \text{const} \;\times \int\limits_{B>0} e^{-\operatorname{tr}(AB)} \, (\det B)^{\beta - \frac{m+1}{2}} \, dB$$

where integration runs over real symmetric positive-definite $m \times m$ matrices $B$. Once again we can differentiate under the integral sign.

Proof for complex hermitian matrices is completely analogous. So the solution to the Lewy–Askey problem involves nothing more than a Gaussian integral!

Quaternionic case requires Jordan theory but is otherwise similar.

Quadratic-form case is similar (and even easier).



Happy Birthday, Dick!