Pre 1975-76 (17)

q-Hypergeometric Series:

$$ {}_{r+1}\phi_r\left(\begin{matrix} a_0, a_1, \ldots, a_r ; q, z \\ b_1, \ldots, b_r \end{matrix}\right) $$

$$ = \sum_{n=0}^{\infty} \frac{(a_0)_n (a_1)_n \cdots (a_r)_n z^n}{(q)_n (b_1)_n \cdots (b_r)_n}, $$

where

$$ (A)_n = (A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1}) $$

q-Appell Series

$$\Phi^{(1)}[a;b,b';c;x,y]$$

$$=\sum_{m,n\geq 0}\frac{(a)_{m+n}(b)_m(b')_n\,x^m y^n}{(q)_m(q)_n(c)_{m+n}}$$

THEOREM.

$$\Phi^{(1)}[a;b,b';c;x,y]$$

$$=\frac{(a)_\infty(bx)_\infty(b'y)_\infty}{(c)_\infty(x)_\infty(y)_\infty}$$

$$\times\,{}_3\phi_2\left(\begin{matrix}c/a,\ x,\ y\ ;\ q,\ a\\ bx,\ b'y\end{matrix}\right)$$

Program

The Seven Hundred Fourth Meeting
Northwestern University
Evanston, Illinois
April 27 – 28, 1973

Reprinted from the Notices

9:45– 9:55 (16) On self-conjugate graphs. Professor JAMES E. SIMPSON, University of Kentucky (704–A10)

10:00–10:10 (17) A scattering operator in the theory of discontinuous Markov processes. Dr. JOSEPH M. COOK, Argonne National Laboratory, Argonne, Illinois (704–F6)

10:15–10:25 (18) Symmetrization inequalities. Professor JOHN W. GAISSER*, Butler University, and Professor SEYMOUR SHERMAN, Indiana University (704–F4)

FRIDAY, 11:00 A. M.

Invited Address, Auditorium, Norris Center

(19) Some differential geometry in PL. Professor HOWARD A. OSBORN, University of Illinois (704–D8)

FRIDAY, 1:45 P. M.

Invited Address, Auditorium, Norris Center

(20 Theorems on counting subgroups of finite p–groups. Professor NORMAN BLACKBURN, University of Illinois, Chicago Circle (704–A6)

FRIDAY, 3:00 P. M.

Special Session on the Four-Color Problem, Room 2G, Norris Center

3:00– 3:20 (21) On the enumeration program for trying to settle the four-color conjecture. Professor FRANK HARARY, University of Michigan (704–A12)

3:30– 3:50 (22) On geographically good configurations. Preliminary report. Professor WOLFGANG R. G. HAKEN, University of Illinois (704–A17)

4:00– 4:20 (23) Non-Hamiltonian cubic planar maps. Dr. G. B. FAULKNER and Dr. DANIEL H. YOUNGER*, University of Waterloo (704–A19)

4:30– 5:00 Informal Session

FRIDAY, 3:00 P. M.

Special Session on Singularities of Varieties and Mappings, Room 2B, Norris Center

3:00– 3:20 (24) Local duality and rational singularities. Preliminary report. Professor JOSEPH LIPMAN, Purdue University (704–A14)

3:30– 3:50 Informal Session

4:00– 4:20 (25) On space curves as complete intersections. Dr. SHREERAM ABHYANKAR* and Mr. AVINASH SATHAYE, Purdue University (704–A8)

4:30– 4:50 Informal Session

5:00– 5:20 (26) Seifert n-manifolds. Professor PETER P. ORLIK*, University of Wisconsin, and Professor PHILIP D. WAGREICH, University of Pennsylvania (704–G1)

FRIDAY, 3:00 P. M.

Special Session on Special Functions, Room 2C, Norris Center

3:00– 3:20 (27) Lie theory and separation of variables. I. Parabolic cylinder coordinates. Professor WILLARD MILLER, JR., University of Minnesota (704–B10)

3:30– 3:50 (28) Convolution structures for Laguerre polynomials. Professor RICHARD A. ASKEY, University of Wisconsin, and Professor GEORGE GASPER, JR.*, Northwestern University (704–B9)

4:00– 4:20 (29) An expansion in ultraspherical polynomials with nonnegative coefficients. Professor CHARLES F. DUNKL, University of Virginia (704–B11)

4:30– 4:50 (30) Some new positive sums and integrals. Professor RICHARD A. ASKEY, University of Wisconsin (704–B5)

5:00– 5:20 (31) Uniform asymptotic expansions of a class of Meijer G-functions. Preliminary report. Professor JERRY L. FIELDS, University of Alberta (704–B29)

FRIDAY, 3:00 P. M.

Special Session on Commutative Harmonic Analysis, Room 2A, Norris Center

3:00– 3:20 (32) Fourier transforms and measure-preserving transformations. Professor O. CARRUTH McGEHEE, Louisiana State University (704–B3)

3:30- 4:20		Recess
4:30- 4:50	(33)	A family of countable compact P_*-hypergroups. Preliminary report. Professor CHARLES F. DUNKL and Professor DONALD E. RAMIREZ*, University of Virginia (704-B23)

FRIDAY, 3:00 P. M.

Special Session on Closed Curves on Surfaces and in Space, Room 2E, Norris Center

3:00- 3:20	(34)	Extensions through codimension one to sense preserving mappings. Professor CHARLES J. TITUS, University of Michigan (704-G11)
3:30- 3:50	(35)	Differential properties of Bernstein polynomial curves and surfaces. Preliminary report. Professor VICTOR T. NORTON, JR., Bowling Green State University (704-D4)
4:00- 4:20	(36)	Curvature measures for complexes. Professor FRANCIS J. FLAHERTY, Oregon State University (704-D5)
4:30- 4:50	(37)	Polygonal methods in global curve theory. Professor THOMAS F. BANCHOFF, Brown University (704-D7)
5:00- 5:20	(38)	Closed curves of constant torsion. Professor JOEL L. WEINER, Michigan State University (704-D1)
5:30- 5:50	(39)	On the Carathèodory conjecture. Preliminary report. Professor MICHAEL MENN, University of Illinois (704-G10)

SATURDAY, 8:30 A. M.

Special Session on Special Functions, Room 2C, Norris Center

8:30- 8:50	(40)	Special functions in combinatorial analysis. Professor LEONARD CARLITZ, Duke University (704-B12)
9:00- 9:20	(41)	Application of basic hypergeometric functions. Professor GEORGE E. ANDREWS, Pennsylvania State University (704-B22)
9:30- 9:50	(42)	Some mean value inequalities for the gamma function. Professor WALTER GAUTSCHI, Purdue University (704-B1)
10:00-10:20	(43)	Nicholson-type integrals for products of Gegenbauer functions. Preliminary report. Professor LOYAL DURAND III, University of Wisconsin (704-B24)
10:30-10:50	(44)	Legendre and Whittaker functions with large parameters. Professor FRANK W. J. OLVER, University of Maryland (704-B14)

SATURDAY, 8:30 A. M.

Special Session on Sample Functions of Stochastic Processes, Room 2G, Norris Center

8:30- 8:50	(45)	Asymptotic maxima of continuous Gaussian processes. Preliminary report. Professor MICHAEL B. MARCUS, Northwestern University (704-F3)
9:00- 9:20	(46)	Local times and supermartingales. Preliminary report. Professor JOSEPH HOROWITZ, University of Massachusetts (704-F5)
9:30- 9:50	(47)	Singular measures and increments of Brownian motion. Professor ROBERT P. KAUFMAN, University of Illinois (704-F1)
10:00-10:20	(48)	The maximal process of a process with stationary, independent increments. Professor BERT E. FRISTEDT, University of Minnesota (704-F2)
10:30-10:50	(49)	A functional form of Chung's law of the iterated logarithm for the maximum absolute partial sums of independent random variables. Preliminary report. Dr. MICHAEL J. WICHURA, University of Chicago (704-F7)

SATURDAY, 8:30 A. M.

Session on Topology and Algebra, Room 2B, Norris Center

8:30- 8:40	(50)	Arbitrary coefficients for cohomology. Preliminary report. Professor PAUL C. KAINEN, Case Western Reserve University (704-G8)
8:45- 8:55	(51)	Characterizations of absolute sets of interior condensation. Preliminary report. Professor HOWARD H. WICKE* and Professor JOHN M. WORRELL, JR., Ohio University (704-G6)
9:00- 9:10	(52)	The class of certain nilpotent semidirect products of p-groups. Preliminary report. Dr. LARRY J. MORLEY, Western Illinois University (704-A9)

9:15– 9:25	(53)	Tensor and direct products. Professor CARY H. WEBB, Chicago State University (704-A1)
9:30– 9:40	(54)	On a locally Cohen-Macauley condition for a graded ring. Preliminary report. Mr. JACOB R. MATIJEVIC, University of Chicago (704-A2)
9:45– 9:55	(55)	Principal ideal domains with specified residue fields. Preliminary report. Mr. RAYMOND C. HEITMANN, University of Wisconsin (704-A15) (Introduced by Professor Lawrence Levy)
10:00–10:10	(56)	Prime regular rings. Professor JOE W. FISHER, University of Texas at Austin, and Professor ROBERT L. SNIDER*, Northwestern University (704-A7)
10:15–10:25	(57)	Anisotropic Lie algebras. Professor JOHAN G. F. BELINFANTE, Carnegie-Mellon University (704-A13)
10:30–10:40	(58)	Automorphisms of quasi-associative algebras. Preliminary report. Professor TAE-IL SUH, East Tennessee State University (704-A18)

SATURDAY, 9:00 A. M.

Special Session on Commutative Harmonic Analysis, Room 2A, Norris Center
(An informal session will be held during the twenty-minute periods between talks.)

9:00– 9:20	(59)	Symmetric maximal ideals in M(G). Professor SADAHIRO SAEKI, Kansas State University (704-B15) (Introduced by Professor Colin C. Graham)
9:40–10:00	(60)	Multipliers of L^p which vanish at infinity. Professor GREGORY F. BACHELIS, Wayne State University (704-B33)
10:20–10:40	(61)	Compact groups with ordered duals. Professor HENRY HELSON, University of California, Berkeley (704-B7)

SATURDAY, 9:00 A. M.

Special Session on Closed Curves on Surfaces and in Space, Room 2E, Norris Center

9:00– 9:20	(62)	A classification of convex immersions of open 2-manifolds in R^3. Mr. EDGAR A. FELDMAN, City University of New York, Graduate Center (704-D6)
9:30– 9:50	(63)	Curves of double tangents on immersed surfaces. Preliminary report. Professor BENJAMIN R. HALPERN, Indiana University (704-D3)
10:00–10:20	(64)	Critical points for the total twist of a closed n-manifold in E^{2n+1}!. Preliminary report. Professor JAMES H. WHITE, University of California, Los Angeles (704-D2)
10:30–10:50		Informal problem session, to be conducted by Professor WILLIAM F. POHL, University of Minnesota

SATURDAY, 11:00 A. M.

Invited Address, Auditorium, Norris Center

(65) On measurability, pointwise convergence, and compactness. Professor ALEXANDRA IONESCU-TULCEA, Northwestern University (704-B21)

SATURDAY, 1:45 P. M.

Invited Address, Auditorium, Norris Center

(66) Quasi-triangular operators and the invariant subspace problem: Some recent progress. Professor CARL M. PEARCY, JR., University of Michigan (704-B20) (704-B20)

SATURDAY, 3:00 P. M.

Special Session on the Four-Color Problem, Room 2G, Norris Center

3:00– 3:20	(67)	The case of equality in the number of admissible boundary colorings. Professor MICHAEL O. ALBERTSON, Swarthmore College (704-A4)
3:30– 3:50	(68)	Symmetries of 3-regular 3-connected planar graphs. Preliminary report. Professor EDWARD F. MOORE, University of Wisconsin (704-G5)
4:00– 4:20	(69)	Computing configurations. Professor KENNETH I. APPEL, University of Illinois (704-A16)
4:30– 5:00		Informal Session

COMBINATORIAL IDENTITIES.

$$\sum_{k\geqq 0}\binom{2n+1}{2p+2k+1}\binom{p+k}{k}=\binom{2n-p}{p}2^{2n-2p}$$

$$\sum_{k\geqq 0}\binom{2n}{2p+2k}\binom{p+k}{k}=\frac{n}{2n-p}\times\text{same}$$

$$\sum_{k\geqq 0}(-1)^k\binom{n}{k}2^{-k}\binom{2k}{k}=\begin{cases}2^{-n}\binom{n}{\frac{1}{2}n}, & n \text{ even}\\ 0, & n \text{ odd}\end{cases}$$

Top 2 are special cases of Gauss's formula for ${}_2F_1[a,b;c;1]$. Last from Gauss's formula ${}_2F_1[a,b;\tfrac{1}{2}(a+b+1);\tfrac{1}{2}]$

$$\sum_{k\geq 0}\binom{2n+1}{2p+2k+1}_q\binom{p+k}{k}_{q^2}q^{k(2p+2k+1)} =$$

$$\binom{2n-p}{p}_{q^2}(-q)_{2n-2p}.$$

$$\sum_{k\geq 0}\binom{2n}{2p+2k}_q\binom{p+k}{k}_{q^2}q^{k(2p+2k-1)} =$$

$$\binom{2n-p}{p}_{q^2}(-q)_{2n-2p}\frac{(1-q^{2n})}{(1-q^{4n-2p})}.$$

$$\sum_{k\geq 0}(-1)^k\binom{n}{k}_q\frac{1}{(-q)_k}\binom{2k}{k}_q q^{\frac{1}{2}(n-k)(n-k-1)} =$$

$$\begin{cases}\dfrac{q^{\frac{1}{2}n^2}}{(-q)_{\frac{1}{2}n}}\cdot\dbinom{n}{\frac{1}{2}n}_q & \text{if } n \text{ even}\\ 0 & \text{if } n \text{ odd.}\end{cases}$$

21

Orthogonal Polynomials and Special Functions

RICHARD ASKEY
University of Wisconsin-Madison

LECTURE 7

Connection Coefficients

The last of the four basic questions to be considered is the problem of connection coefficients between two sequences of functions. If $\{p_n(x)\}$ and $\{q_n(x)\}$ are given $(n = 0, 1, \cdots)$ we wish to find the coefficients $c_{k,n}$ satisfying

$$q_n(x) = \sum_{k=0}^{\infty} c_{k,n} p_k(x). \tag{7.1}$$

Usually (but not always) $p_n(x)$ and $q_n(x)$ are polynomials of degree n, in which case there is no question of the existence of $c_{k,n}$. In this degree of generality nothing useful can be said about the connection coefficients, and in all instances I know, very little of any interest can be said unless the sets of functions are similar, a notion which will not be made precise. For example, when considering orthogonal polynomials the true intervals of orthogonality should be the same.

The first instance of connection coefficients goes back to Stirling [1]. He defined two sets of numbers, Stirling numbers of the first and second kind:

$$x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} S_n^{(k)} x^k \tag{7.2}$$

gives those of the first kind, and

$$x^n = \sum_{k=0}^{n} \mathcal{S}_n^{(k)} x(x-1)\cdots(x-k+1), \tag{7.3}$$

those of the second kind. There is no agreement on a standard notation, so that in the standard handbook of Abramowitz and Stegun [1] is the notation used above. These numbers are useful in combinatorial problems, but they do not play a role in the problems in these lectures and so will not be mentioned further.

For the classical polynomials there are two very old results which can be derived easily from generating functions:

$$L_n^{\alpha+\beta+1}(x) = \sum_{k=0}^{n} \frac{(\beta+1)_{n-k}}{(n-k)!} L_k^{\alpha}(x), \tag{7.4}$$

$$C_n^{\lambda}(\cos\theta) = \sum_{k=0}^{n} \frac{(\lambda)_{n-k}(\lambda)_k}{(n-k)!\,k!} \cos(n-2k)\theta. \tag{7.5}$$

The required generating functions are

$$(1-r)^{-\alpha-1} \exp(-xr/(1-r)) = \sum_{n=0}^{\infty} L_n^{\alpha}(x) r^n \tag{7.6}$$

In more standard notation these sums are

$$\frac{\Gamma(n+q)}{\Gamma(n)\Gamma(q+1)}\sum_{k=0}^{n-1}\frac{(-n+1)_k(p+1)_k}{(-n-q+1)_k k!}=\sum_{k=0}^{n-1}\frac{(p+q+1)_k}{k!}$$

and

$$\sum_{k=0}^{n-1}\frac{(p+1)_k}{k!}=\frac{\Gamma(n+p+1)}{\Gamma(n)\Gamma(p+2)}.$$

Putting them together gives

$$\sum_{k=0}^{n-1}\frac{(-n+1)_k(p+1)_k}{(-n-q+1)_k k!}=\frac{(p+2)_{n-1}}{(q+1)_{n-1}}=\frac{\Gamma(n+p+1)\Gamma(q+1)}{\Gamma(p+2)\Gamma(n+q)}.$$

It is likely that Chu only had this sum for integer values of p and q, but it is easy for us to conclude the same equality for complex p and q from his result. For both sides are rational functions of p and q which agree infinitely often. Thus Chu really had the value of the general polynomial ${}_2F_1$ when $x = 1$. He also had a special case of Saalschütz's formula (see Takács [1] and Carlitz [1]). Since most mathematical historians have missed these important results in Chu [1], this book should be translated so that mathematicians who cannot read Chinese can see what other treasures are contained in it. The fact that Chu had the "Pascal triangle" property of binomial coefficients is not surprising. It is a fairly obvious fact once the binomial coefficients are discovered. The Chu–Vandermonde sum (7.16) is much deeper, and not at all obvious. The distinction between these two results is really the difference between

$$(1+x)^a(1+x)=(1+x)^{a+1} \tag{7.17}$$

and

$$(1+x)^a(1+x)^b=(1+x)^{a+b}. \tag{7.18}$$

This seems a small difference, but to obtain (7.16) from this sum one must also know how to multiply polynomials of arbitrary degree and collect terms. This is far from obvious until adequate notation has been developed. And the special case of Saalschütz's formula that Chu had was absolutely incredible. To see this one only need look at the contortions some very good mathematicians went through to prove this in the middle of the twentieth century (see the papers in the bibliography of Takács [1]). Chu did not have the benefit of integral or differential calculus, the tools used by most of these people. He must have been a remarkable mathematician.

My favorite proof of (7.13) is first to calculate the coefficients in

$$P_n^{(\gamma,\beta)}(x)=\sum_{k=0}^{n}a_{k,n}P_k^{(\alpha,\beta)}(x) \tag{7.19}$$

and then use the quadratic transformations

$$\frac{P_n^{(\alpha,-1/2)}(2x^2-1)}{P_n^{(\alpha,-1/2)}(1)}=\frac{P_{2n}^{(\alpha,\alpha)}(x)}{P_{2n}^{(\alpha,\alpha)}(1)}=\frac{C_{2n}^{\alpha+1/2}(x)}{C_{2n}^{\alpha+1/2}(1)} \tag{3.13}$$

and

$$(3.14)\qquad \frac{xP_n^{(\alpha,1/2)}(2x^2-1)}{P_n^{(\alpha,1/2)}(1)} = \frac{C_{2n+1}^{\alpha+1/2}(x)}{C_{2n+1}^{\alpha+1/2}(1)}$$

on the series (7.14) when $\beta = \pm\frac{1}{2}$ to derive (7.13). The connection coefficients in (7.19) are very easy to derive by orthogonality and Rodriques' formula (2.1). Explicitly they will be given in (7.33). This method has the disadvantage of having to break a problem into two cases when it should not be necessary to do this, but that is a minor objection.

When x is set equal to one in (7.19) the resulting formula gives a special case of Dougall's formula. It is

$$(7.20)\qquad {}_5F_4\left(\begin{matrix}2a, a+1, b+a+1, c+a+1, -n \\ a, a-b, a-c, n+2a+1\end{matrix}; 1\right) = \frac{(1+2a)_n(1-b-c)_n}{(1+a-b)_n(1+a-c)_n}.$$

This gives a partial explanation of a fact which has interested me for years. Well-poised series are series

$$(7.21)\qquad \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(1)_n(a_1-a_2+1)_n\cdots(a_1-a_p+1)_n} x^n$$

in which numerator and denominator factors can be paired so that their sums are constant. After Kummer's sum of the well-poised ${}_2F_1$ at $x=-1$ and Dixon's sum of the well-poised ${}_3F_2$ at $x=1$, most of the well-poised series which can be summed are what I like to call "very well-poised", one of the numerator parameters is one more than the corresponding denominator parameter. This comes in very naturally from the orthogonality relation for Jacobi polynomials. For

$$\int_{-1}^{1} [P_n^{(\alpha,\beta)}(x)]^2(1-x)^\alpha(1+x)^\beta\,dx = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}$$

$$(7.22)\qquad = \frac{(\alpha+1)_n(\beta+1)_n((\alpha+\beta+1)/2)_n 2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{(1)_n(\alpha+\beta+1)_n((\alpha+\beta+3)/2)_n\Gamma(\alpha+\beta+2)}.$$

In a Jacobi series this appears in the denominator, so factors of the form $(2a)_n(a+1)_n/(a)_n$, where $a = (\alpha+\beta+1)/2$, tend to occur. This is only a partial explanation. These sums are fundamental results which can be approached from many ways and there is probably an explanation from each of these ways (see Burchnall–Lakin [1] for a partial explanation of this from the point of view of differential equations).

The third method generalizes the second in that one calculates the coefficients in

$$(7.23)\qquad P_n^{(\gamma,\delta)}(x) = \sum_{k=0}^{n} a_{k,n}P_k^{(\alpha,\beta)}(x).$$

This can be done in many different ways, two of them being a use of Rodrigues' formula and the expansion first in terms of $(1-x)^j$ and then expanding that in terms of $P_k^{(\alpha,\beta)}(x)$. The resulting coefficients are ${}_3F_2$'s and when $\delta = \beta$ or $\gamma = \alpha$

Advanced Seminar
on
SPECIAL FUNCTIONS

March 31-April 2, 1975

Sponsored by
The Mathematics Research Center
at
University of Wisconsin–Madison

SUNDAY, MARCH 30, 1975

p.m.

8:00-10:00	**Registration and Open House,** Blue Lounge, The Wisconsin Center, 702 Langdon Street	

MONDAY, MARCH 31, 1975

a.m.

8:00	**Registration,** first floor, The Wisconsin Center	
8:45	**Welcome,** *Robert M. Bock,* Dean, Graduate School, University of Wisconsin–Madison, and *R. Creighton Buck,* Acting Director, Mathematics Research Center	

SESSION I Chaired by *B. C. Carlson,* Iowa State University

9:00	Speaker:	*Willard Miller, Jr.,* University of Minnesota
	Topic:	Symmetry, separation of variables, and special functions
10:00	Coffee, Exhibit Gallery	
10:30	Speaker:	*K. M. Case,* Rockefeller University
	Topic:	Orthogonal polynomials revisited
11:30	Speaker:	*L. Durand,* University of Wisconsin–Madison
	Topic:	Nicholson-type integrals for products of Legendre functions and related topics
12:30	Lunch	

p.m.

SESSION II Chaired by *Yudell Luke,* University of Missouri, Kansas City

2:15	Speaker:	*George Gasper,* Northwestern University and Technical University Aachen
	Topic:	Positivity and special functions
3:15	Coffee, Exhibit Gallery	
3:30	Speaker:	*James McGregor,* Stanford University
	Topic:	Orthogonal polynomial systems in several variables
4:30	End of Session	
6:30	Cocktails (cash bar), Alumni Lounge, The Wisconsin Center, 702 Langdon Street	
7:30	Dinner, The Wisconsin Center Dining Room	

TUESDAY, APRIL 1, 1975

a.m.

SESSION III Chaired by *A. Erdélyi,* University of Edinburgh

9:00	Speaker:	*Samuel Karlin,* Stanford University and Weizmann Institute
	Topic:	Some applications of orthogonal

polynomials in several variables to stochastic processes

10:00 Coffee, Exhibit Gallery

10:30 Speaker: *Tom Koornwinder*, Mathematical Centre, Amsterdam
Topic: Two-variable analogues of the classical orthogonal polynomials

11:30 Speaker: *Alan James*, University of Adelaide
Topic: Special functions of matrix and single argument in statistics

12:30 Lunch

p.m.

SESSION IV Chaired by *L. Carlitz*, Duke University

2:00 Speaker: *N. J. A. Sloane*, Bell Telephone Laboratories
Topic: Krawtchouk polynomials in coding theory and combinatorics

3:00 Coffee, Exhibit Gallery

3:15 Speaker: *George Andrews*, Pennsylvania State University
Topic: Problems and prospects for basic hypergeometric functions

4:15 Speaker: *G.-C. Rota*, Massachusetts Institute of Technology
Topic: Some relationships between commutative algebra and special functions

5:15 End of Session

WEDNESDAY, APRIL 2, 1975

a.m.

SESSION V Chaired by *W. J. Cody, Jr.*, Argonne National Laboratory

9:00 Speaker: *F. W. J. Olver*, University of Maryland
Topic: Unsolved problems in the asymptotic estimation of special functions

10:00 Coffee, Exhibit Gallery

10:30 Speaker: *Bruce Berndt*, University of Illinois
Topic: Periodic Bernoulli numbers, summation formulas, and applications

11:30 Speaker: *Walter Gautschi*, Purdue University
Topic: Computational methods in special functions

12:30 Lunch

p.m.

Q-ANALOG OF EXTENDED MEIJER'S G-FUNCTION

$$G^{n,\nu_1,\nu_2,m_1,m_2}_{p,t,s,r}\left[\begin{matrix} x \\ \\ y \end{matrix}\,\middle|\,\begin{matrix}(\epsilon_p)\\(\gamma_t);(\gamma'_t)\\(\delta_s)\\(\beta_r);(\beta'_r)\end{matrix}\,\middle|\,q\right] =$$

$$\sum_{h=1}^{m_1}\sum_{k=1}^{m_2} x^{\beta_h} y^{\beta'_k} \frac{\prod_{j=\nu_1+1}^{t}(q/\gamma_j\cdot\beta_h)_\infty \prod_{j=\nu_2+1}^{t}(q/\gamma'_j\beta'_k)_\infty}{\prod_{j=1}^{\nu_1}(\gamma_j\beta_h)_\infty \prod_{j=1}^{\nu_2}(\gamma'_j\beta'_k)_\infty}$$

$$\cdot\frac{\prod_{j=m_1+1}^{r}(q\beta_h/\beta_j)_\infty \prod_{j=m_2+1}^{r}(q\beta'_k/\beta'_j)_\infty}{\prod_{\substack{j=1\\j\neq k}}^{m_2}(\beta'_j/\beta'_k)_\infty \prod_{j=1}^{n}(q\beta_h\beta'_k/\epsilon_j)_\infty}$$

$$\cdot\frac{\prod_{j=n+1}^{p}(\epsilon_j/\beta_h\beta'_k)_\infty \prod_{j=1}^{s}(\delta_j\cdot\beta_h\beta'_k)_\infty}{\prod_{\substack{j=1\\j\neq h}}^{m_1}(\beta_j/\beta_h)_\infty}$$

$$\cdot\,\Phi\left[\begin{array}{c|l|l} p & (q\beta_h\beta_k'/\epsilon_p) & (-1)^{m_1+p-n+t-\nu_1}x \\ t & (\gamma_t\beta_h);\ (\gamma_t'\beta_k') & \\ s & (\delta_s\beta_h\beta_k') & (-1)^{m_2+p-n+t-\nu_2}y \\ r-1 & (q\beta_h/\beta_r)_{h\neq r};\ (q\beta_k'/\beta_r')_{k\neq r} & \end{array}\right]_q$$

where

$$\Phi\left[\begin{array}{c|l|c} p & \epsilon_1,\epsilon_2,\ldots,\epsilon_p & \\ t & \gamma_1,\gamma_1',\ldots,\gamma_t,\gamma_t' & x \\ s & \delta_1,\delta_2,\ldots,\delta_s & y \\ r & \beta_1,\beta_1',\ldots,\beta_r,\beta_r' & \end{array}\right]_q$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\epsilon_1)_{m+n}\cdots(\epsilon_p)_{m+n}(\gamma_1)_m(\gamma_1')_n\cdots(\gamma_t)_m(\gamma_t')_n\,x^m y^n}{(\delta_1)_{m+n}\cdots(\delta_s)_{m+n}(\beta_1)_m(\beta_1')_n\cdots(\beta_r)_m(\beta_r')_n(q)_m(q)_n},$$

where

$$(a)_n=(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1}),$$

$$(a)_\infty=\lim_{n\to\infty}(a)_n.$$

APRIL FOOL

INFORMAL SESSION

2:00 This session will be concerned with the problems of computing special functions and the future of handbooks of special functions.

PROGRAM COMMITTEE

Richard Askey, Chairman
Loyal Durand
Joseph Hirschfelder
Frank W. J. Olver
Gladys G. Moran, Secretary

Proceedings

A proceedings of the Advanced Seminar will be published by Academic Press. The volume will be available approximately six months following the meeting and can be ordered directly from the publisher.

Advanced Registration

Registration by mail before March 25 is recommended to avoid congestion at the time of the Advanced Seminar and to assure the possibility of attendance and accommodations.

Please give a *complete* mailing address on the registration card. Include the name and address of employer if *different* from the address given.

The registration fee is $12.50, which includes the cost of the dinner on March 31. A check in this amount payable to *Mathematics Research Conference Fund* should accompany the registration card.

Registration on Arrival

The registration desk will be open during the sessions. The desk will be located at the entrance of the auditorium of The Wisconsin Center on the first floor. There will also be a registration desk at the Open House on Sunday evening in the Blue Lounge of The Wisconsin Center. If you do not register in advance by mail, you are urged to bring the completed registration card when you register on arrival.

Location

All sessions will be held in the auditorium of The Wisconsin Center building, Lake and Langdon streets, Madison. There is no parking space at this location. Visitors' parking is available at street level of the Helen C. White Hall, 600 N. Park Street (across from the Memorial Union), with entrance on Park Street (with ten-hour meters), and some ten-hour meter parking is available at the Memorial Union, with entrance on Langdon Street. A public parking ramp (Lake Street Ramp) is located on N. Lake Street near State Street. Also, persons coming in automobiles may park their cars in University Lot No. 60, located west of the campus on Walnut Street, for fifty cents per day (no overnight parking). Campus buses travel the mile from

1975-76 leading to RR (5)

W. Hahn (1949) found orthogonal polynomials that are q-analogs of the Jacobi polynomials:

$$p_n(x;\alpha,\beta|q)$$

$$= {}_2\phi_1\left(\begin{matrix} q^{-n}, \alpha\beta q^{n+1} \\ \alpha q \end{matrix}; q, qx\right)$$

Hahn's paper was the starting point for the 1975-76 seminar.

THEOREM (with ASKEY)

$$p_n(x;\gamma,\delta:q) = \sum_{k=0}^{n} a_{k,n}\, p_k(x;\alpha,\beta:q),$$

where

$$a_{k,n} = \frac{(-1)^k q^{\binom{k+1}{2}} (\gamma\delta q^{n+1})_k (q^{-n})_k (\alpha q)_k}{(q)_k (\gamma q)_k (\alpha\beta q^{k+1})_k}$$

$$\times\, {}_3\phi_2\left(\begin{matrix} q^{-n+k}, \gamma\delta q^{n+k+1}, \alpha q^{k+1} \\ \gamma q^{k+1}, \alpha\beta q^{2k+2} \end{matrix}; q, q\right).$$

WATSON'S q-ANALOG OF WHIPPLE'S THEOREM

$$ {}_8\phi_7\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{-n} ; q, X \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, aq^{n+1} \end{matrix}\right) $$

$$ = \frac{(aq)_n \left(\frac{aq}{de}\right)_n}{\left(\frac{aq}{d}\right)_n \left(\frac{aq}{e}\right)_n} {}_4\phi_3\left(\begin{matrix} \frac{aq}{bc}, d, e, q^{-n} ; q, q \\ \frac{aq}{b}, \frac{aq}{c}, \frac{deq^{-n}}{a} \end{matrix}\right) $$

where $X = \dfrac{a^2 q^{2+n}}{bcde}$.

5th and 7th Order (11)

WP74

On the *Expansion of some Infinite Products.* By Prof. L. J. ROGERS.

Received June 5th, 1893. Read June 8th, 1893.

1. It is a well-known theorem that, if $q < 1$, then

$$1/(1-\lambda)(1-\lambda q)(1-\lambda q^2)\ldots = 1+\frac{\lambda}{1-q}+\frac{\lambda^2}{(1-q)(1-q^2)}+\ldots \quad \ldots(1).$$

It will be found convenient to use the symbol (λ) for the infinite product $(1-\lambda)(1-\lambda q)(1-\lambda q^2)\ldots$, and to write the above equation in the form

$$1/(\lambda) = 1+\Sigma\frac{\lambda^r}{(1-q^r)!},$$

where r is to receive all positive integral values, and where $(1-q^r)!$ denotes the product $(1-q)(1-q^2)\ldots(1-q^r)$.

The following abbreviations will also be used in the following pages:—

$H_r(\lambda_1, \lambda_2, \lambda_3, \ldots)$ will denote the coefficient of x^r in

$$1/(\lambda_1 x)(\lambda_2 x)(\lambda_3 x)\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

while $h_r(\lambda_1, \lambda_2, \ldots)$ will be used for $H_r(\lambda_1, \lambda_2, \ldots)(1-q^r)!$ Moreover $H_r(\mu_1, \mu_2, \ldots/\lambda_1, \lambda_2, \ldots)$ will be written for the coefficient of x^r in

$$(\mu_1 x)(\mu_2 x)\ldots \div (\lambda_1 x)(\lambda_2 x)\ldots,$$

while $h_r(\mu_1, \mu_2\ldots/\lambda_1, \lambda_2, \ldots)$ will $= (1-q^r)!\,H_r(\mu_1, \mu_2, \ldots/\lambda_1, \lambda_2, \ldots)$.

WE BEGIN WITH A BRIEF RECAP OF ONE WAY TO PROVE THE ROGERS-RAMANUJAN IDENTITIES:

$$1+\frac{q}{1-q}+\frac{q^4}{(1-q)(1-q^2)}+\frac{q^9}{(1-q)(1-q^2)(1-q^3)}+\cdots$$

$$=\frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})\cdots}$$

AND

$$1+\frac{q^2}{1-q}+\frac{q^6}{(1-q)(1-q^2)}+\frac{q^{12}}{(1-q)(1-q^2)(1-q^3)}+\cdots$$

$$=\frac{1}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)(1-q^{12})(1-q^{13})\cdots}$$

ONE USES THIS GENERAL REDUCTION THEOREM (a.k.a. THE WEAK FORM OF BAILEY'S LEMMA):

IF

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}},$$

THEN

$$\sum_{n=0}^{\infty} q^{n^2} a^n \beta_n = \frac{1}{(aq)_\infty} \sum_{n=0}^{\infty} q^{n^2} a^n \alpha_n,$$

WHERE

$$(A)_n = (A;q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1})$$

(α_n, β_n) IS CALLED A BAILEY PAIR.

TO PROVE ROGERS-RAMANUJAN ONE PROVES THAT WHEN $a=1$, (α_n, β_n) is a BAILEY PAIR with

$$\beta_n = \frac{1}{(q)_n}, \quad \alpha_n = \begin{cases} 1 & \text{if } n=0 \\ (-1)^n q^{n(3n-1)/2}(1+q^n) & \text{if } n>0 \end{cases}$$

AND, WHEN $a=q$, (α_n, β_n) is a BAILEY PAIR with

$$\beta_n = \frac{1}{(q)_n}, \quad \alpha_n = (-1)^n q^{n(3n+1)/2}(1-q^{2n+1})$$

SP4 (No slide)

There are now two ways to prove this. One way is by q basic hypergeometric series (Bailey-Slater), the other is by recurrences (L J Rogers).

FIFTH ORDER MOCK THETA FUNCTIONS

EX.

$$f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n} =$$

$$\frac{1}{(q)_\infty} \sum_{\substack{n=0 \\ |j| \le n}}^{\infty} q^{n(5n+1)/2 - j^2} (-1)^j (1 - q^{4n+2})$$

ASSOCIATED BAILEY PAIR:

$$\alpha_n = \begin{cases} 1 & \text{if } n = 0 \\ q^{n(3n-1)/2} \left(q^n \sum_{j=-n}^{n} (-1)^j q^{-j^2} - \sum_{j=-n+1}^{n-1} (-1)^j q^{-j^2} \right) & \text{if } n > 0 \end{cases}$$

$$\beta_n = \frac{1}{(-q;q)_n}$$

SEVENTH ORDER MOCK THETA FUNCTIONS

EX.

$$\mathcal{F}_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n+1};q)_n}$$

$$= \frac{1}{(q)_\infty}\left(\sum_{\substack{n=0 \\ |j|\le n}}^{\infty} q^{7n^2+n-j^2}(1-q^{12n+6})\right.$$

$$\left. -2q\sum_{\substack{n=0 \\ 0\le j\le n}}^{\infty} q^{7n^2+8n-j^2-j}(1-q^{12n+13})\right)$$

ASSOCIATED BAILEY PAIR

$$\alpha_0 = 1$$

$$\alpha_{2n} = q^{3n^2+n} \sum_{|j| \le n} q^{-j^2} - q^{3n^2-n} \sum_{|j|<n} q^{-j^2}$$

if $n > 0$

$$\alpha_{2n+1} = -2q^{3n^2+4n+1} \sum_{j=0}^{n} q^{-j^2-j}$$

$$+2q^{3n^2+2n} \sum_{j=0}^{n-1} q^{-j^2-j}$$

$$\beta_n = \frac{1}{(q^{n+1};q)_n}$$

Finally we suggest a further study of the Bailey pair $(\alpha_n(b), \beta_n(b))$, where $\beta_n(b) = (bq)_n/(q)_{2n}$. The first few $\alpha_n(b)$ are

$$\begin{aligned}
\alpha_0(b) &= 1, \\
\alpha_1(b) &= -bq - b, \\
\alpha_2(b) &= b^2q^3 + bq^4 + bq^3 - q^2, \\
\alpha_3(b) &= -b^3q^6 - b^2q^8 - b^2q^7 - b^2q^6 - bq^8 + bq^5 + q^7 + q^5.
\end{aligned}$$

Furthermore $\Sigma_{n\geq 0} q^{n^2}\beta_n(b)$ is an important theta series for $b = 0$, $q^{-1/2}$, -1, and is the first seventh order mock series for $b = 1$. Can a representation of $\alpha_n(b)$ be found that directly yields these facts as special cases?

Orth Polys (5)

AT THE END OF THE PAPER

PARITY IN PARTITION IDENTITIES

WE FIND

$$\sum_{n\geq 0} \frac{q^{n^2} a^n (-bq;q)_n}{(q^2;q^2)_n}$$

$$= \frac{1}{(aq)_\infty} \sum_{n\geq 0} \frac{(-1)^n a^n q^{2n^2} (a^2;q^2)_n (1-aq^{2n})}{(q^2;q^2)_n (1-a)}$$

$$\times p_n\left(b; -\frac{a}{q}, -1; q\right)$$

where

$$p_n(y;A,B:q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, ABq^{n+1} \\ Aq \end{matrix}; q, yq\right)$$

(the litte q-Jacobi polynomials).

THIS APPROACH WAS RECENTLY PURSUED IN q-ORTHOGONAL POLYNOMIALS, ROGERS-RAMANUJAN IDENTITIES, AND MOCK THETA FUNCTIONS.

MAIN **THEOREM**. IF $\frac{qabc}{efg} = 1$, THEN

$$
{}_5\phi_4\left(\begin{matrix} q^{-N}, \rho_1, \rho_2, b, c; q, q \\ \frac{\rho_1\rho_2 q^{-N}}{a}, e, f, g \end{matrix}\right)
$$

$$
= \frac{\left(\frac{aq}{\rho_1}\right)_N \left(\frac{aq}{\rho_2}\right)_N}{(aq)_N \left(\frac{aq}{\rho_1\rho_2}\right)_N} \sum_{n=0}^{N} \frac{(\rho_1)_n (\rho_2)_n (q^{-N})_n (a)_n (1-aq^{2n})}{\left(\frac{aq}{\rho_1}\right)_n \left(\frac{aq}{\rho_2}\right)_n (aq^{N+1})_n (q)_n (1-a)}
$$

$$
\times \left(\frac{aq^{N+1}}{\rho_1\rho_2}\right)^n P_n(a, b, c, e, f, g; q)
$$

where

$$P_n(a,b,c,e,f,g;q)$$

$$= {}_4\phi_3\left(\begin{matrix} q^{-n}, aq^n, b, c; q, q \\ e, f, g \end{matrix}\right)$$

(related to Askey-Wilson polynomials).

SOME MOCK THETA TYPE RESULTS FOLLOW, BUT NOTHING ON THE 7^{th} ORDER MOCK THETA FUNCTIONS.

The Recurrence Work

$5^{th} + 7^{th} + k^{th}$

MAIN IDEA:

TREAT

$$\beta_n = \frac{(bq;q)_n}{(q^2;q^2)_n} \quad \text{(FIFTH ORDER)}$$

AND

$$\beta_n = \frac{(bq;q)_n}{(q;q)_{2n}} \quad (7^{\text{th}} \text{ ORDER})$$

USING

RECURRENCES

(i.e. GO BACK TO L. J. ROGERS)

SD $10\frac{1}{2}$

Brigg back SD ~~10~~5 fifth

6 & 7 seventh

[illegible]

SO WE NEED THE α_n IN TERMS OF THE β_n.

INVERSION YIELDS:

$$\alpha_n = \frac{(1-aq^{2n})}{(1-a)} \sum_{j=0}^{n} \frac{(a)_{n+j} (-1)^{n-j} q^{\binom{n-j}{2}} \beta_j}{(q)_{n-j}}$$

ALSO MANY HELPFUL LEMMAS CAN BE PROVED IF WE ASSUME β_n IS INDEPENDENT OF a.

IN FACT, ROGERS ONLY CONSIDERED $a=1$ AND $a=q$

IT TURNS OUT THAT TO UNDERSTAND α_n WHEN $a=1$ AND WHEN $a=q$, WE NEED ONLY OBTAIN RECURRENCES FOR

$$\alpha_0(n) := \frac{(1-q)}{(1-q^{2n+1})}\alpha_n$$

where $a=q$.

THUS IN THE FIFTH ORDER CASE:

$$\beta_n = \frac{(bq;q)_n}{(q^2;q^2)_n},$$

AND

$$\alpha_0(n) + bq^n \alpha_0(n-1)$$
$$= bq^{3n-1}\alpha_0(n-1) + q^{4n-4}\alpha_0(n-2)$$

with initial values

$$\alpha_0(0) = 1$$
$$\alpha_0(-1) = -q$$

AND IN THE SEVENTH ORDER CASE

$$\beta_n = \frac{(bq;q)_n}{(q)_{2n}}$$

AND

$$\alpha_0(n) + bq^n \alpha_0(n-1)$$

$$= bq^{3n-3}\alpha_0(n-2) + q^{4n-7}\alpha_0(n-3)$$

with initial values

$$\alpha_0(0) = 1$$

$$\alpha_0(-1) = -q$$

$$\alpha_0(-2) = bq^4$$

SD 15

Show SD13 & SD14 back and forth a few times. The object is to note

1) the constant LHS

2) the linear shift in entries on RHS

3) the simultaneous initial values.

THUS LINEARITY SUGGESTS THAT WE CONSIDER:

$$\alpha_0(n) + bq^n \alpha_0(n-1)$$

$$= bq^{3n-2k+1} \alpha_0(n-k)$$

$$+ q^{4n-3k-1} \alpha_0(n-k-1).$$

NEXT QUESTION: WHAT ARE THE INITIAL VALUES?

WE DO NOT KNOW THE β_n'S WHEN $k>2$.

THE $k=1$ AND $k=2$ EXAMPLES SUGGEST

$$\alpha_0(0)=1$$

$$\alpha_0(-1)=-q$$

$$\alpha_0(-2)=bq^4$$

COMPUTER ALGEBRA LEADS TO

$$\alpha_0(-n)=(-1)^n b^{n-1} q^{\binom{n+2}{2}-2} \quad \text{for } n>0$$

WITH THESE CHOICES OF INITIAL VALUES, WE FIND FOR $k \geq 4$

CASE I: $b = 0$

$$\alpha_0(n) = \begin{cases} (-1)^{k-1} q^{(2k+2)\nu^2 - (k-1)\nu} & \text{if } n = (k+1)\nu \\ (-1)^k q^{(2k+2)\nu^2 + (3k+1)\nu + k} & \text{if } n = (k+1)\nu + k \\ 0 & \text{otherwise} \end{cases}$$

CASE II: $b = -1/q$

$$\alpha_0(n) = q^{\binom{n}{2}} \sum_{|kj| \leq n} (-1)^j q^{-k(k-3)n^2/2}$$

WHEN $k=3$, WE FIND

$$\beta_n = \begin{cases} 1 & \text{if } n=0 \\ \dfrac{(1-bq)}{(q)_{2n}} \displaystyle\prod_{j=2}^{n} (1-bq^j+q^{2j-2}) \end{cases}$$

YIELDING A VARIETY OF ROGERS-RAMANUJAN TYPE IDENTITIES.

E.G.

$$1+3\sum_{n=1}^{\infty} \frac{(-q;q)_{n-1}^2 \, q^{n^2}}{(q)_{2n}}$$

$$= \frac{1}{(q)_\infty} \sum_{n=0}^{\infty} \left\lfloor \frac{3n+2}{2} \right\rfloor q^{n(3n-1)/2} (1-q^{4n+2}).$$

WHEN $k=4$, WE FIND

$$\beta_n = \begin{cases} 1 & \text{if } n=0 \\ \dfrac{\sum_{j=0}^{n-1} \begin{bmatrix} 2n-2-j \\ j \end{bmatrix} q^{j^2+j}}{(q)_{2n}} & \text{if } n>0 \end{cases}$$

(where $b=0$)

CONSEQUENTLY

$$1+\sum_{n,j\geq 0} \frac{q^{j^2+j+(n+j+1)^2}}{(q)_j\,(q)_{2n}\,(q^{2n+j+1})_{j+2}}$$

$$= \frac{1}{(q)_\infty} \sum_{-\infty}^{\infty} q^{35n^2-3n}\left(1-q^{20n+2}\right).$$

SD20

Conclusion

β_n for $k > 4$

Combinatorics

~~Follow~~ Rogers

WZ : deriv recurr from q

? Rever : deriv q from recurrence

Happy Birthday, Dick!

and many thanks for all you have done and all you continue to do!