THE DIRICHLET DIVISOR PROBLEM, BESSEL SERIES EXPANSIONS IN RAMANUJAN'S LOST NOTEBOOK, AND RIESZ SUMS

Bruce Berndt

University of Illinois
at Urbana-Champaign

IN CELEBRATION OF THE 80TH BIRTHDAY OF RICHARD ASKEY

December 6, 2013

Thanks

Research Conducted by the Speaker with

Sun Kim

Ohio State University

and

Alexandru Zaharescu

University of Illinois at Urbana-Champaign

I have shown you today the highest secret of my own realization. It is supreme and most mysterious indeed.

Verse 575, Vivekachudamani of Adi Shankaracharya
Sixth Century, A.D.

Ramanujan's Passport Photo

The Dirichlet Divisor Problem

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Theorem (Dirichlet, 1849)

For $x > 0$, set

$$D(x) := \sideset{}{'}\sum_{n \le x} d(n) = x(\log x + 2\gamma - 1) + \frac{1}{4} + \Delta(x), \tag{1}$$

where the prime on the summation sign on the left-hand side indicates that if x is an integer then only $\frac{1}{2}d(x)$ is counted, γ is Euler's constant, and $\Delta(x)$ is the "error term." Then, as $x \to \infty$,

$$\Delta(x) = O(\sqrt{x}). \tag{2}$$

The Dirichlet Divisor Problem

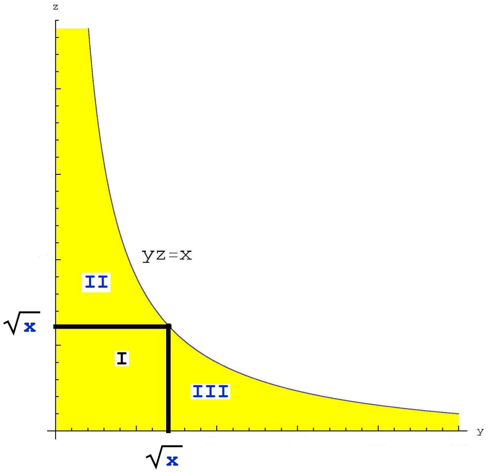

Figure: The Dirichlet Divisor Problem

The Dirichlet Divisor Problem

Conjecture For each $\epsilon > 0$, as $x \to \infty$,

$$\Delta(x) = O(x^{1/4+\epsilon}).$$

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$$\frac{131}{416} = 0.3149\ldots$$

The Dirichlet Divisor Problem

Theorem (Voronoï, 1904)

If $x > 0$,

$$\sum_{n \le x}{}' d(n) = x\left(\log x + 2\gamma - 1\right) + \frac{1}{4} + \sum_{n=1}^{\infty} d(n)\left(\frac{x}{n}\right)^{1/2} I_1(4\pi\sqrt{nx}),$$

where $I_1(z)$ *is defined by*

$$I_\nu(z) := -Y_\nu(z) - \frac{2}{\pi}K_\nu(z). \tag{3}$$

A Page From Ramanujan's Lost Notebook

$$0 < \theta < 1.$$

$$\left[\frac{x}{1}\right]\sin 2\pi\theta + \left[\frac{x}{2}\right]\sin 4\pi\theta + \left[\frac{x}{3}\right]\sin 6\pi\theta + \left[\frac{x}{4}\right]\sin 8\pi\theta + \cdots$$

$$= \pi x\left(\tfrac{1}{2}-\theta\right) - \tfrac{1}{4}\cot\pi\theta + \tfrac{1}{2}\sqrt{x}\sum_{\lambda=1}^{\lambda=\infty}\left\{\frac{J_1(4\pi\sqrt{\lambda\theta x})}{\sqrt{\lambda\theta}} - \frac{J_1(4\pi\sqrt{\lambda(1-\theta)x})}{\sqrt{\lambda(1-\theta)}} + \frac{J_1(4\pi\sqrt{\lambda(1+\theta)x})}{\sqrt{\lambda(1+\theta)}} - \frac{J_1(4\pi\sqrt{\lambda(2-\theta)x})}{\sqrt{\lambda(2-\theta)}} + \frac{J_1(4\pi\sqrt{\lambda(2+\theta)x})}{\sqrt{\lambda(2+\theta)}} - \cdots\cdots\right\}$$

where $[x]$ denotes the greatest integer in x if x is not an integer and $x-\frac{1}{2}$ if x is an integer.

$$\left[\frac{x}{1}\right]\cos 2\pi\theta + \left[\frac{x}{2}\right]\cos 4\pi\theta + \left[\frac{x}{3}\right]\cos 6\pi\theta + \left[\frac{x}{4}\right]\cos 8\pi\theta + \cdots$$

$$= -x\log(2\sin\pi\theta) + \tfrac{1}{4} + \tfrac{1}{2}\sqrt{x}\sum_{\lambda=1}^{\lambda=\infty}\left\{\frac{I_1(4\pi\sqrt{\lambda\theta x})}{\sqrt{\lambda\theta}} + \frac{I_1(4\pi\sqrt{\lambda(1-\theta)x})}{\sqrt{\lambda(1-\theta)}} + \frac{I_1(4\pi\sqrt{\lambda(1+\theta)x})}{\sqrt{\lambda(1+\theta)}} + \frac{I_1(4\pi\sqrt{\lambda(2-\theta)x})}{\sqrt{\lambda(2-\theta)}} + \frac{I_1(4\pi\sqrt{\lambda(2+\theta)x})}{\sqrt{\lambda(2+\theta)}} + \cdots\cdots\right\},$$

where $I_1(x) = H_1(x) - Y_1(x).$

Second Bessel Function Series Claim

Define

$$F(x) = \begin{cases} [x], & \text{if } x \text{ is not an integer,} \\ x - \frac{1}{2}, & \text{if } x \text{ is an integer,} \end{cases} \tag{4}$$

where, as customary, $[x]$ is the greatest integer less than or equal to x.

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$$I_\nu(z) := -Y_\nu(z) - \frac{2}{\pi} K_\nu(z). \tag{5}$$

Second Bessel Function Series Claim

Entry

Let $F(x)$ be defined by (4), and let $I_1(x)$ be defined by (5). For $x > 0$ and $0 < \theta < 1$,

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\cos(2\pi n\theta) = \frac{1}{4} - x\log(2\sin(\pi\theta))$$
$$+\frac{1}{2}\sqrt{x}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{I_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} + \frac{I_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\}.$$

Second Bessel Function Series Claim

As $x \to \infty$,

$$Y_\nu(x) = \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{x^{3/2}}\right),$$
$$K_\nu(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x} + O\left(e^{-x}\frac{1}{x^{3/2}}\right).$$

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$$\frac{1}{\pi\sqrt{2}x^{1/4}m^{3/4}}$$

$$\left(\frac{\sin\left(4\pi\sqrt{m(n+\theta)x} - \frac{3}{4}\pi\right)}{(n+\theta)^{3/4}} + \frac{\sin\left(4\pi\sqrt{m(n+1-\theta)x} - \frac{3}{4}\pi\right)}{(n+1-\theta)^{3/4}}\right).$$

Equivalent Theorem with the Order of Summation Reversed

Theorem

Fix $x > 0$ and set $\theta = u + \frac{1}{2}$, where $-\frac{1}{2} < u < \frac{1}{2}$. Recall that $F(x)$ is defined in (4). If the identity below is valid for at least one value of θ, then it exists for all values of θ, and

$$\sum_{1 \le n \le x} (-1)^n F\left(\frac{x}{n}\right) \cos(2\pi n u) - \frac{1}{4} + x \log(2\cos(\pi u))$$

$$= \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{1}{n + \frac{1}{2} + u} \lim_{M \to \infty} \left\{ \sum_{m=1}^{M} \sin\left(\frac{2\pi(n + \frac{1}{2} + u)x}{m}\right) - \int_0^M \sin\left(\frac{2\pi(n + \frac{1}{2} + u)x}{t}\right) dt \right\}$$

Equivalent Theorem with the Order of Summation Reversed

Theorem (Continued)

$$+\frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{1}{n+\frac{1}{2}-u}\lim_{M\to\infty}\left\{\sum_{m=1}^{M}\sin\left(\frac{2\pi(n+\frac{1}{2}-u)x}{m}\right)\right.$$
$$\left.-\int_0^M \sin\left(\frac{2\pi(n+\frac{1}{2}-u)x}{t}\right)dt\right\}. \tag{6}$$

Moreover, the series on the right-hand side of (6) *converges uniformly on compact subintervals of* $(-\frac{1}{2},\frac{1}{2})$.

Entry

Let $F(x)$ be defined by (4), *and let $I_1(x)$ be defined by* (5). *For $x > 0$ and $0 < \theta < 1$,*

$$\sum_{n=1}^{\infty} F\left(\frac{x}{n}\right)\cos(2\pi n\theta) = \frac{1}{4} - x\log(2\sin(\pi\theta))$$
$$+\frac{1}{2}\sqrt{x}\sum_{m\geq 1, n\geq 0}\left\{\frac{I_1\left(4\pi\sqrt{m(n+\theta)x}\right)}{\sqrt{m(n+\theta)}} + \frac{I_1\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\sqrt{m(n+1-\theta)}}\right\}.$$

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If χ denotes any character modulo q, we define

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$$\tau(\chi) := \sum_{h=1}^{q-1} \chi(h) e^{2\pi i h/q}.$$

Functional Equation for Nonprincipal Even Primitive Characters

$$
\begin{aligned}
&\left(\frac{\pi}{\sqrt{q}}\right)^{-2s} \Gamma^2(s)\zeta(2s)L(2s,\chi) \\
&\quad = \frac{\tau(\chi)}{\sqrt{q}} \left(\frac{\pi}{\sqrt{q}}\right)^{-2(\frac{1}{2}-s)} \Gamma^2\left(\tfrac{1}{2}-s\right)\zeta(1-2s)L(1-2s,\overline{\chi}).
\end{aligned}
$$

Hark Back to My Ph.D. Thesis

Theorem

If χ is a nonprincipal even primitive character modulo q, then

$$\sum_{n\le x}{}' d_\chi(n) = \frac{\sqrt{q}}{\tau(\overline{\chi})}\sum_{n=1}^{\infty} d_{\overline{\chi}}(n)\sqrt{\frac{x}{n}}\, I_1\left(4\pi\sqrt{nx/q}\right) - \frac{x}{\tau(\overline{\chi})}\sum_{h=1}^{q-1}\overline{\chi}(h)\log\left(2\sin(\pi h/q)\right). \qquad (7)$$

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- We have proved Ramanujan's First Identity under all three interpretations.
- Road Blocks: The plus sign between the two Bessel functions; singularities at 0.
- The proofs under different interpretations are completely different.
- New methods of estimating trigonometric sums are introduced.

Riesz Sums

$$\sum_{n \leq x} a(n)(x-n)^a$$

Analogue of a Theorem of Dixon and Ferrar

Theorem

Let a denote a positive integer, and let χ be an even primitive nonprincipal character of modulus q. Set for $x > 0$,

$$D_\chi(a;x) := \frac{1}{\Gamma(a+1)} \sum_{n\leq x} d_\chi(n)(x-n)^a.$$

Then, for $a \geq 2$,

$$D_\chi(a-1;x) = \frac{L(1,\chi)x^a}{\Gamma(1+a)} + \frac{\tau(\chi)q^{a-1}}{(2\pi)^{a-1}} \sum_{n=1}^{\infty} d_{\overline{\chi}}(n) \left(\frac{x}{nq}\right)^{a/2}$$
$$\times \left\{ -Y_a\left(4\pi\sqrt{\frac{nx}{q}}\right) + \frac{2}{\pi}\cos(\pi a)K_a\left(4\pi\sqrt{\frac{nx}{q}}\right)\right\} + S_a,$$

Analogue of a Theorem of Dixon and Ferrar

Theorem (Continued)

where S_a *is the sum of the residues at the poles* $-2m-1$*,*
$0 \le m \le [\frac{1}{2}a] - 1$*, of*

$$\frac{\Gamma(s)\zeta(s)L(s,\chi)}{\Gamma(s+a)}x^{s+a-1}.$$

The Riesz Sum Generalization of Ramanujan's Entry 5

Theorem

Let $x > 0$, $0 < \theta < 1$, and a be a positive integer. Then,

$$\frac{1}{(a-1)!}\sideset{}{'}\sum_{n\le x}(x-n)^{a-1}\sum_{r|n}\cos(2\pi r\theta) = \frac{x^{a-1}}{4(a-1)!} - \frac{x^a}{a!}\log\left(2\sin(\pi\theta)\right)$$

$$+\frac{x^{\alpha/2}}{2(2\pi)^{a-1}}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{I_a\left(4\pi\sqrt{m(n+\theta)x}\right)}{\left(m(n+\theta)\right)^{a/2}} + \frac{I_a\left(4\pi\sqrt{m(n+1-\theta)x}\right)}{\left(m(n+1-\theta)\right)^{a/2}}\right\}$$

$$-\sum_{k=1}^{[a/2]}\frac{(-1)^k\zeta(1-2k)\Big(\zeta(2k,\theta)+\zeta(2k,1-\theta)\Big)x^{a-2k}}{(a-2k)!(2\pi)^{2k}}, \qquad (8)$$

where $I_\nu(x)$ is defined in (3), and $\zeta(s,\alpha)$ denotes the Hurwitz zeta function.

Theorem

Let $I_1(x)$ *be defined by* (5). *If* $0 < \theta, \sigma < 1$ *and* $x > 0$, *then*

$$\sum_{nm\leq x}{}' \cos(2\pi n\theta)\cos(2\pi m\sigma) = \frac{1}{4} + \frac{\sqrt{x}}{4}\sum_{n,m\geq 0}$$

$$\times \left\{ \frac{I_1(4\pi\sqrt{(n+\theta)(m+\sigma)x})}{\sqrt{(n+\theta)(m+\sigma)}} + \frac{I_1(4\pi\sqrt{(n+1-\theta)(m+\sigma)x})}{\sqrt{(n+1-\theta)(m+\sigma)}} \right.$$

$$\left. + \frac{I_1(4\pi\sqrt{(n+\theta)(m+1-\sigma)x})}{\sqrt{(n+\theta)(m+1-\sigma)}} + \frac{I_1(4\pi\sqrt{(n+1-\theta)(m+1-\sigma)x})}{\sqrt{(n+1-\theta)(m+1-\sigma)}} \right\}.$$

An Analogue of Ramanujan's Entry 5

Define, for Dirichlet characters χ_1 modulo p and χ_2 modulo q,

$$d_{\chi_1,\chi_2}(n) = \sum_{d|n} \chi_1(d)\chi_2(n/d).$$

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The Functional Equation

$$(\pi^2/(pq))^{-s}\Gamma^2(s)L(2s,\chi_1)L(2s,\chi_2)$$
$$= \frac{\tau(\chi_1)\tau(\chi_2)}{\sqrt{pq}}(\pi^2/(pq))^{-(\frac{1}{2}-s)}\Gamma^2(\tfrac{1}{2}-s)L(1-2s,\overline{\chi_1})L(1-2s,\overline{\chi_2})$$

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$$\sum_{n\le x}{}' d_{\chi_1,\chi_2}(n) = \frac{\tau(\chi_1)\tau(\chi_2)}{\sqrt{pq}} \sum_{n=1}^{\infty} d_{\overline{\chi_1},\overline{\chi_2}}(n)\left(\frac{x}{n}\right)^{\frac{1}{2}} I_1\left(4\pi\sqrt{\frac{nx}{pq}}\right)$$

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T. J. Kaczynski (The Unabomber), *Boundary functions for bounded harmonic functions*, Trans. Amer. Math. Soc. **137** (1969), 203–209.

Photograph, Shanghai, July 31, 2013

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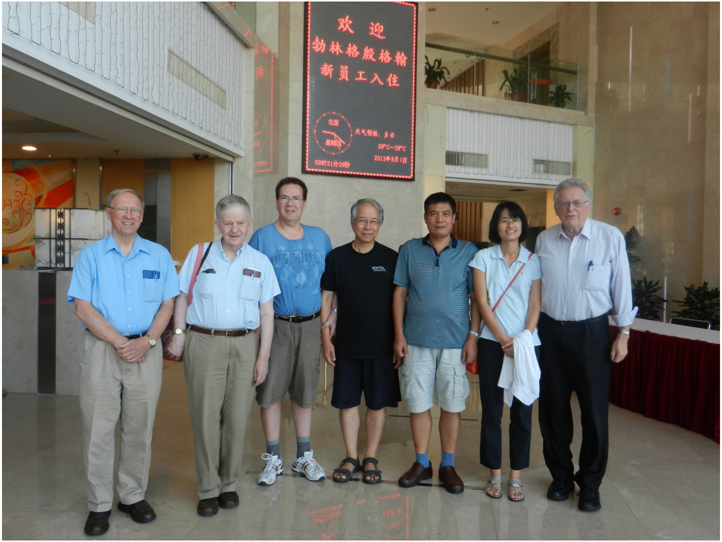

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Happy 80th Birthday, Dick

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and **Many** more