Geometry and Hypergeometry

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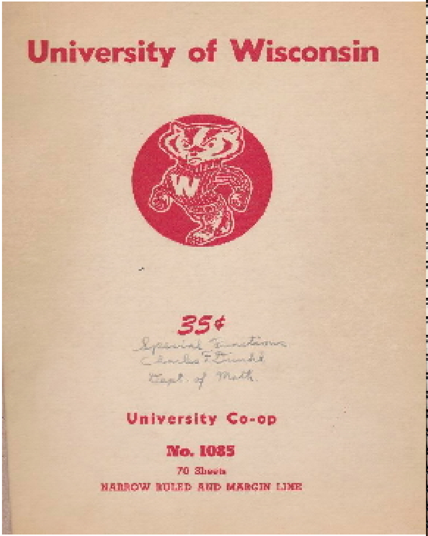
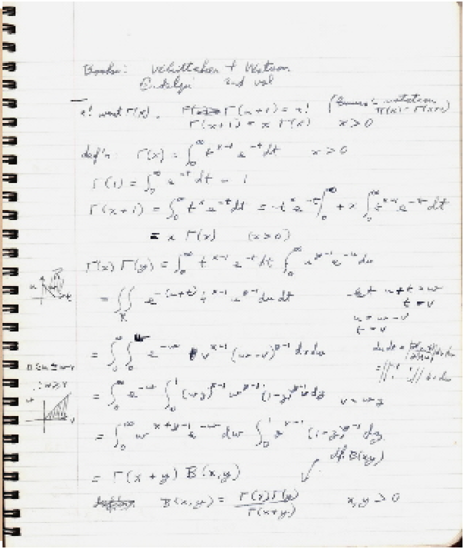
Historical Documents





Books: Whittaker + Watson
Erdélyi 2nd vol

$n!$ and $\Gamma(x)$. $\Gamma(n+1) = n!$ (Gauss' notation $\Pi(x) = \Gamma(x+1)$)

$\Gamma(x+1) = x\,\Gamma(x)$ $\quad x>0$

def'n $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt$ $\quad x>0$

$\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1$

$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt = -t^x e^{-t}\Big|_0^\infty + x\int_0^\infty t^{x-1}e^{-t}\,dt$

$= x\,\Gamma(x)$ $\quad (x>0)$

$\Gamma(x)\Gamma(y) = \int_0^\infty t^{x-1}e^{-t}\,dt \int_0^\infty u^{y-1}e^{-u}\,du$

$= \iint_K e^{-(u+t)} t^{x-1} u^{y-1}\,du\,dt$ $\quad$ let $u+t = w$, $t = v$; $u = w-v$, $t = v$

$0 \le v \le w$

$= \int_0^\infty \int_0^w e^{-w} v^{x-1}(w-v)^{y-1}\,dv\,dw$ $\quad du\,dt = \left|\frac{\partial(u,t)}{\partial(v,w)}\right| dv\,dw$

$w \ge v$

$= \int_0^\infty e^{-w}\int_0^1 (wz)^{x-1} w^{y-1}(1-z)^{y-1} w\,dz$ $\quad v = wz$

$= \int_0^\infty w^{x+y-1} e^{-w}\,dw \int_0^1 z^{x-1}(1-z)^{y-1}\,dz$ $\quad$ def. $B(x,y)$

$= \Gamma(x+y)\,B(x,y)$

$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ $\quad x,y>0$

The Symmetry Group of the Regular m-gon

Consider the symmetry group $I_2(m)$ of the regular m-gon in $\mathbb{R}^2$; it consists of the identity, m reflections and $m-1$ rotations

$$\begin{bmatrix} \cos\frac{2\pi j}{m} & \sin\frac{2\pi j}{m} \\ \sin\frac{2\pi j}{m} & -\cos\frac{2\pi j}{m} \end{bmatrix}, 0 \le j < m; \begin{bmatrix} \cos\frac{2\pi j}{m} & \sin\frac{2\pi j}{m} \\ -\sin\frac{2\pi j}{m} & \cos\frac{2\pi j}{m} \end{bmatrix}, 1 \le j < m.$$

It is often useful to employ complex coordinates for $x = (x_1, x_2) \in \mathbb{R}^2$, $z = x_1 + \mathrm{i}x_2, \bar{z} = x_1 - \mathrm{i}x_2$, $\omega := \exp\frac{2\pi\mathrm{i}}{m}$. Then the reflections are expressed as $z\sigma_j := \bar{z}\omega^j$ $(0 \le j < m)$ and the rotations are $z\varrho_j := z\omega^j\,(1 \le j < m)$.

Fix a real parameter κ. We consider differential-difference operators on polynomials, first on ordinary (scalar) then on vector-valued polynomials in $z, \overline{z}$. (Notation: $\partial_u := \frac{\partial}{\partial u}$)

$$\mathcal{D}f(z,\overline{z}) := \partial_z f(z,\overline{z}) + \kappa \sum_{j=0}^{m-1} \frac{f(z,\overline{z}) - f\left(\overline{z}\omega^j, z\omega^{-j}\right)}{z - \overline{z}\omega^j},$$

$$\overline{\mathcal{D}}f(z,\overline{z}) := \partial_{\overline{z}} f(z,\overline{z}) - \kappa \sum_{j=0}^{m-1} \frac{f(z,\overline{z}) - f\left(\overline{z}\omega^j, z\omega^{-j}\right)}{z - \overline{z}\omega^j}\omega^j.$$

Then $\overline{\mathcal{D}}\mathcal{D} = \mathcal{D}\overline{\mathcal{D}}$ and the Laplacian is $4\mathcal{D}\overline{\mathcal{D}}$. A polynomial f such that $\mathcal{D}\overline{\mathcal{D}}f = 0$ is called harmonic. Harmonic homogenous polynomials of different degrees are orthogonal with respect to the measure $\left|\sin m\theta\right|^{2\kappa} d\theta$ on the unit circle for $\kappa > -\frac{1}{2}$ and the measure

$$\left|z^m - \overline{z}^m\right|^{2\kappa} e^{-|z|^2/2} dm_2(z)$$

on $\mathbb{C} \cong \mathbb{R}^2$ for $\kappa > -\frac{1}{m}$.

For $n = 0, 1, 2, \ldots$ let

$$C_n^{(\kappa,\kappa+1)}(w) := \sum_{j=0}^{n} \frac{(\kappa+1)_{n-j}(\kappa)_j}{(n-j)!\, j!} w^{n-j}\overline{w}^j,$$

then

$$\overline{\mathcal{D}} z^r C_n^{(\kappa,\kappa+1)}(z^m) = 0, 0 \leq r < m, n \geq 0.$$

There is a two-term recurrence for the monic $c_n(w) := \dfrac{n!}{(\kappa+1)_n} C_n^{(\kappa,\kappa+1)}(w)$; let $\sigma_0 p(w, \overline{w}) := p(\overline{w}, w)$ then

$$c_0(w) = 1,$$
$$c_{n+1}(w) = w\ c_n(w) + \frac{\kappa}{\kappa+n+1} \overline{w}\ \sigma_0 c_n(w).$$

Furthermore (with $z = re^{\mathrm{i}\theta}, -\pi < \theta \leq \pi$) there is an expression in Gegenbauer polynomials:

$$C_n^{(\kappa,\kappa+1)}(z^m) = \frac{n+2\kappa}{2\kappa} r^{mn} C_n^{\kappa}(\cos m\theta) + \mathrm{i} r^{mn} \sin m\theta\ C_{n-1}^{\kappa+1}(\cos m\theta).$$

Group-invariant Hermitian Forms on Polynomials

- There is a Hermitian (*contravariant*) form $\langle\cdot,\cdot\rangle$ on polynomials given by

$$c\langle p,q\rangle=\langle\bar{c}p,q\rangle=\langle p,cq\rangle,\langle p,q\rangle=\overline{\langle q,p\rangle},$$
$$\langle zp,q\rangle=\langle p,2\mathcal{D}q\rangle,\langle\bar{z}p,q\rangle=\left\langle p,2\overline{\mathcal{D}}q\right\rangle,\langle 1,1\rangle=1.$$

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- Define the Gaussian form

$$\langle p, q \rangle_G := \left\langle e^{2\mathcal{D}\overline{\mathcal{D}}} p, e^{2\mathcal{D}\overline{\mathcal{D}}} q \right\rangle ;$$

it satisfies

$$\langle \mathcal{D}p, q \rangle_G = \left\langle p, \left(\frac{z}{2} - \overline{\mathcal{D}} \right) q \right\rangle_G , \langle zp, q \rangle_G = \langle p, \overline{z}q \rangle_G ,$$

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- and for $\kappa > -\frac{1}{m}$

$$\langle p, q \rangle_G = \gamma \int_{\mathbb{R}^2} \overline{p(z)} q(z) \left| z^m - \overline{z}^m \right|^{2\kappa} e^{-|z|^2/2} dm_2(z),$$

(normalizing constant γ). Note $e^{2\mathcal{D}\overline{\mathcal{D}}}$ is invertible on polynomials. The condition $\kappa > -\frac{1}{m}$ also appears as $\langle z, z \rangle_G = 2(1 + m\kappa) > 0$

- Observe the special property of harmonic polynomials with respect to the Gaussian form, namely $\langle p,q\rangle_G = \langle p,q\rangle$. Harmonic homogeneous polynomials of different degrees are trivially orthogonal to each other. For $0 \leq r < m$, and $n \geq 0$ set

$$p_{mn+r}(z) := z^r C_n^{(\kappa,\kappa+1)}(z^m)$$

then $\{p_{mn+r}, \overline{p_{mn+r}}\}$ is a basis for the harmonic homogeneous polynomials of degree $nm+r$, orthogonal except for $r=0$, in which case $\{\mathrm{Re}\, p_{mn}, \mathrm{Im}\, p_{mn}\}$ is orthogonal.

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- The weight function $|z^m - \overline{z}^m|^{2\kappa}$ equals $\overline{L}L$ where L satisfies

$$\partial_z L = \kappa L \sum_{j=0}^{m-1} \frac{1}{z - \overline{z}\omega^j} = \kappa L \frac{m z^{m-1}}{z^m - \overline{z}^m},$$

$$\partial_{\overline{z}} L = \kappa L \sum_{j=0}^{m-1} \frac{-\omega^j}{z - \overline{z}\omega^j} = \kappa L \frac{-m \overline{z}^{m-1}}{z^m - \overline{z}^m}.$$

Vector-valued Polynomials

We extend these concepts to vector-valued polynomials. The (pairwise nonequivalent) 2-dimensional irreducible representations of the group are given by

$$\tau_\ell(\sigma_j) = \begin{bmatrix} 0 & \omega^{-j\ell} \\ \omega^{j\ell} & 0 \end{bmatrix}, 1 \leq \ell \leq \left\lfloor \frac{m-1}{2} \right\rfloor, 0 \leq j < m.$$

Henceforth fix ℓ.
We consider the standard module $\mathcal{P}_{m,\ell}$ consisting of polynomials

$$f(z, \overline{z}, t, \overline{t}) = f_1(z, \overline{z})\, t + f_2(z, \overline{z})\, \overline{t}$$

with the group action

$$\sigma_j f(z, \overline{z}, t, \overline{t}) = f_2\left(\overline{z}\omega^j, z\omega^{-j}\right) \omega^{-\ell j} t + f_1\left(\overline{z}\omega^j, z\omega^{-j}\right) \omega^{\ell j} \overline{t}.$$

There is a representation of the rational Cherednik algebra - the abstract algebra generated by $\left\{z, \overline{z}, \mathcal{D}, \overline{\mathcal{D}}\right\} \cup I_2(m)$ (with certain relations) - on the space $\mathcal{P}_{m,\ell}$.

The Dunkl operators are defined by

$$\begin{aligned}\mathcal{D}f\left(z,\overline{z},t,\overline{t}\right) &:= \partial_z f\left(z,\overline{z},t,\overline{t}\right)\\ &\quad + \kappa\sum_{j=0}^{m-1}\frac{f\left(z,\overline{z},\omega^{\ell j}\overline{t},\omega^{-\ell j}t\right)-f\left(\overline{z}\omega^{j},z\omega^{-j},\omega^{\ell j}\overline{t},\omega^{-\ell j}t\right)}{z-\overline{z}\omega^{j}},\\ \overline{\mathcal{D}}f\left(z,\overline{z},t,\overline{t}\right) &:= \partial_{\overline{z}} f\left(z,\overline{z},t,\overline{t}\right)\\ &\quad - \kappa\sum_{j=0}^{m-1}\frac{f\left(z,\overline{z},\omega^{\ell j}\overline{t},\omega^{-\ell j}t\right)-f\left(\overline{z}\omega^{j},z\omega^{-j},\omega^{\ell j}\overline{t},\omega^{-\ell j}t\right)}{z-\overline{z}\omega^{j}}\omega^{j}.\end{aligned}$$

As before they satisfy $\overline{\mathcal{D}}\mathcal{D}=\mathcal{D}\overline{\mathcal{D}}$.

Harmonic Polynomials

- The *leading term* of a homogeneous polynomial $\sum_{j=0}^{n} a_j z^{n-j}\overline{z}^j t + \sum_{j=0}^{n} b_j z^{n-j}\overline{z}^j \overline{t}$ is defined to be $\left(a_0 z^n + a_n \overline{z}^n\right) t + \left(b_0 z^n + b_n \overline{z}^n\right) \overline{t}$.

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- There are 4 linearly independent harmonic polynomials of each degree ≥ 1, with bases given by two sequences $\left\{p_n^{(1)}, \sigma_0 p_n^{(1)} : n \geq 1\right\}$ and $\left\{p_n^{(2)}, \sigma_0 p_n^{(2)} : n \geq 1\right\}$ $\left(\sigma_0 p\left(z, \overline{z}, t, \overline{t}\right) := p\left(\overline{z}, z, \overline{t}, t\right)\right)$. The leading terms of $p_n^{(1)}, \sigma_0 p_n^{(1)}, p_n^{(2)}, \sigma_0 p_n^{(2)}$ are $z^n t$, $\overline{z}^n \overline{t}$, $z^n \overline{t}$, $\overline{z}^n t$ respectively.

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- There are 4 linearly independent harmonic polynomials of each degree ≥ 1, with bases given by two sequences $\left\{p_n^{(1)}, \sigma_0 p_n^{(1)} : n \geq 1\right\}$ and $\left\{p_n^{(2)}, \sigma_0 p_n^{(2)} : n \geq 1\right\}$ $\left(\sigma_0 p\left(z, \overline{z}, t, \overline{t}\right) := p\left(\overline{z}, z, \overline{t}, t\right)\right)$. The leading terms of $p_n^{(1)}, \sigma_0 p_n^{(1)}, p_n^{(2)}, \sigma_0 p_n^{(2)}$ are $z^n t$, $\overline{z}^n \overline{t}$, $z^n \overline{t}$, $\overline{z}^n t$ respectively.
- Let λ be a parameter with $\lambda > 0$ and define polynomials $Q_n^{(1)}\left(\kappa, \lambda; w, \overline{w}\right)$, $Q_n^{(2)}\left(\kappa, \lambda; w, \overline{w}\right)$ by

$$Q_0^{(1)}\left(\kappa, \lambda; w, \overline{w}\right) = 1, Q_0^{(2)}\left(\kappa, \lambda; w, \overline{w}\right) = \frac{\kappa}{\lambda},$$

$$Q_{n+1}^{(1)}\left(\kappa, \lambda; w, \overline{w}\right) = w Q_n^{(1)}\left(\kappa, \lambda; w, \overline{w}\right) + \frac{\kappa}{\lambda + n + 1} \overline{w} Q_n^{(2)}\left(\kappa, \lambda; \overline{w}, w\right),$$

$$Q_{n+1}^{(2)}\left(\kappa, \lambda; w, \overline{w}\right) = \frac{\kappa}{\lambda + n + 1} \overline{w} Q_n^{(1)}\left(\kappa, \lambda; \overline{w}, w\right) + w Q_n^{(2)}\left(\kappa, \lambda; w, \overline{w}\right).$$

Now let

$$P_n^{(1)} := z^{m-\ell+1} Q_n^{(1)}\left(\kappa, \frac{m-\ell}{m}; z^m, \overline{z}^m\right) t$$
$$+ z\overline{z}^{m-\ell} Q_n^{(2)}\left(\kappa, \frac{m-\ell}{m}; z^m, \overline{z}^m\right) \overline{t},$$
$$P_n^{(2)} := z^{\ell+1} Q_n^{(1)}\left(\kappa, \frac{\ell}{m}; z^m, \overline{z}^m\right) \overline{t} + z\overline{z}^{\ell} Q_n^{(2)}\left(\kappa, \frac{\ell}{m}; z^m, \overline{z}^m\right) t.$$

Then the following are harmonic polynomials ($\mathcal{D}\overline{\mathcal{D}}p = 0$):

1. $p_r^{(1)} = z^r t$ for $1 \leq r \leq m-\ell$ and $p_s^{(1)} = z^r P_n^{(1)}$ for $s = m(n+1) - \ell + 1 + r$ and $0 \leq r < m$;
2. $p_r^{(2)} = z^r \overline{t}$ for $1 \leq r \leq \ell$ and $p_s^{(2)} = z^r P_n^{(2)}$ for $s = nm + \ell + 1 + r$ and $0 \leq r < m$.

There are properties similar to the scalar case such as $\overline{\mathcal{D}} p_r^{(j)} = 0$ for certain r, j.

There are closed expressions for the polynomials $Q_n^{(1)}$, $Q_n^{(2)}$:

$$Q_n^{(1)}(\kappa,\lambda;w,\overline{w}) = w^n + \sum_{j=1}^{n} \frac{\kappa^2(n-j+1)}{\lambda(\lambda+n)}\, {}_4F_3\left(\begin{matrix}1-j, j-n, 1-\kappa, 1+\kappa \\ 2, \lambda+1, -\lambda-n+1\end{matrix};1\right) w^{n-j}\overline{w}^j,$$

$$Q_n^{(2)}(\kappa,\lambda;w,\overline{w}) = \sum_{j=0}^{n} \frac{\kappa}{\lambda+j}\, {}_4F_3\left(\begin{matrix}-j, j-n, -\kappa, +\kappa \\ 1, \lambda, -\lambda-n\end{matrix};1\right) w^{n-j}\overline{w}^j,$$

(terminating and balanced!) and an evaluation formula:

$$Q_n^{(1)}(\kappa,\lambda;1,1) = \frac{(\lambda+\kappa)_{n+1} + (\lambda-\kappa)_{n+1}}{2(\lambda)_{n+1}},$$

$$Q_n^{(2)}(\kappa,\lambda;1,1) = \frac{(\lambda+\kappa)_{n+1} - (\lambda-\kappa)_{n+1}}{2(\lambda)_{n+1}}.$$

The Contravariant Form

- Consider $t = s_1 + \mathrm{i}s_2$ with $\langle s_j, s_k \rangle = \delta_{jk}$, then $\langle t, t \rangle = 2 = \langle \overline{t}, \overline{t} \rangle$ and $\langle t, \overline{t} \rangle = 0$. For a scalar polynomial $p = \sum_{j,k} a_{jk} z^j \overline{z}^k$ let $p^* \left(2\mathcal{D}, 2\overline{\mathcal{D}}\right) = \sum_{j,k} \overline{a_{jk}} 2^{j+k} \mathcal{D}^j \overline{\mathcal{D}}^k)$. Then for $p, q \in \mathcal{P}_{m,\ell}$

$$\begin{aligned} & \left\langle p_1 \left(z, \overline{z}\right) t + p_2 \left(z, \overline{z}\right) \overline{t}, q \left(z, \overline{z}, t, \overline{t}\right) \right\rangle \\ = & \left\langle t, p_1^* \left(2\mathcal{D}, 2\overline{\mathcal{D}}\right) q \right\rangle |_{z=0} + \left\langle \overline{t}, p_2^* \left(2\mathcal{D}, 2\overline{\mathcal{D}}\right) q \right\rangle |_{z=0}, \end{aligned}$$

  where $p\left(z, \overline{z}, t, \overline{t}\right) = p_1 \left(z, \overline{z}\right) t + p_2 \left(z, \overline{z}\right) \overline{t}$. This form has the properties $\langle zp, q \rangle = \langle p, 2\mathcal{D}q \rangle$ and $\langle \sigma_j p, \sigma_j q \rangle = \langle p, q \rangle$ (group invariance).

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$$\begin{aligned} & \langle p_1 (z, \overline{z}) \, t + p_2 (z, \overline{z}) \, \overline{t}, q (z, \overline{z}, t, \overline{t}) \rangle \\ = & \left\langle t, p_1^* \left(2\mathcal{D}, 2\overline{\mathcal{D}} \right) q \right\rangle |_{z=0} + \left\langle \overline{t}, p_2^* \left(2\mathcal{D}, 2\overline{\mathcal{D}} \right) q \right\rangle |_{z=0}, \end{aligned}$$

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- For what values of κ is the form positive-definite? The polynomial $P_0^{(2)}$ is harmonic of degree $\ell + 1$ and

$$\left\langle P_0^{(2)}, P_0^{(2)} \right\rangle = 2^{\ell+2} (\ell + 1)! \left(\frac{m}{\ell} \right)^2 \left(\frac{\ell}{m} - \kappa \right) \left(\frac{\ell}{m} + \kappa \right),$$

  thus $-\frac{\ell}{m} < \kappa < \frac{\ell}{m}$ is necessary - and sufficient, as will be seen.

The Gaussian Form

- The polynomials $\left\{p_n^{(1)}, \sigma_0 p_n^{(1)}, p_n^{(2)}, \sigma_0 p_n^{(2)} : n \geq 1\right\}$ form an orthogonal basis for the harmonic homogeneous polynomials with exceptions when $n \pm \ell \equiv 0 \bmod m$.
  If $n + \ell \equiv 0 \bmod m$ then $p_n^{(1)} + \sigma_0 p_n^{(1)} \perp p_n^{(1)} - \sigma_0 p_n^{(1)}$.
  If $n - \ell \equiv 0 \bmod m$ then $p_n^{(2)} + \sigma_0 p_n^{(2)} \perp p_n^{(2)} - \sigma_0 p_n^{(2)}$.

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  If $n - \ell \equiv 0 \bmod m$ then $p_n^{(2)} + \sigma_0 p_n^{(2)} \perp p_n^{(2)} - \sigma_0 p_n^{(2)}$.
- As before define the Gaussian form

$$\langle p, q \rangle_G = \left\langle e^{2\mathcal{D}\overline{\mathcal{D}}} p, e^{2\mathcal{D}\overline{\mathcal{D}}} q \right\rangle;$$

  which satisfies

$$\langle \mathcal{D} p, q \rangle_G = \left\langle p, \left(\frac{z}{2} - \overline{\mathcal{D}}\right) q \right\rangle_G, \langle zp, q \rangle_G = \langle p, \overline{z} q \rangle_G .$$

  Note that the adjoint of multiplication by z is multiplication by $\overline{z}$.

We want to find a 2×2 positive-definite matrix function $K(z)$ such that

$$\langle p, q \rangle_G = \int_{\mathbb{C}} q(z) K(z) p(z)^* e^{-|z|^2/2} dm_2(z),$$

where $q = q_1(z) t + q_2(z) \overline{t}$ is considered as the vector (q_1, q_2), similarly for p. This leads to the requirements $K(z)^* = K(z)$ and $K(zw) = \tau_\ell(w)^* K(z) \tau_\ell(w)$ for $w \in I_2(m)$. The key condition

$$\langle \mathcal{D}p, q \rangle_G = \left\langle p, \left(\frac{z}{2} - \overline{\mathcal{D}} \right) q \right\rangle_G$$

(and its complex conjugate) leads to a differential equation and boundary-value problem for K.

- The idea is to express $K(z) = L(z)^* ML(z)$ where M is a constant positive-definite matrix and L satisfies $L(zw) = L(z)\,\tau_\ell(w)$ $(\forall w \in l_2(m))$ and the differential system.

$$\partial_z L(z,\overline{z}) = \kappa L(z,\overline{z}) \sum_{j=0}^{m-1} \frac{1}{z-\overline{z}\omega^j} \tau_\ell(\sigma_j),$$

$$\partial_{\overline{z}} L(z,\overline{z}) = \kappa L(z,\overline{z}) \sum_{j=0}^{m-1} \frac{-\omega^j}{z-\overline{z}\omega^j} \tau_\ell(\sigma_j).$$

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$$\partial_{\overline{z}} L(z,\overline{z}) = \kappa L(z,\overline{z}) \sum_{j=0}^{m-1} \frac{-\omega^j}{z - \overline{z}\omega^j} \tau_\ell(\sigma_j).$$

- Note

$$(z\partial_z + \overline{z}\partial_{\overline{z}}) L = 0,$$

thus L is positively homogeneous of degree 0 and depends only on $e^{\mathrm{i}\theta}$ for $z = re^{\mathrm{i}\theta}$ $(r > 0, -\pi < \theta \leq \pi)$. Further $\partial_\theta = \mathrm{i}(z\partial_z - \overline{z}\partial_{\overline{z}})$ and thus

$$\partial_\theta L = \mathrm{i}\kappa L \sum_{j=0}^{m-1} \frac{z + \overline{z}\omega^j}{z - \overline{z}\omega^j} \tau_\ell(\sigma_j).$$

- Write $L = L_1 t + L_2 \overline{t}$, then

$$\partial_\theta L_1(\theta) = \frac{m\kappa}{\sin m\theta} e^{\mathrm{i}\theta(m-2\ell)} L_2(\theta),$$
$$\partial_\theta L_2(\theta) = \frac{m\kappa}{\sin m\theta} e^{-\mathrm{i}\theta(m-2\ell)} L_1(\theta).$$

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- Let $\delta := \frac{1}{2} - \frac{\ell}{m}$, (note: $0 < \delta < \frac{1}{2}$), $\widetilde{L_1} := e^{-\mathrm{i}m\delta\theta} L_1$, $\widetilde{L_2} := e^{\mathrm{i}m\delta\theta} L_2$, then

$$\partial_\theta \widetilde{L}_1(\theta) = -\mathrm{i}m\delta \widetilde{L}_1(\theta) + \frac{m\kappa}{\sin m\theta} \widetilde{L}_2(\theta),$$
$$\partial_\theta \widetilde{L}_2(\theta) = \frac{m\kappa}{\sin m\theta} \widetilde{L}_1(\theta) + \mathrm{i}m\delta \widetilde{L}_2(\theta).$$

Now changing $\{t, \overline{t}\}$ to real coordinates $t = s_1 + \mathrm{i}s_2$ we write

$$\widetilde{L}_1 t + \widetilde{L}_2 \overline{t} = \left(\widetilde{L}_1 + \widetilde{L}_2\right) s_1 + \mathrm{i}\left(\widetilde{L}_1 - \widetilde{L}_2\right) s_2 =: g_1 s_1 + g_2 s_2$$

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- By introducing the variables $\phi = m\theta$ and

$$v = \sin^2 \frac{\phi}{2}$$

we can transform the equation to a hypergeometric form and solve it.

We find a fundamental solution in terms of

$$f_1(\kappa,\delta;v) := v^{\kappa/2}(1-v)^{-\kappa/2}\ {}_2F_1\left(\begin{matrix}\delta,-\delta\\ \frac{1}{2}+\kappa\end{matrix};v\right),$$

$$f_2(\kappa,\delta;v) := \frac{\delta}{\frac{1}{2}+\kappa} v^{(\kappa+1)/2}(1-v)^{(1-\kappa)/2}\ {}_2F_1\left(\begin{matrix}1+\delta,1-\delta\\ \frac{3}{2}+\kappa\end{matrix};v\right),$$

indeed (in the real coordinate system)

$$L(\phi) = \begin{bmatrix} f_1(\kappa,\delta;v) & f_2(\kappa,\delta;v) \\ -f_2(-\kappa,\delta;v) & f_1(-\kappa,\delta;v)\end{bmatrix} \times \begin{bmatrix}\cos\delta\phi & -\sin\delta\phi \\ \sin\delta\phi & \cos\delta\phi\end{bmatrix}$$

for $0<\phi=m\theta<\pi$, extended to the whole circle by $L(zw)=L(z)\,\tau_\ell(w)$ for $w\in I_2(m)$. The Wronskian $\det L(\phi)=1$.

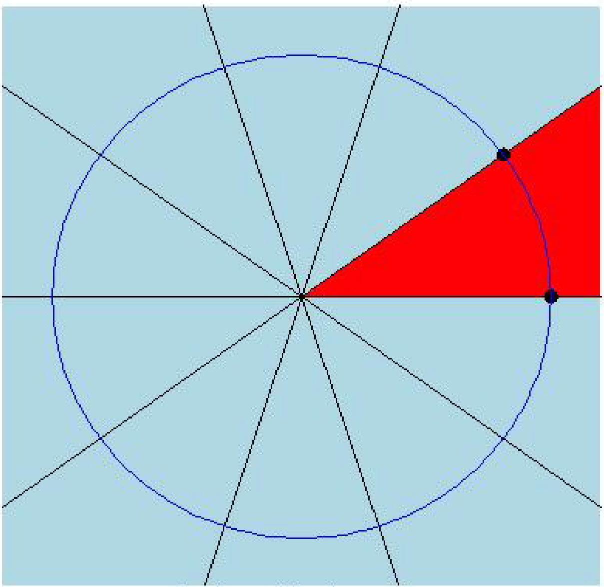
The next task is to determine the constant matrix M so that

$$K(z) = L(z)^* M L(z)$$

satisfies $\langle \mathcal{D}p, q \rangle_G = \left\langle p, \left(\frac{z}{2} - \overline{\mathcal{D}}\right) q \right\rangle_G$.In addition to the differential equation, to make the integration by parts argument work it is required that

$$\lim_{z \to z_0, z \in \mathcal{C}} \left(K(z) - \tau_\ell(\sigma) K(z) \tau_\ell(\sigma)\right) = 0,$$

where $\mathcal{C}$ is the fundamental chamber $(0 < \theta < \frac{\pi}{m})$, σ corresponds to one of the walls (that is, the line fixed by σ) and $z_0 \neq 0$ is a boundary point of $\mathcal{C}$ with $z_0\sigma = z_0$. Here the walls are $\theta = 0$ fixed by σ_0 and $\theta = \frac{\pi}{m}$ fixed by σ_1.



Fundamental chamber, m = 5

- The condition for $\theta = 0$ is satisfied if M is diagonal (fairly straightforward). For the wall $\theta = \frac{\pi}{m}$ we analyze the functions f_1 and f_2 as $v \to 1_-$ (basic facts about ${}_2F_1$-series). Let

$$H(\kappa, \delta) := \frac{\Gamma\left(\frac{1}{2} + \kappa\right)^2}{\Gamma\left(\frac{1}{2} + \kappa + \delta\right)\Gamma\left(\frac{1}{2} + \kappa - \delta\right)},$$

then

$$\begin{aligned} f_1(\kappa, \delta; v) &= H(\kappa, \delta) f_1(-\kappa, \delta; 1 - v) + \frac{\sin \pi\delta}{\cos \pi\kappa} f_2(\kappa, \delta; 1 - v) \\ f_2(\kappa, \delta; v) &= \frac{\sin \pi\delta}{\cos \pi\kappa} f_1(\kappa, \delta; 1 - v) - H(\kappa, \delta) f_2(-\kappa, \delta; 1 - v). \end{aligned}$$

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then

$$f_1(\kappa, \delta; v) = H(\kappa, \delta) f_1(-\kappa, \delta; 1 - v) + \frac{\sin \pi\delta}{\cos \pi\kappa} f_2(\kappa, \delta; 1 - v)$$
$$f_2(\kappa, \delta; v) = \frac{\sin \pi\delta}{\cos \pi\kappa} f_1(\kappa, \delta; 1 - v) - H(\kappa, \delta) f_2(-\kappa, \delta; 1 - v).$$

- We find that the solution (unique up to multiplication by a scalar)

$$L(\phi)^* \begin{bmatrix} H(-\kappa, \delta) & 0 \\ 0 & H(\kappa, \delta) \end{bmatrix} L(\phi)$$

satisfies both boundary conditions.

Observe

$$H(\kappa,\delta)\,H(-\kappa,\delta) = \frac{\cos\pi(\kappa+\delta)\cos\pi(\kappa-\delta)}{\cos^2\pi\kappa} = \frac{\sin\pi\left(\frac{\ell}{m}+\kappa\right)\sin\pi\left(\frac{\ell}{m}-\kappa\right)}{\cos^2\pi\kappa},$$

the condition for K to be positive-definite is $-\frac{\ell}{m}<\kappa<\frac{\ell}{m}$. The normalizing constant (so that $\langle t,t\rangle_G=2$) is found to be $\dfrac{\cos\pi\kappa}{2\pi\cos\pi\delta}$. Note that K is integrable for a larger interval $-\frac{1}{2}<\kappa<\frac{1}{2}$.

The details are in arXiv:1306.6599 (C.D. *Vector polynomials and a matrix weight associated to dihedral groups*).