Askey-Wilson polynomials and the tetrahedron index

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Work in progress.

Partly joint with Ilmar Gahramanov (Humboldt University of Berlin).

Outline

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Encounters with Dick Askey

The tetrahedron index

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You should look at

the ${}_{10}W_9$!

Askey 70: Bexbach, 2003

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Then, the coefficients in

$$h_k(x;a)h_{N-k}(x;b) = \sum_{l=0}^{N} C_l^k h_l(x;c)h_{N-l}(x;d)$$

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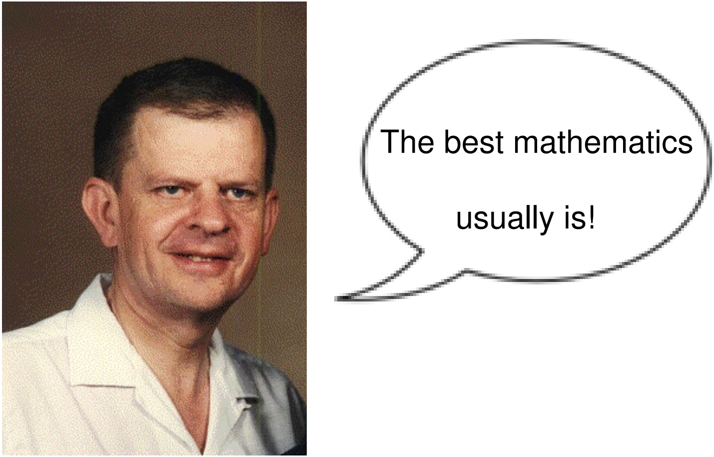

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$$\mathcal{I}_q[m, z] = \frac{(q^{1-m/2}/z; q)_\infty}{(q^{-m/2}z; q)_\infty}$$

where $m \in \mathbb{Z}$, $z \in \mathbb{C}$, $(a; q)_\infty = \prod_{j=0}^{\infty}(1 - aq^j)$.

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Doesn't look very exciting yet...

Sample result: "Pentagon identity"

$$\sum_{m=-\infty}^{\infty} \oint \frac{(-1)^m}{z^{3m}} \prod_{j=1}^{3} \frac{(q^{1+\frac{m}{2}}/a_j z, q^{1-\frac{m}{2}} z/b_j; q)_\infty}{(q^{\frac{m}{2}} a_j z, q^{-\frac{m}{2}} b_j/z; q)_\infty} \frac{dz}{2\pi i z} = \prod_{j,k=1}^{3} \frac{(q/a_j b_k; q)_\infty}{(a_j b_k; q)_\infty},$$

where $|q| < 1$ and $a_1 a_2 a_3 = b_1 b_2 b_3 = q^{1/2}$.
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Sketch of proof

Shrink contour to zero. Pick up residues at $z = q^{-\frac{m}{2}+k} b_j$, $k \geq \max(0, m)$. For $j = 1$, the sum of residues is

$$\text{Const} \cdot \sum_{m=-\infty}^{\infty} \sum_{k=\max(0,m)}^{\infty} \frac{(a_1 b_1, a_2 b_1, a_3 b_1; q)_k}{(q, qb_1/b_2, qb_1/b_3; q)_k} q^k \times \frac{(a_1 b_1, a_2 b_1, a_3 b_1; q)_{k-m}}{(q, qb_1/b_2, qb_1/b_3; q)_{k-m}} q^{k-m}$$

$$= \text{Const} \cdot {}_3\phi_2 \left(\begin{matrix} a_1 b_1, a_2 b_1, a_3 b_1 \\ qb_1/b_2, qb_1/b_3 \end{matrix} ; q, q \right)^2 .$$

We need to prove an identity like

$$C_1 \cdot {}_3\phi_2^2 + C_2 \cdot {}_3\phi_2^2 + C_3 \cdot {}_3\phi_2^2 = D$$

(with C_j and D explicit infinite products).

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Continued proof

Start from

$$(x_2 - x_1)(x_3 - x_1)(x_3 - x_2) = \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}$$
$$= (x_3 - x_2)x_1^2 + (x_1 - x_3)x_2^2 + (x_2 - x_1)x_3^2.$$

Substitute

$$x_1 = \frac{b_1(qb_1/b_2, qb_1/b_3; q)_\infty}{(a_1b_1, a_2b_1, a_3b_1; q)_\infty} \, {}_3\phi_2\left(\begin{matrix} a_1b_1, a_2b_1, a_3b_1 \\ qb_1/b_2, qb_1/b_3 \end{matrix}; q, q\right),$$

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Extended pentagon identity

$$\sum_{m=-\infty}^{\infty} \oint \prod_{j=1}^{3} \frac{(q^{1+m/2}/a_j z, q^{1-m/2} z/b_j; q)_\infty}{(q^{M_j+m/2} a_j z, q^{N_j-m/2} b_j/z; q)_\infty} \frac{dz}{2\pi \mathrm{i} z}$$

$$= \prod_{j,k=1}^{3} \frac{(q/a_j b_k; q)_\infty}{(a_j b_k q^{M_j+N_k}; q)_\infty},$$

where $a_1 a_2 a_3 = b_1 b_2 b_3 = q^{1/2}$, M_j and N_j are integers with $M_1 + M_2 + M_3 = N_1 + N_2 + N_3 = 0$.

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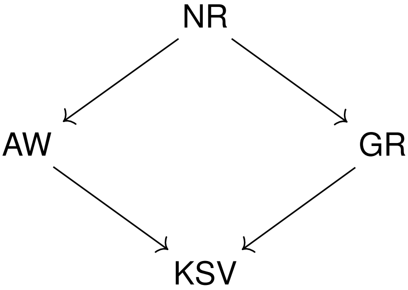

NR: New “Nassrallah–Rahman-type” identity.
AW: New “Askey–Wilson-type” identity.
GR: Identity discussed above.
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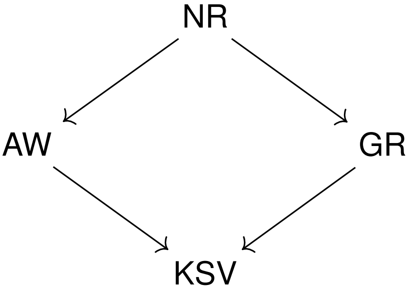

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Nassrallah–Rahman type

$$\sum_{m=-\infty}^{\infty} \oint \prod_{j=1}^{6} \frac{(q^{1+m/2}/a_j z, q^{1-m/2} z/a_j; q)_\infty}{(q^{N_j+m/2} a_j z, q^{N_j-m/2} a_j/z; q)_\infty}$$

$$\times \frac{(1-q^m z^2)(1-q^m z^{-2})}{q^m z^{6m}} \frac{dz}{2\pi \mathrm{i} z}$$

$$= \frac{2}{\prod_{j=1}^{6} q^{\binom{N_j}{2}} a_j^{N_j}} \prod_{1 \le j < k \le 6} \frac{(q/a_j a_k; q)_\infty}{(a_j a_k q^{N_j+N_k}; q)_\infty},$$

$a_1 \cdots a_6 = q$ and $N_1 + \cdots + N_6 = 0$.

Askey–Wilson-type

$$\sum_{m=-\infty}^{\infty} \oint \prod_{j=1}^{4} \frac{(q^{1+m/2}/a_j z, q^{1-m/2} z/a_j; q)_\infty}{(q^{m/2} b_j z, q^{-m/2} b_j/z; q)_\infty} \times \frac{(1-q^m z^2)(1-q^m z^{-2})}{q^m z^{4m}} \frac{dz}{2\pi \mathrm{i} z} = \frac{2(b_1 b_2 b_3 b_4; q)_\infty}{(q/a_1 a_2 a_3 a_4; q)_\infty} \prod_{1\le j<k\le 4} \frac{(q/a_j a_k; q)_\infty}{(b_j b_k; q)_\infty}.$$

Need $|q/a_1 a_2 a_3 a_4| < 1$ for convergence.

Compare Askey–Wilson integral

$$\oint \frac{(z^2, z^{-2}; q)_\infty}{\prod_{j=1}^{4} (b_j z, b_j/z; q)_\infty} \frac{dz}{2\pi \mathrm{i} z} = \frac{2(b_1 b_2 b_3 b_4; q)_\infty}{\prod_{1\le j<k\le 4} (b_j b_k; q)_\infty}.$$

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Proof of Askey–Wilson-type identity

Much easier than what we did before!
Poles at $z = q^{k-m/2}/b_j$, $k \geq 0$. Shift $z \mapsto q^{-m/2}z$.

$$\sum_{m=-\infty}^{\infty} \oint \prod_{j=1}^{4} \frac{(q^{1+m}/a_j z, q^{1-m}z/a_j; q)_\infty}{(b_j z, b_j/z; q)_\infty} \times \frac{(1-z^2)(1-q^{2m}z^{-2})}{q^m z^{4m}} \frac{dz}{2\pi \mathrm{i} z}$$

Interchange sum and integral. Sum is summable ${}_6\psi_6$.
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What can we integrate?

Replacing $a_1 \mapsto a_1 q^k$, $b_1 \mapsto b_1 q^l$ gives
(recall that $h_k(x + x^{-1}; a) = (ax, a/x; q)_k$)

$$\sum_{m=-\infty}^{\infty} \oint \prod_{j=1}^{4} \frac{(q^{1+m/2}/a_j z, q^{1-m/2} z/a_j; q)_\infty}{(q^{m/2} b_j z, q^{-m/2} b_j/z; q)_\infty}$$

$$\times\, h_k(zq^{-m/2} + z^{-1} q^{m/2}; a_1) h_l(zq^{m/2} + z^{-1} q^{-m/2}; b_1)$$

$$\times \frac{(1 - q^m z^2)(1 - q^m z^{-2})}{q^m z^{4m}} \frac{dz}{2\pi \mathrm{i} z}$$

$$= \frac{2(b_1 b_2 b_3 b_4; q)_\infty}{(q/a_1 a_2 a_3 a_4; q)_\infty} \prod_{1 \le j < k \le 4} \frac{(q/a_j a_k; q)_\infty}{(b_j b_k; q)_\infty}$$

$$\times \frac{\prod_{j=2}^{4} (a_1 a_j; q)_k (b_1 b_j; q)_l}{(a_1 a_2 a_3 a_4; q)_k (b_1 b_2 b_3 b_4; q)_l}.$$

Decoupling phenomenon

On the other hand,

$$\frac{\prod_{j=2}^{4}(a_1 a_j;q)_k (b_1 b_j;q)_l}{(a_1 a_2 a_3 a_4;q)_k (b_1 b_2 b_3 b_4;q)_l}$$

$$= \mathrm{Const} \cdot \oint \frac{(z^2, z^{-2};q)_\infty}{\prod_{j=1}^{4}(a_j z, a_j/z;q)_\infty} h_k(z+z^{-1};a_1) \frac{dz}{2\pi \mathrm{i} z}$$

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Our $\sum \oint$ “decouples” as product of two Askey–Wilson integrals.

Askey-Wilson polynomials

Polynomials orthogonal with respect to Askey–Wilson integral, denoted $p_n((z+z^{-1})/2; a_1, a_2, a_3, a_4; q)$.

By decoupling phenomenon, the polynomials

$$p_k\left(\frac{zq^{-m/2}+z^{-1}q^{m/2}}{2}; a_1, a_2, a_3, a_4; q\right) p_l\left(\frac{zq^{m/2}+z^{-1}q^{-m/2}}{2}; b_1, b_2, b_3, b_4; q\right)$$

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Caveat: For convergence, we need $|q^{1-k}/a_1a_2a_3a_4| < 1$, so k runs only over a finite set.

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What about the ${}_{10}W_9$?

For the top level “Nasrallah-Rahman”-type $\sum \oint$, there is no convergence problem.

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Final slide for today

Happy birthday Dick!