Zeros of polynomial special functions

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3 a.m.

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A paper written by Richard Askey

" Graphs as an aid to understanding special functions" R Askey 1989

Zeros of Jacobi polynomials $P_n^{(\alpha,\beta)}$ and $P_n^{(\alpha+t,\beta)}$

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Conjecture that interlacing holds for $t = 2$

Interlacing of zeros of orthogonal polynomials from different families

Zeros of Laguerre polynomials L_n^{α} and $L_n^{\alpha+t}$, $\alpha > -1$

For what (continuous) values of t are the zeros interlacing?

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Look at three term recurrence relations linking classical orthogonal polynomials from different sequences

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Interlacing of zeros of Jacobi polynomials with varying parameters α, β

Kerstin Jordaan, Norbert Mbuyi, KD 2008

Zeros of Jacobi polynomials $P_n^{(\alpha,\beta)}$ and $P_n^{(\alpha+t,\beta-k)}$ interlace for all $0 < t, k \leq 2$

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Similar results on interlacing of zeros of $P_n^{(\alpha,\beta)}$ and $P_{n-1}^{(\alpha+t,\beta-k)}$

"Stieltjes" interlacing of zeros of OP's

Definition

Let p and q be two real polynomials with real, simple, distinct zeros, deg $(p) >$deg (q). The zeros of p and q *interlace if each zero of q lies between two successive zeros of p and there is at most one zero of q between any two successive zeros of p*

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Stieltjes proved that within an orthogonal sequence, the zeros of p_n and p_{n-k} interlace for all $k \geq 1$, **provided** p_n and p_{n-k} have no common zeros

"Stieltjes" interlacing of zeros of OP's, different parameters

KD 2012 Stieltjes interlacing holds between the positive (negative) zeros of ultraspherical C_n^{λ} and $C_{n-2}^{\lambda+t}$ for $0 \leq t \leq 2$, $\lambda > -\frac{1}{2}$.

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Fix $k \in 0, 1, 2, 3$. If C_n^{λ} and $C_{n-3}^{\lambda+k}$ have no common zeros, the zeros of $C_{n-3}^{\lambda+k}$ plus two (symmetric) identified points interlace with the zeros of C_n^{λ}

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Since common zeros cannot occur at the largest zero of C_n^λ these "extra" points give good lower bounds for the largest zero of C_n^λ

Stieltjes interlacing and continuous variation in parameter(s)

2013 Martin Muldoon and KD
Laguerre polynomials

Graphs as an aid to understanding special functions Askey 1988

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Common zeros of $L_n^{(\alpha)}$ and $L_{n-k}^{(\alpha+t)}$ as functions of t

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If $\alpha \geq 0$, k a positive integer with $1 \leq k \leq n-2$, then for each t in the interval $0 \leq t \leq 2k$, excluding the values of t for which $L_n^{(\alpha)}$ and $L_{n-k}^{(\alpha+t)}$ have a common zero, the zeros of these two polynomials interlace.

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The interval $0 \leq t \leq 2k$ is largest possible such that interlacing holds for all n.

Asymptotic zero distribution of hypergemetric polynomials, Peter Duren and KD 1988

P Borwein and W Chen 1995

Asymptotic zero distribution as $n \to \infty$

$$\int_0^1 [t^k(1-t)^s f_z(t)]^n dt$$

where $f_z(t)$ is polynomial in t, analytic in z.

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Euler integral representation of ${}_2F_1(-n, b; c; z), Re(c) > Re(b) > 0$

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^n dt$$

Let $b = n+1, c = 2n+2$. As $n \to \infty$, zeros of ${}_2F_1(-n, n+1; 2n+2; z)$ cluster on the arc of the circle $|z-1| = 1, Re(z) > \frac{1}{2}$

Calculations of the zeros of ${}_2F_1(-n, n+1; 2n+2; z)$

For each $n \in \mathbb{N}$, zeros of ${}_2F_1(-n, n+1; 2n+2; z)$ lie on the circle

$$\{z : |z-1| = 1, Re(z) > \tfrac{1}{2}\}$$

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For each $n \in \mathbb{N}$ and all $b > -\frac{1}{2}$, zeros of ${}_2F_1(-n, b; 2b; z)$ lie on the circle

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R Askey

$${}_2F_1(-n, b; 2b; 1-e^{2i\theta}) = \tfrac{n!e^{in\theta}}{(2b)_n} C_n^{(b)}(cos\theta)$$

Ultraspherical polynomials $C_n^{(b)}(x)$, $b > -\frac{1}{2}$

$\{C_n^{(b)}\}_{n=0}^{\infty}$ is orthogonal on $(-1, 1)$, weight function $(1 - x^2)^{b-\frac{1}{2}}$, $b > -\frac{1}{2}$

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For $b > -\frac{1}{2}$, zeros of $C_n^{(b)}(x)$ lie in $(-1, 1) \Rightarrow$ zeros of ${}_2F_1(-n, b; 2b; z)$ lie on the circle $\{z : |z - 1| = 1\}$

Another identity linking $C_n^{(\lambda)}$ and ${}_2F_1(-n, b; 2b; -)$

$${}_2F_1(-n, b; 2b; z) = \frac{n! 2^{-2n} z^n}{(b+\frac{1}{2})_n} C_n^{(\lambda)}(1 - \frac{2}{z}) \text{ where } \lambda = \frac{1}{2} - b - n$$

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For $\lambda < 1 - n$, zeros of $C_n^{(\lambda)}(w)$ satisfy $|\frac{2}{1-w} - 1| = 1$

For $\lambda < 1 - n$, zeros of $C_n^{(\lambda)}$ lie on the imaginary axis

Pseudo ultraspherical polynomials $\tilde{C}_n^{(\lambda)}$

Define: $\mathcal{C}_n^{(\lambda)}(x) := (-i)^n C_n^{(\lambda)}(ix)$

For $\lambda < 1 - n$ all zeros of $\mathcal{C}_n^{(\lambda)}$ are real.

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Askey 1988 "An integral of Ramanujan and orthogonal polynomials" J. Indian Math.Soc.

The (complex) orthogonality of Jacobi polynomials in the special case $\alpha = \beta = \lambda - \frac{1}{2}$

Zeros of Pseudo ultraspherical polynomials $\mathcal{C}_n^{(\lambda)}$

Martin Muldoon and KD 2013

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Monotonicity properties of the real zeros

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Identification of sub-intervals of the real line that contain all the zeros (depends on n and λ)

Suppose $\lambda \leq -(2+\sqrt{2})n + \frac{1}{2}$. The zeros of $\mathcal{C}_n^{(\lambda)}$ lie in $[-1, 1]$

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Interlacing (yes and no) across different families as λ varies continuously

Thank you Dick

The most enjoyable 3 of many Askey Moments