The Askey–Wilson Polynomials
A Personal Journey

December 5, 2013

For a sequence of orthogonal polynomials you would like to find

1. Orthogonality measure.

2. Raising and lowering operators. Rodrigues formulas.

3. A second order Sturm–Liouville type equation.

4. Connection relations.

5. Coefficients in the linearization of products

$$p_m(x)p_n(x) = \sum_k c(m,n,k)p_k(x).$$

6. Generating functions $\sum_n p_n(x)\lambda_n t^n$, for suitable λ_n.

7. Poisson kernel $\sum p_n(x)p_n(y)r^n$, p_n are orthonormal.

8. Asymptotics as the degree $\to \infty$.

9. Plancherel–Rotach asymptotics if applicable (unbounded support).

Notation:

$$(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j), \quad n = 0, 1, \cdots, \text{or } \infty,$$

$$(a_1, a_2, \cdots, a_m; q)_n = \prod_{k=1}^{m}(a_k;q)_n$$

The Askey–Wilson memoir:

$$w(\cos\theta;\mathbf{a})dx = \frac{(e^{2i\theta}, e^{-2i\theta};q)_\infty}{\prod_{j=1}^4 (a_j e^{i\theta}, a_j e^{-i\theta};q)_\infty}\, d\theta.$$

Notation: $x = \cos\theta \quad$ versus $\quad z = e^{i\theta}$.

$\breve{f}(z) = f((z+1/z)/2)$.

For $e(x) = x, \quad \breve{e}(z) = (z+1/z)/2$.

$$(\mathcal{D}_q f)(x) = \frac{\breve{f}(q^{1/2}z) - \breve{f}(q^{-1/2}z)}{(q^{1/2}-q^{-1/2})(z-1/z)/2}.$$

For example

$$\mathcal{D}_q T_n(x) = \frac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}}\, U_{n-1}(x).$$

I tried to understand the Askey–Wilson operator and the Askey–Wilson integral. I want to share some of the findings with you. Most of the work is joint work.

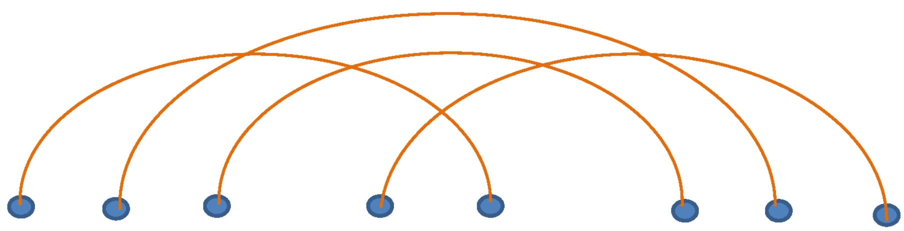
$$\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\prod_{j=1}^4 (a_j e^{i\theta}, a_j e^{-i\theta}; q)_\infty}\, d\theta = \frac{2\pi (a_1 a_2 a_3 a_4; q)_\infty}{(q;q)_\infty \prod_{1 \le j < k \le 4} (a_j a_k; q)_\infty}.$$

In 1893–1895 Rogers studied the q-Hermite polynomials

$$\sum_{n=0}^\infty H_n(\cos\theta|q) \frac{t^n}{(q;q)_n} = \frac{1}{(te^{i\theta}, te^{-i\theta}; q)_\infty}.$$

He proved the linearization formula

$$H_m(x|q) H_n(x|q) = \sum_{k=0}^{m\wedge n} \frac{(q;q)_m (q;q)_n}{(q;q)_k (q;q)_{m-k} (q;q)_{n-k}}\, H_{m+n-2k}(x|q)$$



Azor–Gillis–Victor and independently Godsil (1982) proved that the number of perfect matchings is

$$\frac{2^{-\sum_j n_j/2}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-x^2} \prod_{j=1}^{m} H_{n_j}(x)dx.$$

This means that the following integral is a q-analogue of the number of perfect matchings whose generating function is

$$\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\prod_{j=1}^m (a_j e^{i\theta}, a_j e^{-i\theta}; q)_\infty} d\theta.$$

the coefficient of $\prod_{j=1}^m a_j^{n_j}/(q;q)_{n_j}$ is

$$\int_0^\pi (e^{2i\theta}, e^{-2i\theta}; q)_\infty \prod_{j=1}^{m} H_{n_j}(\cos\theta|q))d\theta.$$

After some scaling we see that this =

$$\text{(1)} \qquad \sum_{\text{Perfect Matchings}} q^{\text{crossing number}}.$$

Ismail–Stanton–Viennot (1987).

Problem: The polynomial (1) is symmetric in $n_1, n_2, \cdots, n_m$. It is

$$\int_{\mathbb{R}} w(x) \prod_{j=1}^{m} \tilde{H}_{n_j}(x|q)dx.$$

Fact:

$$\frac{(q;q)_\infty}{2\pi} \int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\prod_{j=1}^5 (a_j e^{i\theta}, a_j e^{-i\theta}; q)_\infty} d\theta$$
$$= \frac{(a_1a_2a_3a_5, a_2a_3a_4a_5, a_1a_4; q)_\infty}{\prod_{1\le j<k\le 5}(a_ja_k; q)_\infty}$$
$$\times {}_3\phi_2 \left(\left. \begin{matrix} a_2a_3,\ a_2a_5,\ a_3a_5 \\ a_1a_2a_3a_5, a_2a_3a_4a_5 \end{matrix} \right| q, a_1a_4 \right).$$

You get ${}_3\phi_2$ transformations.

The Askey–Wilson operators: Null space is nontrivial, no two-sided inverse. What about one sided inverse. $\frac{d}{dx}\int_a^x f(y)dy = f(x)$. Analogue of $\int_a^x$. Also we need an analogue of integration by parts. This is essentially computing the adjoint of $\mathcal{D}_q$.

We shall use the inner product (Chebyshev)

$$\langle f, g\rangle := \int_{-1}^{1} f(x)\,\overline{g(x)}\,\frac{dx}{\sqrt{1-x^2}}.$$

We require $\breve{f}(z)$ to be defined for $\left|q^{\pm 1/2}z\right| = 1$ as well as for $|z| = 1$. Thus $\mathcal{D}_q$ is defined on:

$$H_\nu := \left\{f : f((z+1/z)/2) \text{ analytic in } \; q^\nu \le |z| \le q^{-\nu}\right\}.$$

Theorem 1. *For f, $g \in H_{1/2}$ we have*

$$\begin{aligned}\langle \mathcal{D}_q f, g\rangle = \frac{\pi\sqrt{q}}{1-q}&[f((q^{1/2}+q^{-1/2})/2)\overline{g(1)}\\ &-f(-(q^{1/2}+q^{-1/2})/2)\overline{g(-1)}]\\ &-\left\langle f, \sqrt{1-x^2}\,\mathcal{D}_q(g(x)(1-x^2)^{-1/2})\right\rangle.\end{aligned}$$

Analyze Sturm–Liouville equations. Brown–Evans–Ismail. Later work with Christiansen, Christainsen and Koelink, Brown–Chriatinsen.

One sided inverse:

$$\mathcal{D}_q T_n(x) = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} U_{n-1}(x).$$

Define $\mathcal{D}_q^{-1}$ by

$$\mathcal{D}_q^{-1} \sum_{n=0}^{\infty} f_n U_n(x) = \sum_{n=1}^{\infty} f_{n-1} \frac{q^{1/2} - q^{-1/2}}{q^{n/2} - q^{-n/2}} T_n(x).$$

This defines a map from $L_2[(1-x^2)^{1/2}, -1, 1]$ into $L_2[(1-x^2)^{-1/2}, -1, 1]$. Indeed it has the representation

$$(\mathcal{D}_q^{-1} g)(\cos\theta)$$

$$= \frac{1-q}{4\pi\sqrt{q}} \int_{-\pi}^{\pi} \frac{\vartheta_4'((\theta+\phi)/2, \sqrt{q})}{\vartheta_4((\theta+\phi)/2, \sqrt{q})} g(\cos\theta) \sin\theta \, d\theta.$$

Moreover $\mathcal{D}_q \mathcal{D}_q^{-1} =$ identity.

One can diagonalize this integral operator. Similarly for q-ultraspherical polynomials, Ismail–Zhang, and the continuous q-Jacobi polynomials, Ismail–Rahman–Zhang. The kernel for the full Askey–Wilson is also known, Ismail–Rahman. This led to a q-analogue od the exponential function.

Taylor Series.

The Askey–Wilson basis is $\{(ae^{i\theta}, ae^{-i\theta}; q)_n\}$. Also

$$\mathcal{D}_q(ae^{i\theta}, ae^{-i\theta}; q)_n$$
$$= -\frac{2a(1-q^n)}{1-q}(aq^{1/2}e^{i\theta}, aq^{1/2}e^{-i\theta}; q)_{n-1}.$$

Theorem 2. *Let f be a polynomial, then*

$$f(x) = \sum_{k=0}^{n} f_k (ae^{i\theta}, ae^{-i\theta}; q)_k,$$

where

$$f_k = \frac{(q-1)^k}{(2a)^k (q;q)_k}\, q^{-k(k-1)/4} \left(\mathcal{D}_q^k f\right)(x_k)$$

with

$$x_k := \frac{1}{2}(aq^{k/2} + q^{-k/2}/a).$$

When $f(x) = (be^{i\theta}, be^{-i\theta}; q)_n$ we get the q-Pfaff–Saalschütz theorem.

Can we build the theory of q-series this way?

Extension to non polynomial cases. This is an interpolation problem.

A classic is to reconstruct an entire function from its values at the integers. Not always possible because $\frac{\sin(\pi z)}{\pi z} = 0$ at $z = \pm 1, \pm 2, \cdots$.

The entire function is unique if $\log M(f, r) = cr[1 + o(1)], c < \pi$. Gelfond and his school considered interpolation at $q^{-n}, |q| < 1$. Our interpolation points are

$$x_k := \frac{1}{2}(aq^{k/2} + q^{-k/2}/a).$$

There is an expansion theorem when $M(f, r)$ has certain growth condition (Ismail–Stanton).

S. Cooper 1996 proved by induction

$$\mathcal{D}_q^n f(x) = \frac{(2z)^n q^{n(3-n)/4}}{(q-1)^n}$$
$$\times \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k(n-k)} z^{-2k} \breve{f}(q^{k-n/2} z)}{\left(q^{n-2k+1} z^{-2}; q\right)_k \left(z^2 q^{2k+1-n}; q\right)_{n-k}}.$$

Application 1: Apply this to

$$f(\cos\theta) = \frac{(\alpha e^{i\theta}, \alpha e^{-i\theta}; q)_\infty}{(\beta e^{i\theta}, \beta e^{-i\theta}; q)_\infty}$$

you get the ${}_6\phi_5$ summation theorem. The Rodrigues formula is

$$w(x; \mathbf{a} \,|\, q)\, p_n(x; \mathbf{a} \,|\, q)$$
$$= (\frac{q-1}{2})^n q^{n(n-1)/4} \mathcal{D}_q^n \left[w(x; q^{n/2}\mathbf{a} \,|\, q) \right].$$

Appiication 2: The Rodrigues formula for the Askey–Wilson polynomials gives the ${}_8\phi_7$ to ${}_4\phi_3$ transformation (Watson).

Expansions in the Askey–Wilson polynomials.

$$f(x) = \sum_{n=0}^{\infty} f_n\, p_n(x; \mathbf{a}).$$

If $\mathcal{D}_q f$ acts nicely on f we can find the coefficients using the Rodrigues formula.

Theorem 3. *(Ismail–Stanton 2013) We have the following expansion*

$$
{}_{p+1}\phi_p\left(\left.\begin{matrix} a_1, \cdots, a_{p-1}, t_4 e^{i\theta}, t_4 e^{-i\theta} \\ t_1 t_4, t_2 t_4, t_3 t_4, b_1, \cdots, b_{p-3} \end{matrix}\right| q, \zeta\right)
$$

$$
= \sum_{k=0}^{\infty} p_k(\cos\theta; \mathbf{t}|q) \frac{(a_1, \cdots, a_{p-1}; q)_k}{(t_1 t_4, t_2 t_4, t_3 t_4, b_1, \cdots, b_{p-3}; q)_k}
$$

$$
\times \frac{(-t_4 \zeta)^k q^{\binom{k}{2}}}{(q, t_1 t_2 t_3 t_4 q^{k-1}; q)_k}
$$

$$
\times {}_{p-1}\phi_{p-2}\left(\left.\begin{matrix} q^k a_1, \cdots, q^k a_{p-1} \\ q^k b_1, \cdots, q^k b_{p-3}, t_1 t_2 t_3 t_4 q^{2k} \end{matrix}\right| q, \zeta\right).
$$

The Andrews formula (2012) is the case $p = 4$ in Theorem 3 with the parameter identification

$$
a_1 = q^{-N}, a_2 = \rho_1, a_3 = \rho_2, b_1 = \rho_1 \rho_2 q^{-N}/a, \zeta = q.
$$

In this case the ${}_3\phi_2$ can be summed by the q-Pfaff–Saalschütz theorem.

Fields and Wimp (1961).

$$\sum_{m=0}^{\infty} a_m b_m \frac{(zw)^m}{m!}$$
$$= \sum_{n=0}^{\infty} \frac{(-w)^n}{n!\,(\gamma+n)_n} \left(\sum_{r=0}^{\infty} \frac{b_{n+r}\, w^r}{r!\,(\gamma+2n+1)_r} \right)$$
$$\times \left[\sum_{s=0}^{n} \frac{(-n)_s (n+\gamma)_s}{s!}\, a_s z^s \right].$$

This version is due to Verma 1972. Lagrange inversion (Gessel–Stanton).

The following general expansion follows from q-Dixon's theorem (${}_4\phi_3$)

$$\sum_{n=0}^{\infty} A_n B_n \frac{(t_4 z, t_4/z; q)_n}{(q;q)_n} \zeta^n$$
$$= \sum_{k=0}^{\infty} \frac{(-\zeta)^k q^{\binom{k}{2}}}{(q, Cq^{k-1}; q)_k} \left[\sum_{n=0}^{\infty} \frac{B_{n+k} \zeta^n}{(q, Cq^{2k}; q)_n} \right]$$
$$\times \left[\sum_{j=0}^{k} \frac{(q^{-k}, Cq^{k-1}; q)_j}{(q;q)_j} A_j (t_4 z, t_4/z; q)_j q^j \right].$$

Moments of the Askey–Wilson weight functions. Used by Corteel–Stanley–Stanton–Williams.

The Stieltjes electrostatic equilibrium leads to system is

$$\frac{\beta+1}{1+x_j} - \frac{\alpha+1}{1-x_j} + \sum_{1\le k\le n,\, k\ne j} \frac{2}{x_j - x_k} = 0,$$

for $1 \le j \le n$. Stieltjes used $y = \prod_{j=1}^{n}(x - x_j)$ and

$$\sum_{1\le k\le n,\, k\ne j} \frac{2}{x_j - x_k} = \frac{y''(x_j)}{y'(x_j)}.$$

He turned this system to

$$(1-x^2)y'' + (\beta - \alpha - x(\alpha+\beta+2))y' + \lambda_n y = 0,$$

at $x = x_1, \cdots, x_n$. By choosing λ this becomes valid for all x. Thus the equilibrium points are at the zeros of a Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$.

For the XXZ model the Bethe Ansatz equations are ($1 \le k \le n$),

$$\left(\frac{\sin\left(\lambda_k + \frac{1}{2}\eta\right)}{\sin\left(\lambda_k - \frac{1}{2}\eta\right)}\right)^{2N}$$

$$= \prod_{j \neq k, j=1}^{n} \frac{\sin\left(\lambda_k + \lambda_j + \eta\right) \sin\left(\lambda_k - \lambda_j + \eta\right)}{\sin\left(\lambda_k + \lambda_j - \eta\right) \sin\left(\lambda_k - \lambda_j - \eta\right)}.$$

Change the system of equations to

$$\prod_{\ell=1}^{2N} \frac{\sin\left(\lambda_k + s_\ell \eta\right)}{\sin\left(\lambda_k - s_\ell \eta\right)}$$

$$= \prod_{j \neq k, j=1}^{n} \frac{\sin\left(\lambda_k + \lambda_j + \eta\right) \sin\left(\lambda_k - \lambda_j + \eta\right)}{\sin\left(\lambda_k + \lambda_j - \eta\right) \sin\left(\lambda_k - \lambda_j - \eta\right)},$$

for $1 \le k \le n$, where s_ℓ's are $2N$ complex numbers. Notation:

$$q = e^{2i\eta}, \quad \theta = 2\lambda, \quad t_j = q^{-s_j}.$$

$$
\begin{aligned}
&w(\cos\theta; t_1, \cdots, t_{2N}) \\
&\qquad := \frac{(e^{iN\theta}, e^{-iN\theta}; q^{N/2})_\infty}{\sin(N\theta/2) \prod_{j=1}^{2N} (t_j e^{i\theta}, t_j e^{-i\theta}; q)_\infty}.
\end{aligned}
$$

Notation $\mathrm{t} = (t_1, t_2, \cdots, t_{2N})$.

$$
\frac{1}{w(x;\mathbf{t})} \mathcal{D}_q \left(w(x; q^{1/2}\mathbf{t}) \mathcal{D}_\mathbf{q} \right) y(x) = r(x) y(x),
$$

where $r(x)$ is a polynomial of degree $\leq N-2$.

$$
\Pi(z;\mathrm{t}) \mathcal{D}_\mathbf{q}^\mathbf{2} \mathbf{y} + \mathbf{\Phi}(\mathbf{z};\mathrm{t}) \mathcal{A}_\mathbf{q} \mathcal{D}_\mathbf{q} \mathbf{y} = \mathbf{r}(\mathbf{x}) \mathbf{y}.
$$

The solution of the Bethe Ansatz equations is at the zeros of the polynomial solutions.

Joint work with Lin and Roan.