The Rogers–Ramanujan identities for $\mathrm{A}_{2n}^{(2)}$

Ole Warnaar

School of Mathematics and Physics

The University of Queensland

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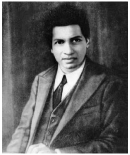
The Rogers–Ramanujan identities, first proved by Rogers in 1894, are the pair of q-series identities



$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)\cdots(1-q^n)} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-1})(1-q^{5n-4})}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)\cdots(1-q^n)} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-2})(1-q^{5n-3})}$$

Schur and MacMahon independently observed that the Rogers–Ramanujan identities have a combinatorial interpretation in terms of integer partitions.

A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is a weakly decreasing sequence of nonnegative integers, such that only finitely many $\lambda_i > 0$.

If $|\lambda| := \lambda_1 + \lambda_2 + \cdots = n$ we say that λ is a partition of n.

For example, there are 7 partitions of 5:

$$(5), \quad (4,1), \quad (3,2), \quad (3,1,1),$$

$$(2,2,1), \quad (2,1,1,1), \quad (1,1,1,1,1)$$

(First) Rogers–Ramanujan identity.

The number of partitions of n such that consecutive parts differ by at least 2 is equinumerous to the number of partitions of n such that parts are congruent to ± 1 modulo 5.

For example, for $n = 9$ both sets of partitions have cardinality of 5:

$$(9), \quad (8,1), \quad (7,2), \quad (6,3), \quad (5,3,1)$$

$$(9), \quad (6,1^3), \quad (4^2,1), \quad (4,1^5), \quad (1^9)$$

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(Second) Rogers–Ramanujan identity.

The number of partitions of n such that consecutive parts differ by at least 2 and such that no 1s occur is equinumerous to the number of partitions of n such that parts are congruent to ± 2 modulo 5.

Gordon generalised the combinatorial version of the Rogers–Ramanujan identities to

Gordon's partition theorem.

The number of partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ of n such that

$$\lambda_j - \lambda_{j+m} \geq 2$$

and such that at most $i-1$ 1s occur is equinumerous to the number of partitions of n such that parts are *not* congruent to $0, \pm i$ modulo $2m+3$.

For $m = 1$ and $i = 2$ this is the first Rogers–Ramanujan identity and for $m = 1$ and $i = 1$ this is the second Rogers–Ramanujan identity.

Andrews discovered the q-series analogue of Gordon's partition theorem.

For $n \in \mathbb{N}_0 \cup \{\infty\}$ let

$$(q)_n = (q;q)_n = (1-q)(1-q^2)\cdots(1-q^n)$$

and define the "theta function"

$$\theta(z;q) = (z;q)_\infty (q/z;q)_\infty = \frac{1}{(q)_\infty} \sum_{n\in\mathbb{Z}} (-z)^n q^{\binom{n}{2}}$$

The Andrews–Gordon identities.

For $1 \le i \le m+1$

$$\sum_{r_1 \ge \cdots \ge r_m \ge 0} \frac{q^{r_1^2+\cdots+r_m^2+r_i+\cdots+r_m}}{(q)_{r_1-r_2}\cdots(q)_{r_{m-1}-r_m}(q)_{r_m}} = \frac{(q^{2m+3};q^{2m+3})_\infty}{(q)_\infty}\,\theta(q^i;q^{2m+3})$$

In the 1980s Lepowsky and Milne observed that the Rogers–Ramanujan and Andrews–Gordon q-series arise in the representation theory of affine Kac–Moody algebras.

Let $\mathrm{A}_1^{(1)}$ be the affine Kac–Moody algebra with Cartan matrix and Dynkin diagram

$$C = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \qquad \underset{\alpha_0}{\bullet} \Longleftrightarrow \underset{\alpha_1}{\bullet}$$

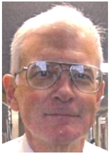
If $V(\Lambda)$ is the integrable highest weight module of $\mathrm{A}_1^{(1)}$ of highest weight $\Lambda = (2m - i + 2)\Lambda_0 + (i-1)\Lambda_1$ and ϕ the homomorphism (principal specialisation)

$$\phi : \mathbb{C}[[\mathrm{e}^{-\alpha_0}, \mathrm{e}^{-\alpha_1}]] \to \mathbb{C}[[q]]$$
$$e^{-\alpha_i} \mapsto q$$

then

$$(q;q^2)_\infty \, \phi\big(\mathrm{e}^{-\Lambda} \mathrm{char}\, V(\Lambda)\big) = \frac{(q^{2m+3};q^{2m+3})_\infty}{(q)_\infty} \, \theta(q^i;q^{2m+3})$$

The previous observation resulted in a new, representation theoretic derivation of the Rogers–Ramanujan identities by Lepowsky and Wilson. A complicating factor in their proof is the occurrence of the unwanted infinite product in



$$(q;q^2)_\infty \, \phi\big(\mathrm{e}^{-\Lambda}\mathrm{char}\, V(\Lambda)\big)$$

and to this day it is not clear how to extend their approach to obtain Rogers–Ramanujan identities for $\mathrm{A}_n^{(1)}$.

Fortunately, there is another Kac–Moody algebra that realises the Rogers–Ramanujan and Andrews–Gordon q-series.

Let $\mathrm{A}_2^{(2)}$ be the twisted affine Kac–Moody algebra with Cartan matrix and Dynkin diagram

$$C = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$$

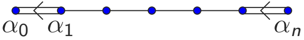
If $V(\Lambda)$ is the integrable highest weight module of $\mathrm{A}_2^{(2)}$ of highest weight $\Lambda = (2m-2i+2)\Lambda_0 + (i-1)\Lambda_1$ and ϕ the specialisation

$$\phi : \mathbb{C}[[\mathrm{e}^{-\alpha_0}, \mathrm{e}^{-\alpha_1}]] \to \mathbb{C}[[q]]$$
$$e^{-\alpha_0} \mapsto -1, \quad e^{-\alpha_1} \mapsto q$$

then

$$\phi\big(\mathrm{e}^{-\Lambda}\mathrm{char}\, V(\Lambda)\big) = \frac{(q^{2m+3}; q^{2m+3})_\infty}{(q)_\infty}\, \theta(q^i; q^{2m+3})$$

It is this second interpretation of the Rogers–Ramaujan and Andrews–Gordon q-series that we will extend to $\mathrm{A}_{2n}^{(2)}$:



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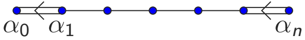

Generalising the specialisation ϕ to arbitrary rank by

$$\mathrm{e}^{-\alpha_0} \mapsto -1 \quad \text{and} \quad \mathrm{e}^{-\alpha_i} \mapsto q \;\; (1 \leq i \leq n)$$

we have

$$\begin{aligned}\phi\big(\mathrm{e}^{-\Lambda}\,\mathrm{ch}\,V(\Lambda)\big) = \frac{(q^{\kappa};q^{\kappa})_{\infty}^{n}}{(q)_{\infty}^{n}} \prod_{i=1}^{n} \theta\big(q^{\lambda_0-\lambda_i+i};q^{\kappa}\big) \\ \times \prod_{1\leq i<j\leq n} \theta\big(q^{\lambda_i-\lambda_j-i+j},q^{\lambda_i+\lambda_j-i-j+2n+1};q^{\kappa}\big),\end{aligned}$$

where $\kappa = 2\lambda_0 + 2n + 1$, $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n$ and

$$\Lambda = 2\lambda_n\Lambda_0 + (\lambda_{n-1}-\lambda_n)\Lambda_1 + \cdots + (\lambda_0-\lambda_1)\Lambda_n$$

But what is the corresponding sum side?

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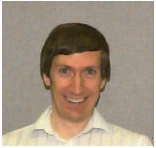
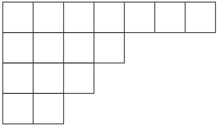


Stembridge noted that the sum side of the Rogers–Ramanujan identities and the $i = m + 1$ and $i = 1$ cases of the Andrews–Gordon identities can be written in a compact form using the Hall–Littlewood symmetric functions $P_\lambda(x_1, x_2, \dots ; q)$.

For λ a partition, let $m_i = m_i(\lambda)$ be the multiplicity of parts of size i.

For example, if $\lambda = (4, 4, 3, 2, 1, 1, 1)$ then $m_1 = 3$, $m_2 = m_3 = 1$, $m_4 = 2$ and $m_i = 0$ for $i \geq 5$.

It is easy to see that $m_i = \lambda'_i - \lambda'_{i+1}$ where λ' is the conjugate of λ.

$\lambda' =$ 

For λ a partition, define the power sum symmetric function $p_\lambda = \prod_{i\geq 1} p_{\lambda_i}$ by

$$p_r(x_1, x_2, \dots) = x_1^r + x_2^r + \cdots$$

The power sums may be used to define a q-analogue of the Hall scalar product on the ring of symmetric function Λ:

$$\langle p_\lambda, p_\mu \rangle_q = \delta_{\lambda\mu} \prod_{i\geq 1} \frac{i^{m_i} m_i!}{1 - q^{\lambda_i}}$$

Then the Hall–Littlewood symmetric functions P_λ are the unique symmetric functions such that

$$P_\lambda(x;q) = m_\lambda(x) + \sum_{\mu<\lambda} u_{\lambda\mu}(q) m_\mu(x)$$

and

$$\langle P_\lambda, P_\mu \rangle_q = 0 \quad \text{for } \lambda \neq \mu$$

Here $<$ refers to the dominance order on partitions and m_λ is the monomial symmetric function.

For example,

$$P_{(2,1)}(x;q) = m_{(2,1)}(x) + (1-q)(q+2)m_{(1,1,1)}(x)$$

so that

$$\begin{aligned} P_{(2,1)}(x_1,x_2,x_3;q) = x_1^2x_2 + x_2^2x_1 + x_1^2x_3 + x_3^2x_1 + x_2^2x_3 + x_3^2x_2 \\ + (1-q)(q+2)x_1x_2x_3 \end{aligned}$$

The principal specialisation formula for Hall–Littlewood polynomials gives

$$P_\lambda(1,q,\dots,q^{n-1};q) = \frac{q^{n(\lambda)}(q)_n}{(q)_{n-l(\lambda)}b_\lambda(q)}$$

where

$$n(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i \quad \text{and} \quad b_\lambda(q) = \prod_{i\geq 1}(q)_{m_i}$$

For example

$$P_{(2,1)}(1,q,q^2;q) = \frac{q(q)_3}{(q)_1^3} = q(1+q)(1+q+q^2)$$

$$P_{(2,1)}(1,q,q^2,\dots;q) = \frac{q}{(q)_1^2} = \frac{q}{(1-q)^2} = q(1+2q+3q^2+\cdots)$$

If we interpret $(r_1, r_2, \ldots, r_m)$ as a partition λ' of length at most m, so that $r_i - r_{i+1} = \lambda'_i - \lambda'_{i+1} = m_i$, then

$$\frac{q^{r_1^2+\cdots+r_m^2}}{(q)_{r_1-r_2}\cdots(q)_{r_{m-1}-r_m}(q)_{r_m}} = q^{|\lambda|}P_{2\lambda}(1, q, q^2, \ldots; q)$$

and

$$\frac{q^{r_1^2+\cdots+r_m^2+r_1+\cdots+r_m}}{(q)_{r_1-r_2}\cdots(q)_{r_{m-1}-r_m}(q)_{r_m}} = q^{2|\lambda|}P_{2\lambda}(1, q, q^2, \ldots; q)$$

Therefore

(Two of the) Andrews–Gordon identities. For $\sigma = 0, 1$,

$$\sum_{\substack{\lambda \\ \lambda_1 \le m}} q^{(\sigma+1)|\lambda|}P_{2\lambda}(1, q, q^2, \ldots; q) = \frac{(q^{2m+3}; q^{2m+3})_\infty}{(q)_\infty}\,\theta\big(q^{(1-\sigma)m+1}; q^{2m+3}\big)$$

We are now ready for our main theorem.

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$\mathrm{A}_{2n}^{(2)}$ Rogers–Ramanujan and Andrews–Gordon identities.

For $\kappa = 2m + 2n + 1$

$$\begin{aligned}
&\sum_{\substack{\lambda \\ \lambda_1 \leq m}} q^{|\lambda|} P_{2\lambda}\big(1, q, q^2, \dots; q^{2n-1}\big) \\
&\qquad = \frac{(q^\kappa; q^\kappa)_\infty^n}{(q)_\infty^n} \prod_{i=1}^{n} \theta\big(q^{i+m}; q^\kappa\big) \prod_{1 \leq i < j \leq n} \theta\big(q^{j-i}, q^{i+j-1}; q^\kappa\big) \\
&\qquad = \frac{(q^\kappa; q^\kappa)_\infty^m}{(q)_\infty^m} \prod_{i=1}^{m} \theta\big(q^{i+1}; q^\kappa\big) \prod_{1 \leq i < j \leq m} \theta\big(q^{j-i}, q^{i+j+1}; q^\kappa\big)
\end{aligned}$$

$\mathrm{A}_{2n}^{(2)}$ RR and AG identities (continued).

For $\kappa = 2m + 2n + 1$

$$\sum_{\substack{\lambda \\ \lambda_1 \le m}} q^{2|\lambda|} P_{2\lambda}\big(1, q, q^2, \dots; q^{2n-1}\big)$$

$$= \frac{(q^\kappa; q^\kappa)_\infty^n}{(q)_\infty^n} \prod_{i=1}^{n} \theta\big(q^i; q^\kappa\big) \prod_{1 \le i < j \le n} \theta\big(q^{j-i}, q^{i+j}; q^\kappa\big)$$

$$= \frac{(q^\kappa; q^\kappa)_\infty^m}{(q)_\infty^m} \prod_{i=1}^{m} \theta\big(q^i; q^\kappa\big) \prod_{1 \le i < j \le m} \theta\big(q^{j-i}, q^{i+j}; q^\kappa\big)$$

The proof of the theorem is long and technical. The key step is to prove that the unspecialised characters

$$\mathrm{e}^{-\Lambda}\mathrm{char}\, V(\Lambda) \quad \text{for} \quad \Lambda = m\Lambda_n \text{ and } \Lambda = 2m\Lambda_0$$

of $\mathrm{A}_{2n}^{(2)}$ can be expressed in terms of the Hall–Littlewood polynomials. Through specialisation the $\mathrm{A}_{2n}^{(2)}$ Rogers–Ramanujan and Andrews–Gordon identities then follow.

Congratulations Dick