Multiple orthogonal polynomials for modified Bessel weights

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Classical orthogonal polynomials

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	w	σ	τ
Hermite	e^{-x^2}	1	$-2x$
Laguerre	$x^\alpha e^{-x}$	x	$-x+\alpha+1$
Jacobi	$(1-x)^\alpha(1+x)^\beta$	$1-x^2$	$-x(\alpha+\beta+2)-\alpha+\beta$

Generalizations

If one replaces the derivative by a difference, q-difference, divided difference (Askey-Wilson) operator then one gets all the orthogonal polynomials in the **Askey table**.

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What about replacing the first derivative in the Pearson equation by a second derivative?

Problem (A.P. Prudnikov, 1992)

Construct the orthogonal polynomials for the weight $\rho_\nu = 2x^{\nu/2}K_\nu(2\sqrt{x})$ on $[0,\infty)$ for $\nu \geq 0$, where K_ν is the **modified Bessel function** *of the second kind, satisfying*

$$x^2y''(x) + xy'(x) - (x^2+\nu^2)y(x) = 0.$$

The problem statement is not natural. It is really a problem for a system of first order differential equations

$$\begin{aligned} (x^{-\nu}\rho_\nu)' &= -x^{-\nu-1}\rho_{\nu+1} \\ \rho_{\nu+1}' &= -\rho_\nu \end{aligned}$$

and involves two weight functions on $[0,\infty)$

$$\rho_\nu(x) = 2x^{\nu/2}K_\nu(2\sqrt{x}), \qquad \rho_{\nu+1}(x) = 2x^{(\nu+1)/2}K_{\nu+1}(2\sqrt{x}).$$

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The natural question is to ask for the **multiple orthogonal polynomials** for the two weights $(x^\alpha\rho_\nu, x^\alpha\rho_{\nu+1})$ on $[0,\infty)$, with $\alpha > -1$ and $\nu \geq 0$.

Multiple orthogonal polynomials of type 2 satisfy

$$\begin{aligned}
\int_0^\infty P_{n,m}(x)x^k x^\alpha \rho_\nu(x)\,dx &= 0, \qquad k = 0,1,\ldots,n-1,\\
\int_0^\infty P_{n,m}(x)x^k x^\alpha \rho_{\nu+1}(x)\,dx &= 0, \qquad k = 0,1,\ldots,m-1,
\end{aligned}$$

where $P_{n,m}$ is a monic polynomial of degree $m+n$.

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where $P_{n,m}$ is a monic polynomial of degree $m+n$.

Type 1 multiple orthogonal polynomials are the pair $(A_{n,m}, B_{n,m})$, with $\deg A_{n,m} = n$ and $\deg B_{n,m} = m$, satisfying

$$\int_0^\infty [A_{n,m}(x)\rho_\nu(x) + B_{n,m}(x)\rho_{\nu+1}(x)]x^\alpha x^k\,dx = 0, \quad k = 0,1,\ldots,n+m.$$

and we often write

$$Q_{n,m}(x) = A_{n,m}(x)\rho_\nu(x) + B_{n,m}(x)\rho_{\nu+1}(x).$$

MOPS for modified Bessel functions K_ν

Theorem (VA + Yakubovich, 2000; Ben Cheikh and Douak, 2000)

The type 1 multiple orthogonal polynomials for the weights $(x^\alpha \rho_\nu, x^\alpha \rho_{\nu+1})$, *where*

$$\rho_\nu(x) = 2x^{\nu/2} K_\nu(2\sqrt{x}), \qquad x \in [0, \infty)$$

are given by

$$x^\alpha Q_{n,n-1}(x) = \frac{d^{2n}}{dx^{2n}}[x^{2n+\alpha}\rho_\nu(x)], \quad x^\alpha Q_{n,n}(x) = \frac{d^{2n+1}}{dx^{2n+1}}[x^{2n+\alpha+1}\rho_\nu(x)],$$

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and the type 2 multiple orthogonal polynomials are

$$\begin{aligned} P_{n,n}(x) &= A_{n,n}(x)B_{n,n-1}(x) - A_{n,n-1}(x)B_{n,n}(x) \\ P_{n+1,n}(x) &= A_{n+1,n}(x)B_{n,n}(x) - A_{n,n}(x)B_{n+1,n}(x) \end{aligned}$$

Properties

Let

$$p_{2n}(x) = P_{n,n}(x), \quad p_{2n+1}(x) = P_{n+1,n}(x),$$

then

$$p_n(x) = (-1)^n(\alpha+1)_n(\alpha+\nu+1)_n \, {}_1F_2\left(\begin{matrix} -n \\ \alpha+1, \alpha+\nu+1 \end{matrix} ; x \right)$$

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Recurrence relation

$$xp_n(x) = p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x) + d_n p_{n-2}(x)$$

with

$$\begin{aligned} b_n &= (n+\alpha+1)(3n+\alpha+2\nu) - (\alpha+1)(\nu-1), \\ c_n &= n(n+\alpha)(n+\alpha+\nu)(3n+2\alpha+\nu), \\ d_n &= n(n-1)(n+\alpha)(n+\alpha-1)(n+\alpha+\nu)(n+\alpha+\nu-1). \end{aligned}$$

Properties

One has

$$\frac{d}{dx} p_n^{\alpha}(x) = n p_{n-1}^{\alpha+1}(x), \qquad \frac{d}{dx} x^{\alpha} q_n^{\alpha}(x) = x^{\alpha-1} q_{n+1}^{\alpha-1}(x).$$

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Differential equation: if $y(x) = p_n(x)$, then

$$x^2 y'''(x) + x(2\alpha+\nu+3) y''(x) + [(\alpha+1)(\alpha+\nu+1) - x] y'(x) + n y(x) = 0.$$

Modified Bessel function of the first kind

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It has no moments, but we can introduce an exponential factor and consider

$$\omega_\nu(x) = x^{\nu/2} I_\nu(2\sqrt{x}) e^{-cx}, \qquad x \in [0,\infty),$$

with $c > 0$ and $\nu \geq 0$. Then

$$\begin{aligned} x\omega_\nu'(x) &= (\nu - cx)\omega_\nu(x) + \omega_{\nu+1}(x) \\ \omega_{\nu+1}'(x) &= \omega_\nu(x) - c\omega_{\nu+1}(x). \end{aligned}$$

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The weight ω_ν is known as the **non-central χ^2 distribution** when 2ν is a positive integer.

MOPS for modified Bessel functions I_ν

Theorem (Douak 1999; Coussement + VA 2003)

$q_{2n}(x) = Q_{n,n}(x),\ q_{2n+1}(x) = Q_{n+1,n}(x),\ \ p_{2n}(x) = P_{n,n}(x),\ p_{2n+1}(x) = P_{n+1,n}(x)$

The type 1 multiple orthogonal polynomials for the weights $(\omega_\nu, \omega_{\nu+1})$ *are given by*

$$q_n(x) = \sum_{k=0}^{n+1} \binom{n+1}{k} (-c)^k x^{(\nu+k)/2} I_{\nu+k}(2\sqrt{x}) e^{-cx},$$

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The type 2 multiple orthogonal polynomials are

$$p_n(x) = \frac{(-1)^n}{c^{2n}} \sum_{k=0}^{n} \binom{n}{k} c^k k! L_k^\nu(cx),$$

where L_k^ν are the Laguerre polynomials.

Properties

The type 2 multiple orthogonal polynomials satisfy the **recurrence relation**

$$xp_n(x) = p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x) + d_n p_{n-2}(x),$$

with

$$\begin{aligned} b_n &= \frac{(\nu+2n+1)c+1}{c^2}, \\ c_n &= \frac{n((\nu+n)c+2)}{c^3}, \\ d_n &= \frac{n(n-1)}{c^4}. \end{aligned}$$

Properties

One has

$$\frac{d}{dx} q_n^{\nu+1}(x) = q_{n+1}^{\nu}(x), \qquad \frac{d}{dx} p_n^{\nu}(x) = n p_{n-1}^{\nu+1}(x).$$

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Differential equation: if $y(x) = p_n(x)$, then

$$xy'''(x)+(-2cx+\nu+2)y''(x)+(c^2x+c(n-\nu-2)-1)y'(x)-c^2ny(x) = 0.$$

Distribution of zeros

For the multiple orthogonal polynomials related to the modified Bessel functions K_ν we have for $n/N \to t > 0$

$$\lim_{n,N\to\infty} \frac{b_n}{N^2} = 3t^2, \quad \lim_{n,N\to\infty} \frac{c_n}{N^4} = 3t^4, \quad \lim_{n,N\to\infty} \frac{d_n}{N^6} = t^6.$$

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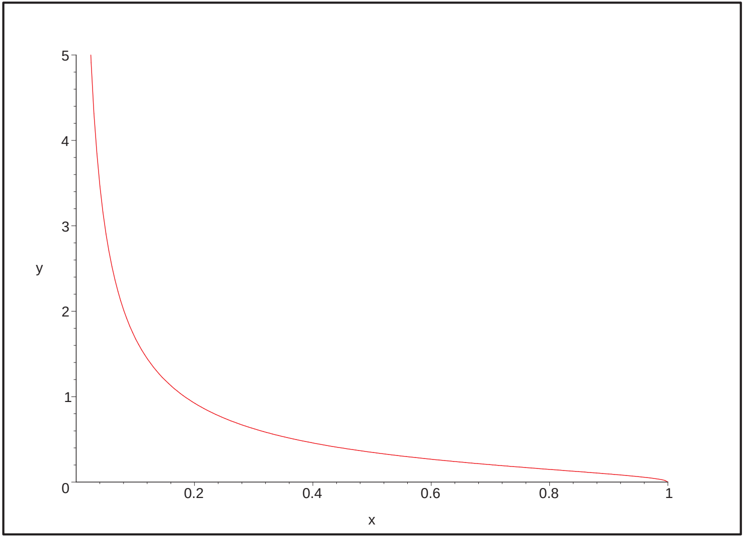
Theorem (E+J Coussement + VA, 2008)

The asymptotic distribution of the (scaled) zeros of p_n is given by

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f\left(\frac{x_{j,n}}{n^2}\right) = \frac{4}{27} \int_0^{27/4} f(x) h(4x/27)\, dx,$$

for every $f \in C([0, \frac{4}{27}])$ and

$$h(y) = \frac{3\sqrt{3}}{4\pi} \frac{(1+\sqrt{1-y})^{1/3} - (1-\sqrt{1-y})^{1/3}}{y^{2/3}}, \qquad y \in [0,1].$$


5
4
3
y
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1
0
0.2
0.4
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1
x

Distribution of zeros

For the multiple orthogonal polynomials related to the modified Bessel functions I_ν we have for $n/N \to t > 0$

$$\lim_{n,N\to\infty} \frac{b_n}{N} = \frac{2t}{c}, \quad \lim_{n,N\to\infty} \frac{c_n}{N^2} = \frac{t^2}{c^2}, \quad \lim_{n,N\to\infty} \frac{d_n}{N^3} = 0.$$

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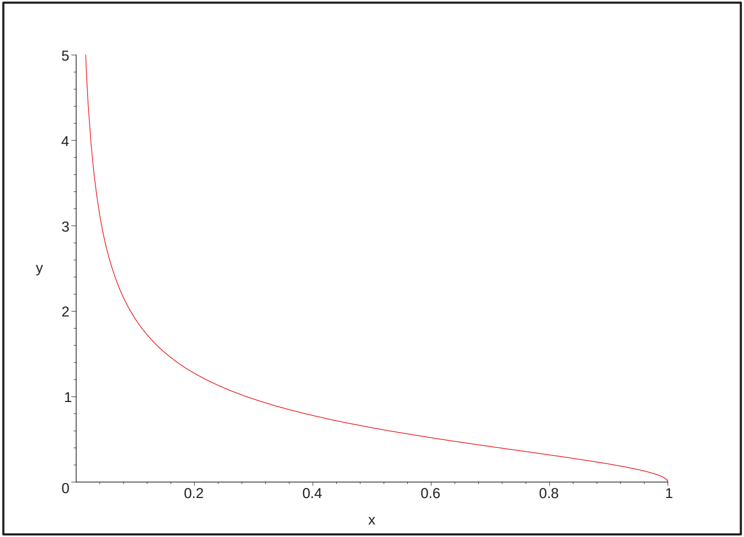
for every $f \in C([0,4/c])$. Furthermore

$$\lim_{n\to\infty} \frac{(-c)^n p_n(x/n)}{n^\nu n!} = e^{1/c} (cx)^{-\nu/2} J_\nu(2\sqrt{cx}).$$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Tools looking for an application

We now have found multiple orthogonal polynomials for two weights involving modified Bessel functions

$$x^{\alpha}\Big(x^{\nu/2}K_{\nu}(2\sqrt{x}), x^{(\nu+1)/2}K_{\nu+1}(2\sqrt{x})\Big)$$

and

$$e^{-cx}\Big(x^{\nu/2}I_{\nu}(2\sqrt{x}), x^{(\nu+1)/2}I_{\nu+1}(2\sqrt{x})\Big).$$

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Are they useful ?

Non-intersecting squared Bessel paths

Let $\{X_1(t), \ldots, X_d(t) : \ t \geq 0\}$ be d independent Brownian motions, then

$$R(t) = \sqrt{X_1^2(t) + X_2^2(t) + \cdots + X_d^2(t)}, \qquad t \geq 0$$

is the **Bessel process**: distance of the d dimensional Brownian motion in $\mathbb{R}^d$ to the origin.

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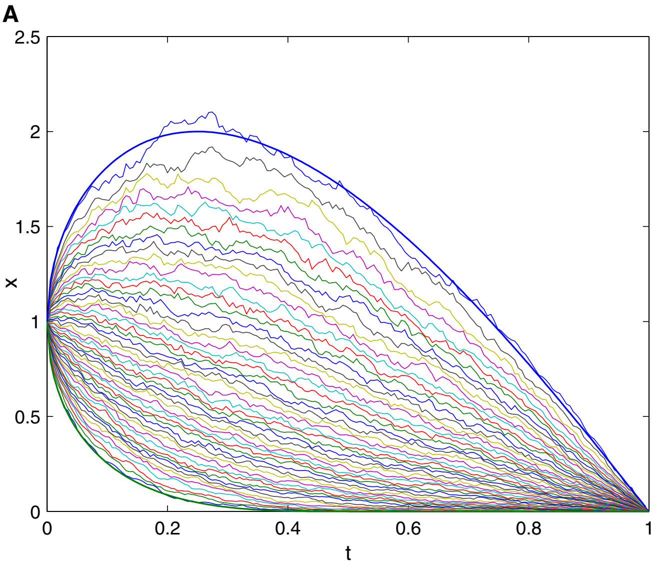
Consider n squared Bessel processes $R_1^2(t), \ldots, R_n^2(t)$ such that

- $R_1^2(0) = R_2^2(0) = \cdots = R_n^2(0) = a > 0$,
- $R_1^2(1) = R_2^2(1) = \cdots = R_n^2(1) = 0$
- The paths do not intersect


A
2.5
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t

Kuijlaars, Martínez-Finkelshtein, Wielonsky, 2009

Let $x_1, x_2, \ldots, x_n$ be the points on the vertical line at time t, then these points form a **determinantal point process**:

$$\mathbb{P}_{n,t}(x_1, x_2, \ldots, x_n) = \frac{1}{Z_{n,t}} \det\big(K_n(x_i, x_j)\big)_{i,j=1,\ldots,n}$$

with

$$K_n(x, y) = \sum_{j=0}^{n-1} p_j(2nax) q_j(2nay),$$

where q_j are the type 1 and p_j are the type 2 multiple orthogonal polynomials for the modified Bessel functions $(I_\nu, I_{\nu+1})$, where $\nu = \frac{d}{2} - 1$.

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$$\mathbb{E}\left(\prod_{i=1}^{n}(z - x_j)\right) = (2an)^{-n} p_n(2anz).$$

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$$\mathbb{E}\left(\prod_{i=1}^{n} (z - x_j)\right) = (2an)^{-n} p_n(2anz).$$

The points $(x_1, \ldots, x_n)$ have for $n \to \infty$ the same asymptotic distribution as the zeros of $p_n(2anx)$

Products of random matrices

Akeman, Kieburg, Wei, J. Phys. A: Math. Theor. **46** (2013).

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Let X_1 and X_2 be independent complex matrices of size n with independent and identically distributed normal random variables $a_{j,k} + ib_{j,k}$, (**Ginibre random matrix**) then the density is, up to a normalization constant

$$\exp[-\mathrm{Tr}\,(X_j^* X_j)], \qquad j = 1, 2$$

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We are interested in the squared singular values of the product $X_2 X_1$, i.e, the eigenvalues of $X_1^* X_2^* X_2 X_1$.

Products of random matrices

Theorem (Lun Zhang, 2013)

The squared singular values of X_2X_1 form a determinantal process with kernel

$$K_n(x,y) = \sum_{j=0}^{n-1} p_j(n^2x)q_j(n^2x)$$

where q_j are the type 1 and p_j the type 2 multiple orthogonal polynomials for the weights $(K_0(2\sqrt{x}), K_1(2\sqrt{x}))$.

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where q_j are the type 1 and p_j the type 2 multiple orthogonal polynomials for the weights $(K_0(2\sqrt{x}), K_1(2\sqrt{x}))$. Furthermore

$$\mathbb{E}\Big(z - X_1^*X_2^*X_2X_1\Big) = n^{-2n}p_n(n^2z)$$

and the squared singular values are asymptotically distributed as the zeros of $p_n(n^2z)$.

Longer products: Kuijlaars and Zhang, 2013

The squared singular values of $X_M X_{M-1} \cdots X_1$ also form a determinantal process with multiple orthogonal polynomials, but now for the M weights $(w_0(x), w_1(x), \ldots, w_{M-1})$, where

$$w_k(x) = G^{M,0}_{0,M}\left(\begin{array}{c} - \\ 0, 0, \ldots, 0, k \end{array} \middle| x \right),$$

with $G^{M,0}_{0,M}$ a **Meijer G-function**

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$$G^{m,n}_{p,q}\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \middle| x \right)$$
$$= \frac{1}{2\pi i} \int_\gamma \frac{\prod_{j=1}^m \Gamma(b_j + u) \prod_{j=1}^n \Gamma(1 - a_j - u)}{\prod_{j=m+1}^q \Gamma(1 - b_j - u) \prod_{j=n+1}^p \Gamma(a_j + u)} \, z^{-u} \, du.$$

The squared singular values of $X_M X_{M-1} \cdots X_1$ also form a determinantal process with multiple orthogonal polynomials, but now for the M weights $(w_0(x), w_1(x), \ldots, w_{M-1})$, where

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(see, e.g., R. Beals and J. Szmigielski, *Meijer G-functions: a gentle introduction*, Notices AMS **60** (2013), 866–872.)

Products of rectangular matrices

Let $Y_M = X_M X_{M-1} \cdots X_1$, where X_j is a $N_j \times N_{j-1}$ random matrix with independent (complex) Gaussian entries and suppose

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Then the squared singular values of Y_M form a determinantal process with multiple orthogonal polynomials for the M weights $(w_0, w_1, \ldots, w_{M-1})$, where

$$w_k(x) = G^{M,0}_{0,M}\left(\begin{array}{c} - \\ \nu_M, \nu_{M-1}, \ldots, \nu_2, \nu_1 + k \end{array} \middle| x \right).$$

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If $M = 2$ then

$$w_0(x) = 2x^{(\nu_1+\nu_2)/2} K_{\nu_1-\nu_2}(2\sqrt{x}), \quad w_1(x) = 2x^{(\nu_1+\nu_2+1)/2} K_{\nu_1-\nu_2+1}(2\sqrt{x})$$

multiple orthogonal polynomials with respect to **modified Bessel functions of the second kind**.

References

W. Van Assche, S.B. Yakubovich, *Multiple orthogonal polynomials associated with Macdonald functions*, Integral Transform and Special Functions **9**, nr. 3 (2000), 229–244.

E. Coussement, W. Van Assche, *Multiple orthogonal polynomials associated with the modified Bessel functions of the first kind*, Constructive Approximation **19** (2003), 237–263.

E. Coussement, J. Coussement, W. Van Assche, *Asymptotic zero distribution for a class of multiple orthogonal polynomials*, Trans. Amer. Math. Soc. **360**, nr. 10 (2008), 5571–5588.

A.B.J. Kuijlaars, A. Martínez-Finkelshtein, F. Wielonsky, *Non-intersecting squared Bessel paths and multiple orthogonal polynomials for modified Bessel weights*, Commun. Math. Phys. **286** (2009), 217–275.

Lun Zhang, *A note on the limiting mean distribution of singular values for products of two Wishart random matrices*, arXiv:1305.0726v2 [math-ph] (May 2013)

A.B.J. Kuijlaars, Lun Zhang, *Singular values of products of Ginibre random variables, multiple orthogonal polynomials and hard edge scalings*, arXiv:1308.1003v1 [math-ph] (August 2013)