

Contractions of 2D 2nd order quantum superintegrable systems and the Askey scheme for hypergeometric orthogonal polynomials

Willard Miller, [Joint with E.G. Kalnins (Waikato), Sarah Post (Hawaii), Eyal Subag (Technion) and Robin Heinonen (Minnesota)]

University of Minnesota

Abstract

We show explicitly that all 2nd order superintegrable systems in 2 dimensions are limiting cases of a single system: the generic 3-parameter potential on the 2-sphere, S_9 in our listing.

Analogously we show that all of the quadratic symmetry algebras of these systems are contractions of that of S_9 .

By contracting function space realizations of irreducible representations of the S_9 algebra (which give the structure equations for Racah/Wilson polynomials) to the other superintegrable systems we obtain the full Askey scheme of orthogonal hypergeometric polynomials.

Amazingly, all of these contractions of superintegrable systems with potential are uniquely induced by Wigner Lie algebra contractions of $so(3, \mathbb{C})$ and $e(2, \mathbb{C})$.

Outline

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- 2 Lie algebra contractions
- 3 Representatives of Nondegenerate Systems
- 4 Representatives of Degenerate Systems
- 5 Contractions of Superintegrable Systems
- 6 Models of Superintegrable Systems
- 7 The S_9 Difference Operator Model
- 8 Derivation of the Contraction Scheme
- 9 Discussion and Conclusions

Superintegrable Systems: $H\Psi = E\Psi$

- A quantum superintegrable system is an integrable Hamiltonian system on an n -dimensional Riemannian/pseudo-Riemannian manifold with potential:

$$H = \Delta_n + V$$

that admits $2n - 1$ algebraically independent partial differential operators commuting with H , the maximum possible.

$$[H, L_j] = 0, \quad L_{2n-1} = H, \quad n = 1, 2, \dots, 2n - 1.$$

- Superintegrability captures the properties of quantum Hamiltonian systems that allow the Schrödinger eigenvalue problem $H\Psi = E\Psi$ to be solved exactly, analytically and algebraically.
- A system is of order K if the maximum order of the symmetry operators, other than H , is K . For $n = 2$, $K = 1, 2$ all systems are known.

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1st order systems

These are the (zero-potential) Laplace-Beltrami eigenvalue equations on constant curvature spaces.

Simplest examples: **Euclidean Helmholtz equation** $(P_1^2 + P_2^2)\phi = -\lambda^2\phi$ (or the **Klein-Gordon equation** $(P_1^2 - P_2^2)\phi = -\lambda^2\phi$), and **Laplace -Beltrami eigenvalue equation on the 2-sphere** $(J_1^2 + J_2^2 + J_3^2)\psi = -j(j+1)\psi$.

Here the symmetry algebras close under commutation to form the Lie algebras $e(2, \mathbb{R})$, $e(1, 1)$ or $o(3, \mathbb{R})$.

One can find 2-variable differential operator models of the irreducible representations of these Lie algebras in which the basis eigenfunctions are the spherical harmonics ($o(3, \mathbb{R})$) or Bessel functions ($e(2, \mathbb{R})$).

Inönü and Wigner (1953)

It was exactly these systems which motivated the pioneering work of Inönü and Wigner on Lie algebra contractions.

While, that paper introduced Lie algebra contractions in general, the motivation and virtually all the examples were of symmetry groups of these systems.

It was shown that $o(3, \mathbb{R})$ contracts to $e(2, \mathbb{R})$. In the physical space this is accomplished by letting the radius of the sphere go to infinity, so that the surface flattens out. Under this limit the Laplace-Beltrami eigenvalue equation goes to the Helmholtz equation.

Lie algebra contractions

Let $(A; [\cdot; \cdot]_A)$, $(B; [\cdot; \cdot]_B)$ be two complex Lie algebras. We say B is a *contraction* of A if for every $\epsilon \in (0; 1]$ there exists a linear invertible map $t_\epsilon : B \rightarrow A$ such that for every $X, Y \in B$,

$$\lim_{\epsilon \rightarrow 0} t_\epsilon^{-1} [t_\epsilon X, t_\epsilon Y]_A = [X, Y]_B.$$

Thus, as $\epsilon \rightarrow 0$ the 1-parameter family of basis transformations can become nonsingular but the structure constants go to a finite limit.

Features of Wigner contractions (Wigner-Talman)

- ‘Saving’ a representation.
- Simple models of irreducible representations.
- Limit relations between special functions.
- Expansion coefficients relating different special function bases.

Contractions of $e(2, \mathbb{C})$ and $o(3, \mathbb{C})$

The contractions of these Lie algebras have long since been classified. There are 6 nontrivial contractions of $e(2, \mathbb{C})$ and 4 of $o(3, \mathbb{C})$.

Example: A Wigner-Inönü contraction of $o(3, \mathbb{C})$. We use the classical realization for $o(3, \mathbb{C})$ acting on the 2-sphere, with basis

$J_1 = s_2 p_3 - s_3 p_2$, $J_2 = s_3 p_1 - s_1 p_3$, $J_3 = s_1 p_2 - s_2 p_1$, commutation relations

$$[J_2, J_1] = J_3, \quad [J_3, J_2] = J_1, \quad [J_1, J_3] = J_2,$$

and Hamiltonian $H = J_1^2 + J_2^2 + J_3^2$. Here $s_1^2 + s_2^2 + s_3^2 = 1$.

Basis change : $\{J'_1, J'_2, J'_3\} = \{\epsilon J_1, \epsilon J_2, J_3\}$, $0 < \epsilon \leq 1$

coordinate implementation $x = \frac{s_1}{\epsilon}$, $y = \frac{s_2}{\epsilon}$, $s_3 \approx 1$, $J = J_3$

New structure relations : $[J'_2, J'_1] = \epsilon^2 J'_3$, $[J'_3, J'_2] = J'_1$, $[J'_1, J'_3] = J'_2$,

Let $\epsilon \rightarrow 0$: $[J'_2, J'_1] = 0$, $[J'_3, J'_2] = J'_1$, $[J'_1, J'_3] = J'_2$, get $e(2, \mathbb{C})$

Wigner-Inonu contractions of $e(2, \mathbf{C})$:

$$\textcircled{1} \quad \{\mathcal{J}', p'_1, p'_2\} = \{\mathcal{J}, \epsilon p_1, \epsilon p_2\} : e(2, \mathbf{C}),$$

coordinate implementation $x' = \frac{x}{\epsilon}, y' = \frac{y}{\epsilon},$

$$\textcircled{2} \quad \{\mathcal{J}', p'_1, p'_2\} = \{\epsilon \mathcal{J}, p_1, \epsilon p_2\} : \text{Heisenberg algebra},$$

coordinate implementation $x' = x, y' = \frac{y}{\epsilon}, \mathcal{J}' = x' p'_2,$

$$\textcircled{3} \quad \{\mathcal{J}', p'_1 + ip'_2, p'_1 - ip'_2\} = \{\epsilon \mathcal{J}, \epsilon(p_1 + ip_2), p_1 - ip_2\} : \text{abelian algebra},$$

$$\textcircled{4} \quad \{\mathcal{J}', p'_1, p'_2\} = \{\epsilon \mathcal{J}, p_1, p_2\} : \text{abelian algebra},$$

$$\textcircled{5} \quad \{\mathcal{J}', p'_1 + ip'_2, p'_1 - ip'_2\} = \{\mathcal{J}, \epsilon(p_1 + ip_2), p_1 - ip_2\} : e(2, \mathbf{C}),$$

coordinate implementation $x' + iy' = x + iy, x' - iy' = \frac{x - iy}{\epsilon},$

The other natural contractions of $e(2, \mathbf{C})$:

$$\textcircled{6} \quad \{\mathcal{J}', p'_1, p'_2\} = \left\{ \mathcal{J} + \frac{p_1}{\epsilon}, p_1, p_2 \right\} : e(2, \mathbf{C}),$$

coordinate implementation $x' = x, y' = y - \frac{1}{\epsilon}$,

$$\textcircled{7} \quad \{\mathcal{J}', p'_1, p'_2\} = \left\{ \mathcal{J} + \frac{p_1 + ip_2}{\epsilon}, p_1, p_2 \right\} : e(2, \mathbf{C}),$$

coordinate implementation $x' = x + \frac{i}{\epsilon}, y' = y - \frac{1}{\epsilon}$.

Wigner-Inonu contractions of $\mathfrak{o}(3, \mathbf{C})$:

1

$$\{\mathcal{J}'_1, \mathcal{J}'_2, \mathcal{J}'_3\} = \{\epsilon \mathcal{J}_1, \epsilon \mathcal{J}_2, \mathcal{J}_3\} : \mathfrak{e}(2, \mathbf{C}),$$

coordinate implementation $x = \mathbf{s}_1/\epsilon, y = \mathbf{s}_2/\epsilon, \mathbf{s}_3 \approx 1, \mathcal{J} = \mathcal{J}_3,$

2

$$\{\mathcal{J}'_1 + i\mathcal{J}'_2, \mathcal{J}'_1 - i\mathcal{J}'_2, \mathcal{J}'_3\} = \{\mathcal{J}_1 + i\mathcal{J}_2, \epsilon(\mathcal{J}_1 - i\mathcal{J}_2), \epsilon \mathcal{J}_3\} : \text{Heisenberg},$$

coordinate implementation $\mathbf{s}_1 = \frac{\cos \phi}{\cosh \psi}, \mathbf{s}_2 = \frac{\sin \phi}{\cosh \psi}, \mathbf{s}_3 = \frac{\sinh \psi}{\cosh \psi}.$

3

$$\{\mathcal{J}'_1 + i\mathcal{J}'_2, \mathcal{J}'_1 - i\mathcal{J}'_2, \mathcal{J}'_3\} = \{\mathcal{J}_1 + i\mathcal{J}_2, \epsilon(\mathcal{J}_1 - i\mathcal{J}_2), \mathcal{J}_3\} : \mathfrak{e}(2, \mathbf{C}).$$

coordinate implementation $\mathbf{s}_1 + i\mathbf{s}_2 = \epsilon z, \mathbf{s}_1 - i\mathbf{s}_2 = \bar{z}, \mathbf{s}_3 \approx 1,$

The other natural contraction of $\mathfrak{o}(3, \mathbf{C})$:

5

$$\{\mathcal{J}'_1 + i\mathcal{J}'_2, \mathcal{J}'_1 - i\mathcal{J}'_2, \mathcal{J}'_3\} = \{\epsilon(\mathcal{J}_1 + i\mathcal{J}_2), \frac{\mathcal{J}_1 - i\mathcal{J}_2}{\epsilon}, \mathcal{J}_3\} : \mathfrak{o}(3, \mathbf{C}),$$

coordinate implementation

$$\mathbf{s}'_1 = \frac{\epsilon + \epsilon^{-1}}{2} \mathbf{s}_1 + i \frac{\epsilon - \epsilon^{-1}}{2} \mathbf{s}_2, \quad \mathbf{s}'_2 = -i \frac{\epsilon - \epsilon^{-1}}{2} \mathbf{s}_1 + \frac{\epsilon + \epsilon^{-1}}{2} \mathbf{s}_2, \quad \mathbf{s}'_3 = \mathbf{s}_3.$$

Free 2nd order superintegrable systems, (no potential, $K = 2$)

We apply these ideas to 2nd order systems in 2D ($2n - 1 = 3$). The complex spaces with Laplace-Beltrami operators admitting at least three 2nd order symmetries were classified by Koenigs (1896). They are:

- **The two constant curvature spaces**, 6 linearly independent 2nd order symmetries and 3 1st order symmetries,
- **The four Darboux spaces**, 4 2nd order symmetries and 1 1st order symmetry,

$$ds^2 = 4x(dx^2 + dy^2), \quad ds^2 = \frac{x^2 + 1}{x^2}(dx^2 + dy^2),$$

$$ds^2 = \frac{e^x + 1}{e^{2x}}(dx^2 + dy^2), \quad ds^2 = \frac{2 \cos 2x + b}{\sin^2 2x}(dx^2 + dy^2),$$

- **Eleven 4-parameter Koenigs spaces**. Example

$$ds^2 = \left(\frac{c_1}{x^2 + y^2} + \frac{c_2}{x^2} + \frac{c_3}{y^2} + c_4 \right) (dx^2 + dy^2).$$

2nd order systems with potential, $K = 2$

- The symmetry operators of each system close under commutation to generate a quadratic algebra, and the irreducible representations of this algebra determine the eigenvalues of H and their multiplicity
- All the 2nd order superintegrable systems are limiting cases of a single system: the generic 3-parameter potential on the 2-sphere, **S9** in our listing. Analogously all quadratic symmetry algebras of these systems are contractions of **S9**.

$$\mathbf{S9} : \quad H = \Delta_2 + \frac{a_1}{s_1^2} + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2}, \quad s_1^2 + s_2^2 + s_3^2 = 1,$$

$$L_1 = (s_2 \partial_{s_3} - s_3 \partial_{s_2})^2 + \frac{a_3 s_2^2}{s_3^2} + \frac{a_2 s_3^2}{s_2^2}, \quad L_2, \quad L_3,$$

$$H = L_1 + L_2 + L_3 + a_1 + a_2 + a_3$$

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$$\mathbf{S9} : \quad H = \Delta_2 + \frac{a_1}{s_1^2} + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2}, \quad s_1^2 + s_2^2 + s_3^2 = 1,$$

$$L_1 = (s_2 \partial_{s_3} - s_3 \partial_{s_2})^2 + \frac{a_3 s_2^2}{s_3^2} + \frac{a_2 s_3^2}{s_2^2}, \quad L_2, \quad L_3,$$

$$H = L_1 + L_2 + L_3 + a_1 + a_2 + a_3$$

3 types of 2nd order superintegrable systems:

1 Nondegenerate:

$$V(\mathbf{x}) = a_1 V_{(1)}(\mathbf{x}) + a_2 V_{(2)}(\mathbf{x}) + a_3 V_{(3)}(\mathbf{x}) + a_4$$

2 Degenerate:

$$V(\mathbf{x}) = a_1 V_{(1)}(\mathbf{x}) + a_2$$

3 Free:

$$V = a_1, \text{ no potential}$$

Nondegenerate systems ($2n - 1 = 3$ generators)

The symmetry algebra generated by H, L_1, L_2 always closes under commutation. Define 3rd order commutator R by $R = [L_1, L_2]$. Then

$$[L_j, R] = A_1^{(j)} L_1^2 + A_2^{(j)} L_2^2 + A_3^{(j)} H^2 + A_4^{(j)} \{L_1, L_2\} + A_5^{(j)} HL_1 + A_6^{(j)} HL_2$$

$$+ A_7^{(j)} L_1 + A_8^{(j)} L_2 + A_9^{(j)} H + A_{10}^{(j)}, \quad \{L_1, L_2\} = L_1 L_2 + L_2 L_1,$$

$$R^2 = b_1 L_1^3 + b_2 L_2^3 + b_3 H^3 + b_4 \{L_1^2, L_2\} + b_5 \{L_1, L_2^2\} + b_6 L_1 L_2 L_1 + b_7 L_2 L_1 L_2$$

$$+ b_8 H \{L_1, L_2\} + b_9 HL_1^2 + b_{10} HL_2^2 + b_{11} H^2 L_1 + b_{12} H^2 L_2 + b_{13} L_1^2 + b_{14} L_2^2 + b_{15} \{L_1, L_2\}$$

$$+ b_{16} HL_1 + b_{17} HL_2 + b_{18} H^2 + b_{19} L_1 + b_{20} L_2 + b_{21} H + b_{22},$$

This structure is an example of a **quadratic algebra**.

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$$+ A_7^{(j)} L_1 + A_8^{(j)} L_2 + A_9^{(j)} H + A_{10}^{(j)}, \quad \{L_1, L_2\} = L_1 L_2 + L_2 L_1,$$

$$R^2 = b_1 L_1^3 + b_2 L_2^3 + b_3 H^3 + b_4 \{L_1^2, L_2\} + b_5 \{L_1, L_2^2\} + b_6 L_1 L_2 L_1 + b_7 L_2 L_1 L_2$$

$$+ b_8 H \{L_1, L_2\} + b_9 HL_1^2 + b_{10} HL_2^2 + b_{11} H^2 L_1 + b_{12} H^2 L_2 + b_{13} L_1^2 + b_{14} L_2^2 + b_{15} \{L_1, L_2\}$$

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Degenerate systems ($2n - 1 = 3$)

There are 4 generators: one 1st order X and 3 second order H, L_1, L_2 .

$$[X, L_j] = C_1^{(j)}L_1 + C_2^{(j)}L_2 + C_3^{(j)}H + C_4^{(j)}X^2 + C_5^{(j)}, \quad j = 1, 2,$$

$$[L_1, L_2] = E_1\{L_1, X\} + E_2\{L_2, X\} + E_3HX + E_4X^3 + E_5X,$$

Since $2n - 1 = 3$ there must be an identity satisfied by the 4 generators. It is of 4th order:

$$c_1L_1^2 + c_2L_2^2 + c_3H^2 + c_4\{L_1, L_2\} + c_5HL_1 + c_6HL_2 + c_7X^4 + c_8\{X^2, L_1\} + c_9\{X^2, L_2\}$$

$$+ c_{10}HX^2 + c_{11}XL_1X + c_{12}XL_2X + c_{13}L_1 + c_{14}L_2 + c_{15}H + c_{16}X^2 + c_{17} = 0$$

Stäckel Equivalence Classes

All such systems are known. There are 59 types, on a variety of manifolds, but under **the Stäckel transform, an invertible structure preserving mapping**, they divide into 12 equivalence classes with representatives on flat space and the 2-sphere, 6 with nondegenerate 3-parameter potentials

$$\{S9, E1, E2, E3', E8, E10\}$$

and 6 with degenerate 1-parameter potentials

$$\{S3, E3, E4, E5, E6, E14\}$$

S9

$$H = J_1^2 + J_2^2 + J_3^2 + \frac{a_1}{s_1^2} + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2}$$

where $J_3 = s_1 \partial_{s_2} - s_2 \partial_{s_1}$ and J_2, J_3 are obtained by cyclic permutations of indices.

Basis symmetries: ($J_3 = s_2 \partial_{s_1} - s_1 \partial_{s_2}, \dots$)

$$L_1 = J_1^2 + \frac{a_3 s_2^2}{s_3^2} + \frac{a_2 s_3^2}{s_2^2}, \quad L_2 = J_2^2 + \frac{a_1 s_3^2}{s_1^2} + \frac{a_3 s_1^2}{s_3^2}, \quad L_3 = J_3^2 + \frac{a_2 s_1^2}{s_2^2} + \frac{a_1 s_2^2}{s_1^2},$$

Structure equations:

$$[L_i, R] = 4\{L_i, L_k\} - 4\{L_i, L_j\} - (8 + 16a_j)L_j + (8 + 16a_k)L_k + 8(a_j - a_k),$$

$$R^2 = \frac{8}{3}\{L_1, L_2, L_3\} - (16a_1 + 12)L_1^2 - (16a_2 + 12)L_2^2 - (16a_3 + 12)L_3^2 + \frac{52}{3}(\{L_1, L_2\} + \{L_2, L_3\} + \{L_3, L_1\}) + \frac{1}{3}(16 + 176a_1)L_1 + \frac{1}{3}(16 + 176a_2)L_2 + \frac{1}{3}(16 + 176a_3)L_3 + \frac{32}{3}(a_1 + a_2 + a_3) + 48(a_1 a_2 + a_2 a_3 + a_3 a_1) + 64a_1 a_2 a_3, \quad R = [L_1, L_2].$$

Interesting 2 variable O.P's: Prorial-Karlin-MacGregor, (right triangle)

$$H = J_1^2 + J_2^2 + J_3^2 + \frac{a_1}{s_1^2} + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2}$$

where $J_3 = s_1 \partial_{s_2} - s_2 \partial_{s_1}$ and J_2, J_3 are obtained by cyclic permutations of indices.

Basis symmetries: ($J_3 = s_2 \partial_{s_1} - s_1 \partial_{s_2}, \dots$)

$$L_1 = J_1^2 + \frac{a_3 s_2^2}{s_3^2} + \frac{a_2 s_3^2}{s_2^2}, \quad L_2 = J_2^2 + \frac{a_1 s_3^2}{s_1^2} + \frac{a_3 s_1^2}{s_3^2}, \quad L_3 = J_3^2 + \frac{a_2 s_1^2}{s_2^2} + \frac{a_1 s_2^2}{s_1^2},$$

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Interesting 2 variable O.P.'s: Prorial-Karlin-MacGregor, (right triangle)

E1

$$H = \partial_x^2 + \partial_y^2 - \omega^2(x^2 + y^2) + \frac{b_1}{x^2} + \frac{b_2}{y^2}$$

Generators:

$$L_1 = \partial_x^2 - \omega^2 x^2 + \frac{b_1}{x^2}, \quad L_2 = \partial_y^2 - \omega^2 y^2 + \frac{b_2}{y^2}, \quad L_3 = (x\partial_y - y\partial_x)^2 + y^2 \frac{b_1}{x^2} + x^2 \frac{b_2}{y^2}$$

Structure relations:

$$[R, L_1] = 8L_1^2 - 8HL_1 - 16\omega^2 L_3 + 8\omega^2,$$

$$[R, L_3] = 8HL_3 - 8\{L_1, L_3\} + (16b_1 + 8)H - 16(b_1 + b_2 + 1)L_1,$$

$$R^2 + \frac{8}{3}\{L_1, L_1, L_3\} - 8H\{L_1, L_3\} + (16b_1 + 16b_2 + \frac{176}{3})L_1^2 - 16\omega^2 L_3^2 - (32b_1 + \frac{176}{3})HL_1 \\ + (16b_1 + 12)H^2 + \frac{176}{3}\omega^2 L_3 + 16\omega^2(3b_1 + 3b_2 + 4b_1 b_2 + \frac{2}{3}) = 0$$

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Generators:

$$L_1 = \partial_x^2 - \omega^2 x^2 + \frac{b_1}{x^2}, \quad L_2 = \partial_y^2 - \omega^2 y^2 + \frac{b_2}{y^2}, \quad L_3 = (x\partial_y - y\partial_x)^2 + y^2 \frac{b_1}{x^2} + x^2 \frac{b_2}{y^2}$$

Structure relations:

$$[R, L_1] = 8L_1^2 - 8HL_1 - 16\omega^2 L_3 + 8\omega^2,$$

$$[R, L_3] = 8HL_3 - 8\{L_1, L_3\} + (16b_1 + 8)H - 16(b_1 + b_2 + 1)L_1,$$

$$R^2 + \frac{8}{3}\{L_1, L_1, L_3\} - 8H\{L_1, L_3\} + (16b_1 + 16b_2 + \frac{176}{3})L_1^2 - 16\omega^2 L_3^2 - (32b_1 + \frac{176}{3})HL_1 \\ + (16b_1 + 12)H^2 + \frac{176}{3}\omega^2 L_3 + 16\omega^2(3b_1 + 3b_2 + 4b_1 b_2 + \frac{2}{3}) = 0$$

E2

$$H = \partial_x^2 + \partial_y^2 - \omega^2(4x^2 + y^2) + bx + \frac{c}{y^2}$$

Generators:

$$L_1 = \partial_x^2 - 4\omega^2 x^2 + bx, \quad L_2 = \partial_y^2 - \omega^2 y^2 + \frac{c}{y^2}, \quad L_3 = \frac{1}{2}\{(x\partial_y - y\partial_x), \partial_y\} + y^2(\omega^2 x - \frac{b}{4}) + \frac{cx}{y^2}$$

Structure equations:

$$[L_1, R] + 2bL_2 - 16\omega^2 L_3 = 0, \quad [L_3, R] + 2L_2^2 - 4L_1 L_2 + 2bL_3 + \omega^2(8c + 6) = 0,$$

$$R^2 = 4L_1 L_2^2 + 16\omega^2 L_3^2 - 2b\{L_2, L_3\} + (12 + 16c)\omega^2 L_1 - 32\omega^2 L_2 - b^2(c + \frac{3}{4})$$

Here, the algebra generators are $H, L_1, L_3, R = [L_1, L_3]$

E3'

$$H = \partial_x^2 + \partial_y^2 - \omega^2(x^2 + y^2) + c_1x + c_2y = L_1 + L_2$$

Generators:

$$L_1 = \partial_x^2 - \omega^2x^2 + c_1x, \quad L_2 = \partial_y^2 - \omega^2y^2 + c_2y, \quad L_3 = \partial_{xy} - \omega^2xy + \frac{c_2x + c_1y}{2}$$

Structure relations:

$$[L_1, R] = 4\omega^2L_3 - c_1c_2, \quad [L_3, R] = -2\omega^2L_1 + 2\omega^2L_2 + \frac{1}{2}(c_1^2 - c_2^2),$$

$$R^2 = 4\omega^2(L_3^2 - L_1L_2) - 2c_1c_2L_3 + c_2^2L_1 + c_1^2L_2 + 4\omega^4$$

The algebra generators are $H, L_1, L_3, R = [L_1, L_3]$.

E10

$$H = \partial_x^2 + \partial_y^2 + \alpha\bar{z} + \beta\left(z - \frac{3}{2}\bar{z}^2\right) + \gamma\left(z\bar{z} - \frac{1}{2}\bar{z}^3\right)$$

Generators:

$$L_1 = (\partial_x - i\partial_y)^2 + \gamma\bar{z}^2 + 2\beta\bar{z},$$

$$L_2 = 2i\{x\partial_y - y\partial_x, \partial_x - i\partial_y\} + (\partial_x + i\partial_y)^2 - 4\beta z\bar{z} - \gamma z\bar{z}^2 - 2\beta\bar{z}^3 - \frac{3}{4}\gamma\bar{z}^4 + \gamma z^2 + \alpha\bar{z}^2 + 2\alpha z$$

Structure equations:

$$[R, L_1] + 32\gamma L_1 + 32\beta^2 = 0, \quad [R, L_2] - 96L_1^2 - 64\beta H + 128\alpha L_1 - 32\gamma L_2 - 32\alpha^2,$$

$$R^2 = 64L_1^3 - 64\gamma H^2 - 128\alpha L_1^2 + 128\beta HL_1 + 32\gamma\{L_1, L_2\} - 128\alpha\beta H + 64\alpha^2 L_1 + 64\beta^2 L_2 - 256\gamma^2$$

$$\text{Here } R = [L_1, L_2], \quad z = x + iy, \quad \bar{z} = x - iy,$$

E8

$$H = \partial_x^2 + \partial_y^2 + \frac{c_1 z}{\bar{z}^3} + \frac{c_2}{\bar{z}^2} + c_3 z \bar{z}$$

Generators:

$$L_1 = (\partial_x - i\partial_y)^2 - \frac{c_1}{\bar{z}^2} + c_3 \bar{z}^2, \quad L_2 = (x\partial_y - y\partial_x)^2 + c_1 \frac{z^2}{\bar{z}^2} + c_2 \frac{z}{\bar{z}}$$

Structure relations:

$$[R, L_1] = 8L_1^2 + 32c_1 c_3, \quad [R, L_2] = -8\{L_1, L_2\} + 8c_2 H - 16L_1,$$

$$R^2 = -\frac{16}{3}\{L_1^2, L_2\} - \frac{16}{3}L_1 L_2 L_1 - \frac{176}{3}L_1^2 + 16c_1 H^2 + 16c_2 L_1 H - 64c_1 c_3 L_2 + 16c_3 \left(\frac{4}{3}c_1 - c_2^2\right).$$

$$\text{Here, } R = [L_1, L_2], \quad z = x + iy, \quad \bar{z} = x - iy,$$

Relations between nondegenerate/degenerate systems

- Every 1-parameter potential can be obtained from some 3-parameter potential by parameter restriction.
- It is not simply a restriction, however, because the structure of the symmetry algebra changes.
- A formally skew-adjoint 1st order symmetry appears and this induces a new 2nd order symmetry.
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S3 (Higgs oscillator)

$$H = J_1^2 + J_2^2 + J_3^2 + \frac{a}{s_3^2}$$

The system is the same as S9 with $a_1 = a_2 = 0$, $a_3 = a$ with the former L_2 replaced by

$$L_2 = \frac{1}{2}(J_1 J_2 + J_2 J_1) - \frac{a s_1 s_2}{s_3^2}$$

and

$$X = J_3 = s_2 \partial_{s_3} - s_3 \partial_{s_2}.$$

Structure relations:

$$[L_1, X] = 2L_2, [L_2, X] = -X^2 - 2L_1 + H - a, [L_1, L_2] = -(L_1 X + X L_1) - \left(\frac{1}{2} + 2a\right)X,$$

$$\frac{1}{3} \left(X^2 L_1 + X L_1 X + L_1 X^2 \right) + L_1^2 + L_2^2 - H L_1 + \left(a + \frac{11}{12} \right) X^2 - \frac{1}{6} H + \left(a - \frac{2}{3} \right) L_1 - \frac{5a}{6} = 0.$$

Interesting 2-variable O.P.'s: Koschmieder, Zerneke, (disk)

E3 (Harmonic oscillator)

$$H = \partial_x^2 + \partial_y^2 - \omega^2(x^2 + y^2)$$

Basis symmetries:

$$L_1 = \partial_x^2 - \omega^2 x^2, \quad L_3 = \partial_{xy} - \omega^2 xy, \quad X = x\partial_y - y\partial_x.$$

Also we set $L_2 = \partial_y^2 - \omega^2 y^2 = H - L_1.$

Structure equations:

$$[L_1, X] = 2L_3, \quad [L_3, X] = H - 2L_1, \quad [L_1, L_3] = 2\omega^2 X,$$

$$L_1^2 + L_3^2 - L_1 H - \omega^2 X^2 + \omega^2 = 0$$

E4

$$H = \partial_x^2 + \partial_y^2 + a(x + iy)$$

Basis Symmetries: (with $M = x\partial_y - y\partial_x$)

$$L_1 = \partial_x^2 + ax, \quad L_2 = \frac{i}{2}\{M, X\} - \frac{a}{4}(x + iy)^2, \quad X = \partial_x + i\partial_y$$

Structure equations:

$$[L_1, X] = a, \quad [L_2, X] = X^2, \quad [L_1, L_2] = X^3 + HX - \{L_1, X\},$$

$$X^4 - 2\{L_1, X^2\} + 2HX^2 + H^2 + 4aL_2 = 0$$

E5

$$H = \partial_x^2 + \partial_y^2 + ax$$

Basis symmetries: (where $M = x\partial_y - y\partial_x$)

$$L_1 = \partial_{xy} + \frac{1}{2}ay, \quad L_2 = \frac{1}{2}\{M, X\} - \frac{1}{4}ay^2, \quad X = \partial_y$$

Structure equations:

$$[L_1, L_2] = 2X^3 - HX, \quad [L_1, X] = -\frac{a}{2}, \quad [L_2, X] = L_1,$$

$$X^4 - HX^2 + L_1^2 + aL_2 = 0$$

E6

$$H = \partial_x^2 + \partial_y^2 + \frac{a}{x^2}$$

Basis symmetries: ($M = x\partial_y - y\partial_x$)

$$L_1 = \frac{1}{2}\{M, \partial_x\} - \frac{ay}{x^2}, \quad L_2 = M^2 + \frac{ay^2}{x^2}, \quad X = \partial_y$$

Structure equations:

$$[L_1, L_2] = \{X, L_2\} + (2a + \frac{1}{2})X, \quad [L_1, X] = H - X^2, \quad [L_2, X] = 2L_1,$$

$$L_1^2 + \frac{1}{4}\{L_2, X^2\} + \frac{1}{2}XL_2X - L_2H + (a + \frac{3}{4})X^2 = 0$$

E14

$$H = \partial_x^2 + \partial_y^2 + \frac{b}{z^2}$$

Basis symmetries: (with $M = x\partial_y - y\partial_x$, $z = x + iy$, $\bar{z} = x - iy$.)

$$X = \partial_x - i\partial_y, \quad L_1 = \frac{i}{2}\{M, X\} + \frac{b}{z}, \quad L_2 = M^2 + \frac{bz}{z}$$

Structure equations:

$$[L_1, L_2] = -\{X, L_2\} - \frac{1}{2}X, \quad [X, L_1] = -X^2, \quad [X, L_2] = 2L_1,$$

$$L_1^2 + XL_2X - bH - \frac{1}{4}X^2 = 0$$

Contractions of nondegenerate systems. 1

Suppose we have a nondegenerate superintegrable system with generators $H, L_1, L_2, R = [L_1, L_2]$ and the usual structure equations, defining a quadratic algebra Q . If we make a change of basis to new generators $\tilde{H}, \tilde{L}_1, \tilde{L}_2$ and parameters $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3$ such that

$$\begin{pmatrix} \tilde{L}_1 \\ \tilde{L}_2 \\ \tilde{H} \end{pmatrix} = \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ 0 & 0 & A_{3,3} \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \\ H \end{pmatrix} + \begin{pmatrix} B_{1,1} & B_{1,2} & B_{1,3} \\ B_{2,1} & B_{2,2} & B_{2,3} \\ B_{3,1} & B_{3,2} & B_{3,3} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix},$$

$$\begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \tilde{a}_3 \end{pmatrix} = \begin{pmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

for some 3×3 constant matrices $A = (A_{i,j}), B, C$ such that $\det A \cdot \det C \neq 0$, we will have the same system with new structure equations of the same form for $\tilde{R} = [\tilde{L}_1, \tilde{L}_2], [\tilde{L}_j, \tilde{R}], \tilde{R}^2$, but with transformed structure constants.

Contraction of nondegenerate systems. 2

- Choose a continuous 1-parameter family of basis transformation matrices $A(\epsilon)$, $B(\epsilon)$, $C(\epsilon)$, $0 < \epsilon \leq 1$ such that $A(1) = C(1)$ is the identity matrix, $B(1) = 0$ and $\det A(\epsilon) \neq 0$, $\det C(\epsilon) \neq 0$.
- Now suppose as $\epsilon \rightarrow 0$ the basis change becomes singular, (i.e., the limits of A , B , C either do not exist or, if they exist do not satisfy $\det A(0) \det C(0) \neq 0$) but the structure equations involving $A(\epsilon)$, $B(\epsilon)$, $C(\epsilon)$, go to a limit, defining a new quadratic algebra Q' .
- We call Q' a contraction of Q in analogy with Lie algebra contractions.

There is a similar definition of a contraction of a degenerate superintegrable system.

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Free triplets

We say that the 2D system without potential,

$$H_0 = \Delta_2$$

and with 3 algebraically independent second-order symmetries is a **2nd order free triplet**. The possible spaces admitting free triplets are just those classified by Koenigs.

Note that every nondegenerate or degenerate superintegrable system defines a free triplet, simply by setting the parameters $a_j = 0$ in the potential. Similarly, this free triplet defines a **free quadratic algebra**, i.e., a quadratic algebra with all $a_j = 0$.

In general, a free triplet cannot be obtained as a restriction of a superintegrable system and its associated algebra does not close to a free quadratic algebra.

Closure Theorems

Theorem

With the single exception of the free nondegenerate quadratic algebra $E0$, a free triplet extends to a superintegrable system if and only if it generates a free quadratic algebra \tilde{Q} .

Theorem

A superintegrable system, degenerate or nondegenerate, with quadratic algebra Q , is uniquely determined by its free quadratic algebra \tilde{Q} .

Remark: These theorems are constructive. Given a free quadratic algebra \tilde{Q} one can compute the potential V and the symmetries of the quadratic algebra Q . The exceptional algebra is the flat space system

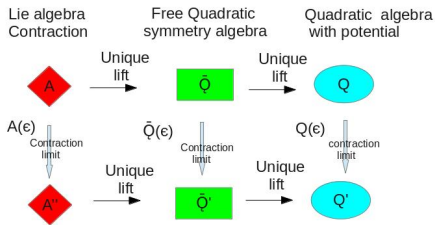
$$E0 : L_1 = (\partial_x + i\partial_y)^2, L_2 = (\partial_x + i\partial_y)(x\partial_y - y\partial_x), H = \partial_x^2 + \partial_y^2,$$

Lie algebra contractions \Rightarrow quadratic algebra contractions

Theorem

Every Lie algebra contraction of $A = e(2, \mathbb{C})$ or $A = o(3, \mathbb{C})$ induces uniquely a contraction of a free quadratic algebra \tilde{Q} based on A , which in turn induces uniquely a contraction of the quadratic algebra Q with potential. This is true for both classical and quantum algebras.

Contraction interplay



Induced Contraction of Quadratic algebras

Example: $\tilde{S}9 \rightarrow \tilde{E}1$

Use $so(3, \mathbb{C})$ contraction: $\{\mathcal{J}'_1, \mathcal{J}'_2, \mathcal{J}'_3\} = \{\epsilon \mathcal{J}_1, \epsilon \mathcal{J}_2, \mathcal{J}_3\} = \{p'_1, p'_2, J'_3\}$.

$$\mathcal{L}'_1 = \mathcal{J}_3^2 = \mathcal{L}_1$$

$$\mathcal{L}'_2 = \epsilon^2 \mathcal{J}_1^2 = p'_1{}^2 = \epsilon^2 \mathcal{L}_2$$

$$\mathcal{H}' = \epsilon^2 \mathcal{H} = \epsilon^2 (\mathcal{J}_1^2 + \mathcal{J}_2^2 + \mathcal{J}_3^2) = p'_1{}^2 + p'_2{}^2.$$

The primed symmetries are a basis for $\tilde{E}1$.

It follows from the closure theorems that under this Lie algebra contraction, the 4-dimensional solution space for the potentials V of $S9$ will deform continuously into the 4-dimensional solution space for the potentials V' of $E1$. Thus the target space of solutions V' is uniquely determined. **However, there is still the freedom of choosing bases for these two spaces.**

Example: $S_9 \rightarrow E_1$

In terms of coordinates ϕ, ψ on the sphere where $s_1 = \frac{\cos \phi}{\cosh \psi}$, $s_2 = \frac{\sin \phi}{\cosh \psi}$, $s_3 = \frac{\sinh \psi}{\cosh \psi}$, the S_9 potential is

$$V = \frac{a_1 \cosh^2 \psi}{\cos^2 \phi} + \frac{a_2 \cosh^2 \psi}{\sin^2 \phi} + \frac{a_3 \cosh^2 \psi}{\sinh^2 \psi} + a_4,$$

where we have made a choice of basis. For E_1 and using polar coordinates $y_1 = R, y_2 = \phi'$ where $x = e^R \cos \phi'$, $y = e^R \sin \phi'$, the general potential is

$$V' = b_1 e^{2R} + \frac{b_2 e^{-2R}}{\cos^2 \phi} + \frac{b_3 e^{-2R}}{\sin^2 \phi} + b_4.$$

The Lie algebra contraction from the sphere to flat space is expressed as $\psi \approx \frac{1}{2} \ln\left(\frac{1}{\epsilon}\right) - R$, $\phi = \phi'$. In the limit the 4 dimensional space of potentials must go to the 4 dimensional vector space of potentials V' .

Example: $S9 \rightarrow E1.2$

However our chosen basis functions for the $S9$ potential,

$$\frac{\cosh^2 \psi}{\cos^2 \phi}, \frac{\cosh^2 \psi}{\sin^2 \phi}, \frac{\cosh^2 \psi}{\sinh^2 \psi}, 1$$

will not go to a new basis in the limit; 2 basis functions become unbounded and 2 go to a constant. There are many ways to choose an ϵ dependent basis so that the limit can be taken. One of the simplest choices is

$$V^{(1)}(\epsilon) = \frac{1}{4\epsilon} \left(\frac{\cosh^2 \psi}{\sinh^2 \psi} - 1 \right) \rightarrow e^{2R},$$

$$V^{(2)}(\epsilon) = \epsilon \frac{\cosh^2 \psi}{\cos^2 \phi} \rightarrow \frac{e^{-2R}}{\cos^2 \phi}, \quad V^{(3)}(\epsilon) = \epsilon \frac{\cosh^2 \psi}{\sin^2 \phi} \rightarrow \frac{e^{-2R}}{\sin^2 \phi}, \quad V^{(4)}(\epsilon) = 1 \rightarrow 1.$$

Hypergeometric polynomials

$${}_4F_3 \left(\begin{matrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3 \end{matrix} ; x \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k (a_3)_k (a_4)_k}{(b_1)_k (b_2)_k (b_3)_k k!} x^k$$

$$(a)_0 = 1, \quad (a)_k = a(a+1)(a+2)\cdots(a+k-1) \text{ if } n \geq 1.$$

Here $k! = (1)_k$. If $a_1 = -n$ for n a nonnegative integer then the sum is finite with $n+1$ terms.

Wilson polynomials (of order n in t^2):

$$\Phi_n(\alpha, \beta, \gamma, \delta; t) = {}_4F_3 \left(\begin{matrix} -n, & \alpha + \beta + \gamma + \delta + n - 1, & \alpha - t, & \alpha + t \\ \alpha + \beta, & \alpha + \gamma, & \alpha + \delta \end{matrix} ; 1 \right)$$

Racah polynomials: If $\alpha + \beta = -m$ for m a nonnegative integer then only finite set $\Phi_0, \Phi_1, \dots, \Phi_m$

The Askey Scheme. 1

The irreducible representations of S_9 have a realization in terms of difference operators in 1 variable, exactly the structure algebra for the Wilson and Racah polynomials! By contracting these representations to obtain the representations of the quadratic symmetry algebras of the other superintegrable systems we obtain the full Askey scheme of orthogonal hypergeometric polynomials. This relationship ties the structure equations directly to physical phenomena.

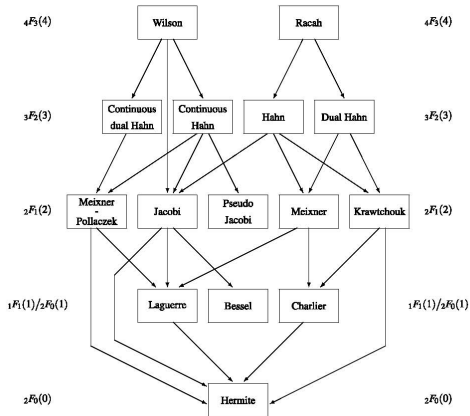
The Askey Scheme. 2

The Askey Scheme organizes the theory of hypergeometric orthogonal polynomials of one variable by exhibiting the relations such that each of these polynomials can be obtained as a sequence of pointwise limits from either the Racah polynomials in the finite dimensional case or the Wilson polynomials in the infinite dimensional case.

$$\lim_{\tau \rightarrow \infty} \Phi_n(\tau) = \Phi'_n$$

The Askey Scheme. 3

ASKEY SCHEME OF HYPERGEOMETRIC ORTHOGONAL POLYNOMIALS



What are models?

- A **representation of a quadratic algebra Q** is a homomorphism of Q into the associative algebra of linear operators on some vector space: products go to products, commutators to commutators, etc.
- A **model M** is a faithful representation of Q in which the vector space is a space of polynomials in one complex variable and the action is via differential/difference operators acting on that space. We study classes of irreducible representations realized by these models.

What are model contractions?

- Suppose a quadratic algebra Q contracts to a algebra Q' via a continuous family of transformations indexed by ϵ . If we have a model M of Q we can try to “save” this representation by passing through a continuous family of models $M(\epsilon)$ of $Q(\epsilon)$ to obtain a model M' of Q' .
- **We will show that as a byproduct of contractions to systems from S9 for which we save representations in the limit, we obtain the Askey Scheme for hypergeometric orthogonal polynomials.**

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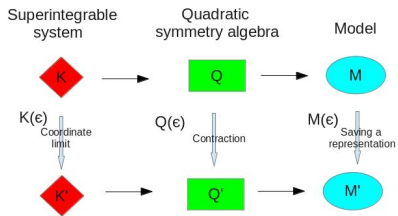
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Model interplay



The Contraction of Models

The S9 difference operator model. 1

$$L_2 f_{n,m} = (-4t^2 - \frac{1}{2} + B_1^2 + B_3^2) f_{n,m},$$

$$L_3 f_{n,m} = (-4\tau^* \tau - 2[B_1 + 1][B_2 + 1] + \frac{1}{2}) f_{n,m},$$

$$H = L_1 + L_2 + L_3 + \frac{3}{4} - (B_1^2 + B_2^2 + B_3^2) = -4(m+1)(B_1 + B_2 + B_3 + m + 1) - 2(B_1 B_2 + B_1 B_3 + B_2 B_3) + \frac{3}{4} - (B_1^2 + B_2^2 + B_3^2).$$

Here $n = 0, 1, \dots, m$ if m is a nonnegative integer and $n = 0, 1, \dots$ otherwise. Also

$$a_j = \frac{1}{4} - B_j^2, \quad \alpha = -(B_1 + B_3 + 1)/2 - m, \quad \beta = (B_1 + B_3 + 1)/2, \quad \gamma = (B_1 - B_3 + 1)/2, \quad \delta = (B_1 + B_3 - 1)/2 + B_2 + m + 2,$$

$$E^A F(t) = F(t+A), \quad \tau = \frac{1}{2t}(E^{1/2} - E^{-1/2}), \quad \tau^* = \frac{1}{2t} \left[(\alpha+t)(\beta+t)(\gamma+t)(\delta+t)E^{1/2} - (\alpha-t)(\beta-t)(\gamma-t)(\delta-t)E^{-1/2} \right],$$

$$w_n(t^2) = (\alpha + \beta)_n (\alpha + \gamma)_n (\alpha + \delta)_n {}_4F_3 \left(\begin{matrix} -n, & \alpha + \beta + \gamma + \delta + n - 1, & \alpha - t, & \alpha + t \\ \alpha + \beta, & \alpha + \gamma, & \alpha + \delta \end{matrix} ; 1 \right)$$

$$= (\alpha + \beta)_n (\alpha + \gamma)_n (\alpha + \delta)_n \Phi_n^{(\alpha, \beta, \gamma, \delta)}(t^2), \quad \Phi_n \equiv f_{n,m},$$

$$\tau^* \tau \Phi_n = n(n + \alpha + \beta + \gamma + \delta - 1) \Phi_n,$$

where $(a)_n$ is the Pochhammer symbol and ${}_4F_3(1)$ is a hypergeometric function of unit argument. The polynomial $w_n(t^2)$ is symmetric in $\alpha, \beta, \gamma, \delta$. For the finite dimensional representations the spectrum of t^2 is $\{(\alpha + k)^2, k = 0, 1, \dots, m\}$ and the orthogonal basis eigenfunctions are Racah polynomials. In the infinite dimensional case they are Wilson polynomials.

The S9 difference operator model. 1

$$L_2 f_{n,m} = (-4t^2 - \frac{1}{2} + B_1^2 + B_3^2) f_{n,m},$$

$$L_3 f_{n,m} = (-4\tau^* \tau - 2[B_1 + 1][B_2 + 1] + \frac{1}{2}) f_{n,m},$$

$$H = L_1 + L_2 + L_3 + \frac{3}{4} - (B_1^2 + B_2^2 + B_3^2) = -4(m+1)(B_1 + B_2 + B_3 + m + 1) - 2(B_1 B_2 + B_1 B_3 + B_2 B_3) + \frac{3}{4} - (B_1^2 + B_2^2 + B_3^2).$$

Here $n = 0, 1, \dots, m$ if m is a nonnegative integer and $n = 0, 1, \dots$ otherwise. Also

$$a_j = \frac{1}{4} - B_j^2, \quad \alpha = -(B_1 + B_3 + 1)/2 - m, \quad \beta = (B_1 + B_3 + 1)/2, \quad \gamma = (B_1 - B_3 + 1)/2, \quad \delta = (B_1 + B_3 - 1)/2 + B_2 + m + 2,$$

$$E^A F(t) = F(t+A), \quad \tau = \frac{1}{2t}(E^{1/2} - E^{-1/2}), \quad \tau^* = \frac{1}{2t} \left[(\alpha+t)(\beta+t)(\gamma+t)(\delta+t)E^{1/2} - (\alpha-t)(\beta-t)(\gamma-t)(\delta-t)E^{-1/2} \right],$$

$$w_n(t^2) = (\alpha + \beta)_n (\alpha + \gamma)_n (\alpha + \delta)_n {}_4F_3 \left(\begin{matrix} -n, & \alpha + \beta + \gamma + \delta + n - 1, & \alpha - t, & \alpha + t \\ \alpha + \beta, & \alpha + \gamma, & \alpha + \delta & \end{matrix} ; 1 \right)$$

$$= (\alpha + \beta)_n (\alpha + \gamma)_n (\alpha + \delta)_n \Phi_n^{(\alpha, \beta, \gamma, \delta)}(t^2), \quad \Phi_n \equiv f_{n,m},$$

$$\tau^* \tau \Phi_n = n(n + \alpha + \beta + \gamma + \delta - 1) \Phi_n,$$

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The S9 difference operator model. 2

The action of L_2 and L_3 on an L_3 eigenbasis is

$$L_2 f_{n,m} = -4K(n+1, n)f_{n+1,m} - 4K(n, n)f_{n,m} - 4K(n-1, n)f_{n-1,m} + (B_1^2 + B_3^2 - \frac{1}{2})f_{n,m},$$

$$L_3 f_{n,m} = -(4n^2 + 4n[B_1 + B_2 + 1] + 2[B_1 + 1][B_2 + 1] - \frac{1}{2})f_{n,m},$$

$$K(n+1, n) = \frac{(B_1 + B_2 + n + 1)(n - m)(-B_3 - m + n)(B_2 + n + 1)}{(B_1 + B_2 + 2n + 1)(B_1 + B_2 + 2n + 2)},$$

$$K(n-1, n) = \frac{n(B_1 + n)(B_1 + B_2 + B_3 + m + n + 1)(B_1 + B_2 + m + n + 1)}{(B_1 + B_2 + 2n)(B_1 + B_2 + 2n + 1)},$$

$$K(n, n) = \left[\frac{B_1 + B_2 + 2m + 1}{2} \right]^2 - K(n+1, n) - K(n-1, n),$$

Example: $S9 \rightarrow E1.1$

Quantum system limit:

$$H_{S9} = J_1^2 + J_2^2 + J_3^2 + \frac{a_1}{s_1^2} + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2}$$

where $J_3 = s_1 \partial_{s_2} - s_2 \partial_{s_1}$ and J_2, J_3 are obtained by cyclic permutations of the indices 1, 2, 3.

$$H_{E1} = \partial_x^2 + \partial_y^2 - \omega^2(x^2 + y^2) + \frac{b_1}{x^2} + \frac{b_2}{y^2}$$

In $S9$ we contract about the north pole of the unit sphere. Set

$$s_1 = \sqrt{\epsilon}x, \quad s_2 = \sqrt{\epsilon}y, \quad s_3 = \sqrt{1 - s_1^2 - s_2^2} \approx 1 - \frac{\epsilon}{2}(x^2 + y^2),$$

$$a'_1 = b_2 = a_1, \quad a'_2 = b_1 = a_2, \quad a'_3 = -\omega^2 = \epsilon^2 a_3,$$

in H_{S9} to get $\epsilon(H_{S9} - a_3) \rightarrow H_{E1}$ as $\epsilon \rightarrow 0$.

Quadratic algebra contraction:

$$L'_1 = \epsilon L_1, \quad L'_2 = \epsilon L_2, \quad L'_3 = L_3, \quad H' = \epsilon(H - a_3)$$

$$R' = \epsilon R, \quad a'_1 = b_2 = a_1, \quad a'_2 = b_1 = a_2, \quad a'_3 = -\omega^2 = \epsilon^2 a_3.$$

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Example: $S9 \rightarrow E1.2$

Saving a representation: We set

$$t = -x + B_3/2 + (B_1 + 1)/2 + m, \quad B_3 = \frac{\omega}{\epsilon} \rightarrow \infty \implies$$

$$f'_{n,m} = {}_3F_2 \left(\begin{matrix} -n, & B_1 + B_2 + n + 1, & -x \\ -m, & B_2 + 1 \end{matrix} ; 1 \right) = Q_n(x; B_2, B_1, m)$$

where the Q_n are Hahn polynomials. We have the model

$$L'_2 f'_{n,m} = 2\omega(2x - 2m - B_1 - 1)f'_{n,m} = -4K'(n+1, n)f'_{n+1,m} - 4K'(n, n)f'_{n,m} - 4K'(n-1, n)f'_{n-1,m},$$

$$L'_3 f'_{n,m} = - \left(4n^2 + 4n[B_1 + B_2 + 1] + 2[B_1 + 1][B_2 + 1] - \frac{1}{2} \right) f'_{n,m} =$$

$$\left[-4(x - m)(x + B_2 + 1)E_x^1 + 4x(x - m - B_1 - 1)E_x^{-1} + 8x^2 + 4x(B_1 + B_2 - 2m) \right. \\ \left. - 4m(B_2 + 1) - 2(B_2 + 1)(B_1 + 1) + \frac{1}{2} \right] f'_{n,m},$$

$$H' = L'_1 + L'_2 = -2\omega(2m + 2 + B_1 + B_2).$$

Here the K' are the appropriate limits of the K as $B_3 \rightarrow \infty$.

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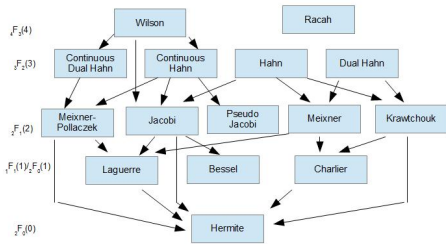
$$L'_2 f'_{n,m} = 2\omega(2x - 2m - B_1 - 1)f'_{n,m} = -4K'(n+1, n)f'_{n+1,m} - 4K'(n, n)f'_{n,m} - 4K'(n-1, n)f'_{n-1,m},$$

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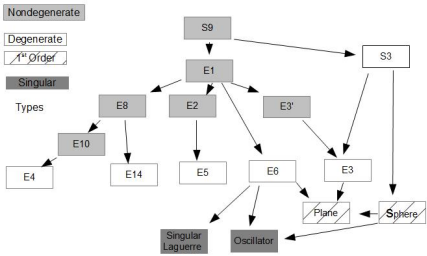
$$\left[-4(x - m)(x + B_2 + 1)E_x^1 + 4x(x - m - B_1 - 1)E_x^{-1} + 8x^2 + 4x(B_1 + B_2 - 2m) \right. \\ \left. - 4m(B_2 + 1) - 2(B_2 + 1)(B_1 + 1) + \frac{1}{2} \right] f'_{n,m},$$

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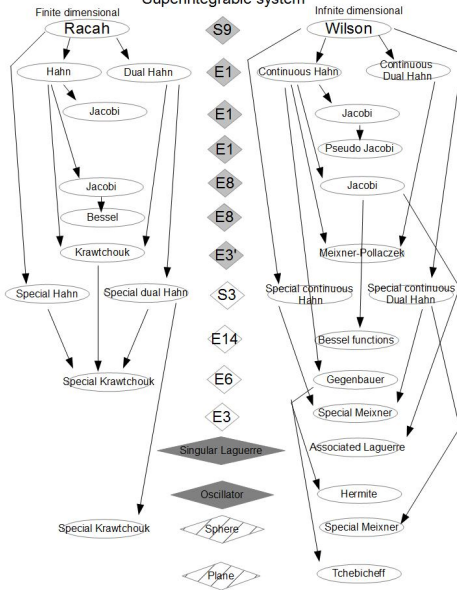


Askey Scheme of Hypergeometric Orthogonal Polynomials



Partial list of contractions of superintegrable systems

Superintegrable system



Wrap-up. 1

- Free quadratic algebras uniquely determine associated superintegrable systems with potential.
- A contraction of a free quadratic algebra to another uniquely determines a contraction of the associated superintegrable systems.
- For a 2D superintegrable systems on a constant curvature space these contractions can be induced by Lie algebra contractions of the underlying Lie symmetry algebra.
- Every 2D superintegrable system is obtained either as a sequence of contractions from S_9 or is Stäckel equivalent to a system that is so obtained.

Wrap-up. 2

- Taking contractions step-by-step from the $S9$ model we can recover the Askey Scheme. However, the contraction method is more general. It applies to all special functions that arise from the quantum systems via separation of variables, not just polynomials of hypergeometric type, and it extends to higher dimensions.
- The special functions arising from the models can be described as the coefficients in the expansion of one separable eigenbasis for the original quantum system in terms of another separable eigenbasis.
- The functions in the Askey Scheme are just those hypergeometric polynomials that arise as the expansion coefficients relating two separable eigenbases that are *both* of hypergeometric type. Thus, there are some contractions which do not fit in the Askey scheme since the physical system fails to have such a pair of separable eigenbases.

Wrap-up. 3

- Even though 2nd order 2D nondegenerate superintegrable systems admit no group symmetry, their structure is determined completely by the underlying symmetry of constant curvature spaces.
- To extend the method to Askey-Wilson polynomials we would need to find appropriate q -quantum mechanical systems with q -symmetry algebras and we have not yet been able to do so.